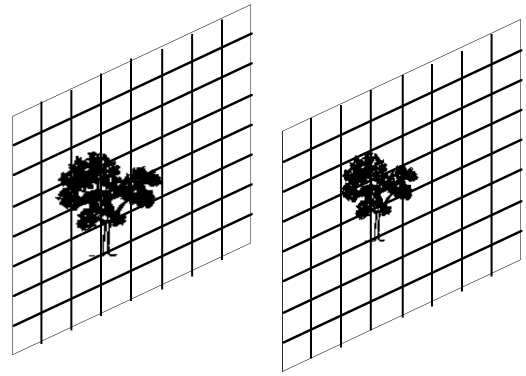
DPCM IN VIDEO CODING

Though not usually called DPCM, the most commonly used methods of video coding use a form of DPCM, e.g. MPEG, H.26X, HDTV, satellite "Direct TV", DVD.



Prediction is done on frame-by-frame basis, with prediction of a frame being the decoded reconstruction of previous frame, or a "motion-compensated" version thereof. In latter case, coder has a forward-adaptive component in addition to the backward adaption.

Example: (Netravali & Haskell, Digital Pictures, p. 331)

Prediction Coefficients				
left pixel, curr. frame	left pixel prev. frame	same pixel prev frame	MSPE	Pred. gain[1]
1			53.1	12.7 dB
1	-1/2	1/2	29.8	15.3 dB
3/4	-1/2	3/4	27.9	15.5 dB
7/8	-5/8	3/4	26.3	15.8 dB

[1]Based on educated guess that $\sigma^2 = 1000$.

DESIGN AND PERFORMANCE -- HIGH RESOLUTION ANALYSIS

Warning: Though this analysis is almost universally accepted, it has never been satisfactorily proven to be correct.

Assume source is stationary, zero-mean random process.

It can be shown that under ordinary conditions $\{(X_i,U_i,V_i,Y_i)\}$ is asymptotically stationary. So we assume it is stationary. Therefore,

$$D = E(X_i-Y_i)^2 = E(U_i-Q(U_i))^2 \qquad \text{(same for all i)}$$

Key assumption:

When R is large and quantizer is well designed

$$D \cong E(\tilde{U}_i - Q(\tilde{U}_i))^2$$

where $\tilde{U}_i = X_i - g(X_{i-1}, X_{i-2}, \ldots)$. $\tilde{U}_i \neq U_i$ but $\tilde{U}_i \cong U_i$

Note: $\tilde{U}$ is a stationary random process.

Key assumption implies

least possible distortion of DPCM with rate R

$\cong$ least possible distortion of SQ with rate R encoding $\tilde{U}$, minimized over choice of predictor g

It follows that for FLC or VLC,

$$\delta_{dpcm}(R) \cong \min_g \delta_{\tilde{U},sq}(R) \cong \frac{1}{12}\left(\min_g \sigma^2_{\tilde{U}} \alpha_{\tilde{U}}\right) 2^{-2R}$$

where $\alpha_{\tilde{U}} = \begin{cases} \beta_{\tilde{U}}, & \text{for FLC} \\ \eta_{\tilde{U}}, & \text{for VLC} \end{cases}$

Conclusions:

A good (but not necessarily optimal) way to design DPCM is to

(a) Choose g to minimize $\sigma^2_{\tilde{U}} \alpha_{\tilde{U}}$

(b) Choose q to achieve $\delta_{\tilde{U},sq}(R)$

Common situation:

The pdf of $\tilde{U}$ is similar to a scaled version of that X, which implies, $\beta_{\tilde{U}} \cong \beta_X$ and $\eta_{\tilde{U}} \cong \eta_X$. Indeed, if X is Gaussian source, then so is $\tilde{U}$, and these approximations become exact. In this case (a) becomes

(a') Choose g to minimize $\sigma^2_{\tilde{U}}$

Therefore,

$$\delta_{dpcm}(R) \cong \frac{1}{12} M \alpha_X 2^{-2R}, \quad \text{where } M = \min_g \sigma^2_{\tilde{U}} = \text{MSPE}$$

$$\cong \frac{M}{\sigma^2} \delta_{sq}(R)$$

$$S_{dpcm}(R) \cong S_{sq}(R) + G_N \text{ dB}, \quad \text{where } G_N = 10 \log_{10} \frac{\sigma^2}{M} \text{ dB} = \text{pred'n gain}$$

That is, gain of DPCM over SQ approximately equals the prediction gain.

Example: First-order autoregressive (AR) Gaussian source.

Assume X is stationary and

$$X_i = \rho X_{i-1} + Z_i \qquad (*)$$

where $-1 < a < 1$, Z_i's are IID Gaussian with zero mean and variances σ_Z^2. Z_i is independent of X_j, $j < i$. ($\rho = .9$ is a typical value.) Then

1) $EX_i = 0$:

Derivation: Let $m = EX_i$. It does not depend on i since X is stationary. Taking expected values of both sides of $(*)$ gives

$m = \rho m + 0$, which implies $m = 0$.

2) $\sigma_X^2 = EX_i^2 = \dfrac{\sigma_Z^2}{1-\rho^2}$ $\qquad$ ($= 5.26\sigma_Z^2$ if $\rho = .9$)

Derivation: Let $s = EX_i^2$. Taking expected square value of both sides of $(*)$ gives, $s = \rho^2 s + 2\rho\, EX_{i-1}Z_i + \sigma_Z^2$.

Since X_{i-1} and Z_i are independent,

$EX_{i-1}Z_i = EX_{i-1}\, EZ_i = 0$.

Therefore, $s = \rho^2 s + \sigma_Z^2 \Rightarrow s = \dfrac{\sigma_Z^2}{(1-\rho^2)}$.

3) $R_X(k)$ = autocorrelation function = $E\, X_i X_{i+k} = \sigma_X^2 \rho^{|k|}$

$\Rightarrow \rho$ = correlation coefficent of X = $\dfrac{\mathrm{cov}(X_i X_{i+1})}{\sigma_X^2}$

Derivation: $R_X(0) = E\, X_i^2 = \sigma_X^2$. We now show $R_X(k) = \rho R_X(k-1)$.

$R_X(k) = EX_iX_{i-k} = E\,(\rho X_{i-1}+Z_i)X_{i-k} = \rho\, EX_{i-1}X_{i-k} + EZ_iX_{i-k} = \rho\, R_X(k) + 0.$

4) Best predictor for X_i based on $X_{i-1}, X_{i-2}, \ldots$ is $\tilde{X}_i = \rho X_{i-1}$ with MMSPE $= \sigma_Z^2$

Derivation: It's not easy to use $\underline{a} = K^{-1}\underline{r}$. However, we may use the orthogonality principle, which indicates that $\tilde{X}_i$ is optimal if and only if

$E\,(X_i-\tilde{X}_i)X_{i-j}$, for $j = 1,\ldots,N$.

We find $E\,(X_i-\rho X_{i-1})X_{i-j} = E\,(\rho X_{i-1}+Z_i-\rho X_{i-1})X_{i-j} = E\, Z_iX_{i-j}$

$= E\, Z_i\, E\, X_{i-j} = 0$, since Z_i indep. of past X's

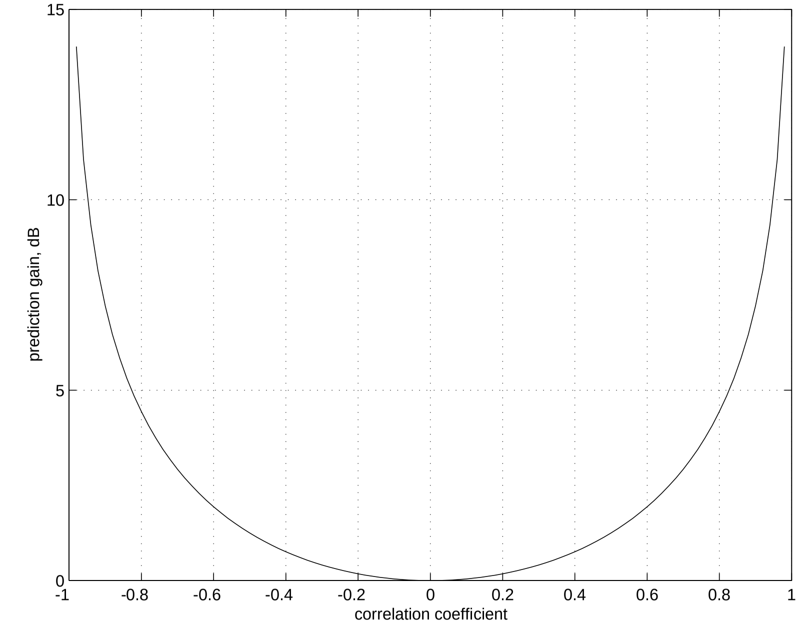
It follows from orthog. principal that $\tilde{X}_i = \rho X_{i-1}$ is optimal.

MMSPE $= E(X_i - \rho X_{i-1})^2 = E(\rho X_{i-1} + Z_i - \rho X_{i-1})^2 = E\, Z_i^2 = \sigma_Z^2$

Therefore, for $N \geq 1$, the prediction gain (which is the gain of DPCM over SQ) is

$$G_N = 10 \log_{10} \frac{\sigma_X^2}{\mathrm{MMSE}} = 10 \log_{10} \frac{1}{1-\rho^2} \qquad (= 7.2 \text{ dB if } \rho = .9)$$

G_N vs. ρ



COMPARISON OF DPCM AND TRANSFORM CODING

Consider a stationary, Gaussian source and large R.

For k-dimensional transform coding:

$$\delta_{tr}(k,R) \cong \frac{1}{12} |K^{(k)}|^{1/k} \alpha_G 2^{-2R}$$

For DPCM with kth-order linear prediction

$$\delta_{dpcm}(k,R) \cong \frac{1}{12} M_k \alpha_G 2^{-2R}$$

where M_k is MSPE of optimal kth-order linear prediction for X_i from $X_{i-k},\ldots,X_{i-1}$.

Fact A: $|K^{(k)}|^{1/k} \geq M_k$.

Proof: $|K^{(k)}|^{1/k} = \left(\sigma_X^2 \prod_{i=1}^{k-1} M_i\right)^{1/k}$ this was proved in the transform coding notes

$\geq M_k$ because all terms being averaged are $\geq M_k$

It follows from this fact that DPCM with kth-order prediction is at least as good as k-dimensional transform coding.

Fact B: $\lim_{k\to\infty} M_k = \lim_{k\to\infty} |K^{(k)}|^{1/k} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S(\omega)\, d\omega\right\}$

Proof: First we note that since the M_k's are nonnegative and nonincreasing, they converge to a limit. Secondly,

$$|K^{(k)}|^{1/k} = \left(\sigma_X^2 \prod_{i=1}^{k-1} M_i\right)^{1/k},$$

and the latter expression converges to $\lim_{k\to\infty} M_k$ because when one takes the geometric average of the first k terms of a convergent sequence (the M_k's), that geometric average converges to the same value as the sequence. Hence,

$$\lim_{k\to\infty} |K^{(k)}|^{1/k} = \lim_{k\to\infty} M_k$$

It follows from this fact that for large k, DPCM and Transform Coding have the same performance.

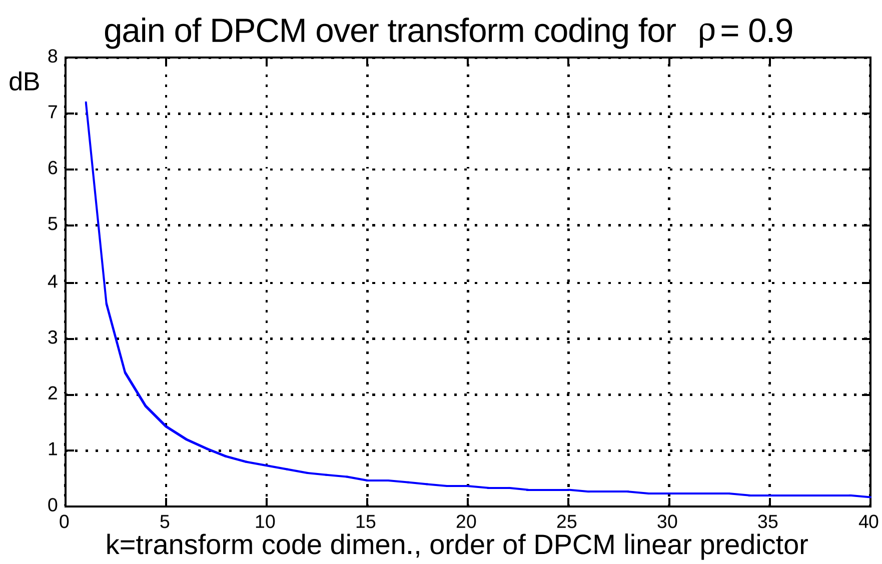
However, consider the example of a first-order AR Gaussian source.

Then, since $M_1 = \lim_{k\to\infty} M_k = Q$ = "one-step prediction error"

$$\delta_{dpcm}(1,R) \cong \delta_{tr}(\infty,R)$$

The SNR gain in dB of DPCM with kth-order linear prediction over k-dimensional transform coding is

$$10 \log_{10} \frac{\delta_{tr}(1,R)}{\delta_{dpcm}(k,R)} = 10 \log_{10} \frac{(1-\rho^2)^{(k-1)/k}}{1-\rho^2} = -\frac{10}{k} \log_{10} (1-\rho^2)$$



Similarly, for an Nth-order AR Gaussian source,

$$\delta_{dpcm}(N,R) \cong \delta_{tr}(\infty,R)$$

WHAT COULD BE WRONG WITH THE HIGH-RESOLUTION ANALYSIS?

For large R, we assumed $E(U-Q(U))^2 \cong E(\tilde{U}_i-Q(\tilde{U}_i))^2$.

This is obviously true because both sides are approximatetly 0.

But it hasn't been shown that $E(U-Q(_N U))^2 / E(\tilde{U}_i-Q_N(\tilde{U}_i))^2 \to 1$ as $N \to \infty$.

So it is not clear that as R increases, the pdf of $\tilde{U}$ becomes so similar to that of U that $\delta_{U,sq}(R)/\delta_{\tilde{U},sq}(R) \to 1$.

Since both terms in the ratio go to zero, it wouldn't take much of a difference between the terms to make the ratio not converge to one.

WHAT IF THE TRADITIONAL HIGH-RESOLUTION ANALYSIS IS WRONG?

I don't think it could be wrong by much.

Nitadori made an analysis of DPCM that takes into account the fact that $\tilde{U}$ contains quantization as well as prediction errors. See Jayant and Noll, p. 278, 279. I don't believe this analysis has been checked thoroughly for correctness.

Naraghi-Pour (UM Thesis, 1987) has shown that the best predictor is not always the predictor that minimizes the $E\tilde{U}^2$.

DESIGN WITHOUT THE HIGH-RESOLUTION ASSUMPTION

A "hypothetical" design algorithm.

Make an initial choice of the predictor and iterate the following two steps.

1. Optimize quantizer for a given predictor
   a) Make an initial choice of the quantizer $Q^{(0)}$.
      Let $k = 1$.
   b) Find the steady-state pdf $f^{(k)}(u)$ of U assuming the quantizer is $Q^{(k-1)}$.
      (There is an iterative algorithm for doing this.)
   c) Let $Q^{(k)}$ be the optimal quantizer for pdf $f^{(k)}(u)$.
   d) Let $k = k+1$ and go to b).
2. Optimize predictor for a given quantizer
   a) Make an initial choice of predictor $\underline{a}^{(0)}$.
      Let $k = 1$.
   b) Find the steady-state moments of $X_i, Y_{i-1}, \ldots, Y_{i-N}$: EX_iY_{i-j}'s, $EY_{i-m}Y_{i-n}$'s.
   c) Let $\underline{a}^{(k)}$ be the optimal predictor for these statistics.
   d) Let $k = k+1$ and go to b).

FURTHER NOTES:

- It is not clear that for an optimal DPCM system, Q is the optimal quantizer for U.

  One might think that if Q weren't optimal for U, then one could improve DPCM by making it optimal for U. However, changing Q, changes the pdf of U, so it's not clear that changing Q would help.

- Similarly, it is not clear that for an optimal DPCM system, g is the optimal predictor for X based on $Y_{i-N},\ldots,Y_{i-1}$.

  One might think that if g weren't optimal for X, then one could improve DPCM by making it optimal. However, changing g, changes the moments of the Y's, so it's not clear that changing g would help.

- DPCM can be made backward adaptive in various ways, e.g. by adapting the quantizer or the predictor.

- Jayant and Noll is an excellent reference on fancier versions of DPCM.

ANALYSIS WITHOUT THE HIGH-RESOLUTION ASSUMPTION

- There are iterative algorithms for computing actual distortion.