High-Resolution Analysis of Quantizer Distortion

For fixed-rate, memoryless VQ, there are two principal results of high-resolution analysis:

Bennett's Integral

A formula for the mean-squared error distortion of a "high-resolution" VQ in terms of its "gross" characteristics.

A "high-resolution" VQ is one with "small" cells, so it has "small" distortion and, usually, "many" cells, and "large" rate. Later we'll see roughly how "small", how "many" and how "large" are adequate.

Question: What "gross" characteristics" distinguish different high-resolution quantizers?

Zador's formula

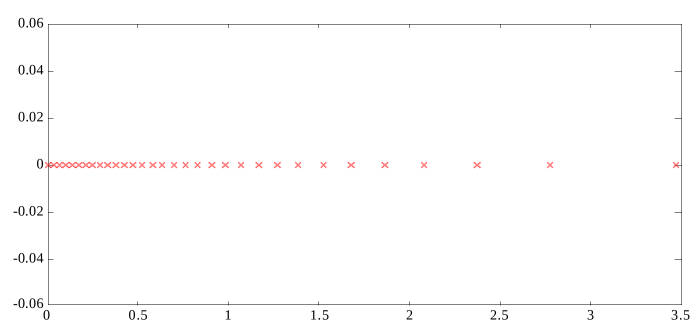
An approximation to the OPTA function $\delta(k,R)$. It applies when R is large.

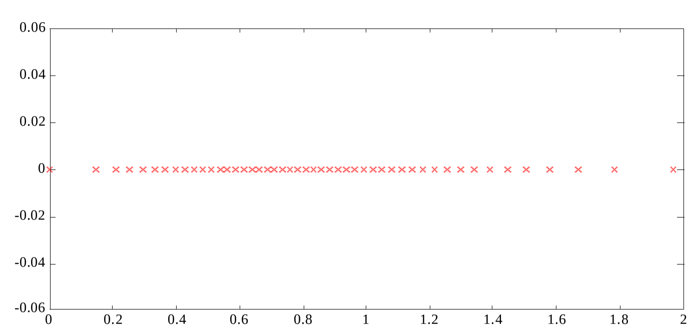
Question: We'd much rather have small rate, than large rate. Why should we be interested in high-resolution formulas, which apply when rate is larger?

Answer: We'll see that these formulas are accurate when $R \geq 3$, and accurate enough even when $R \cong 2$ to provide excellent insight.

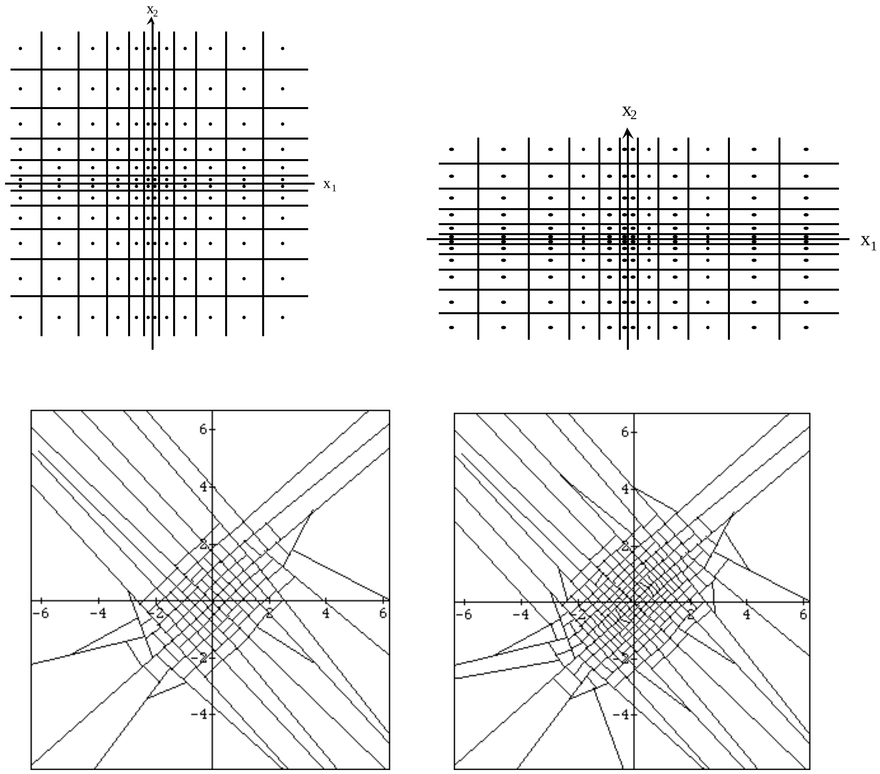
Examples of High-Resolution Quantizers:

k=1 (scalar quantizers)

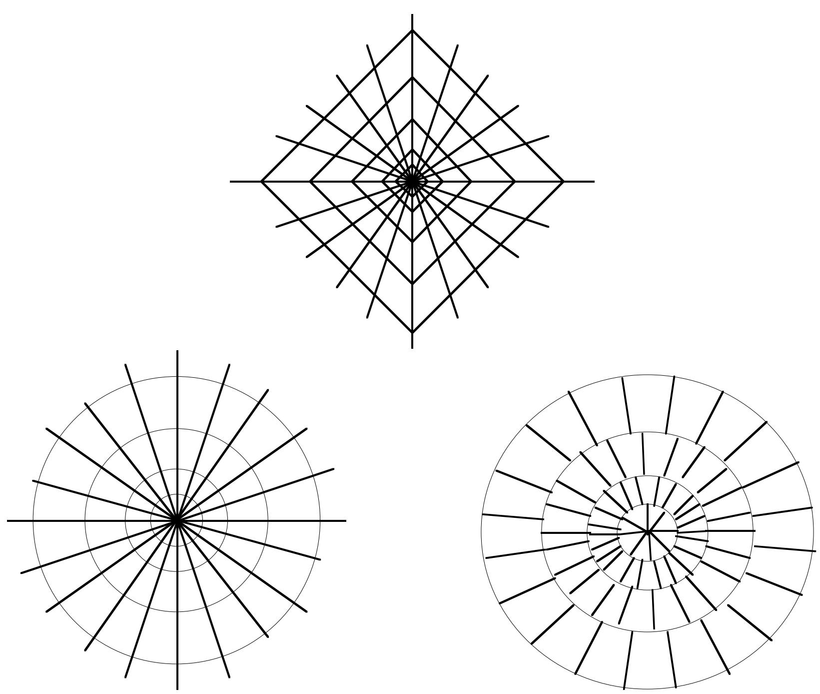




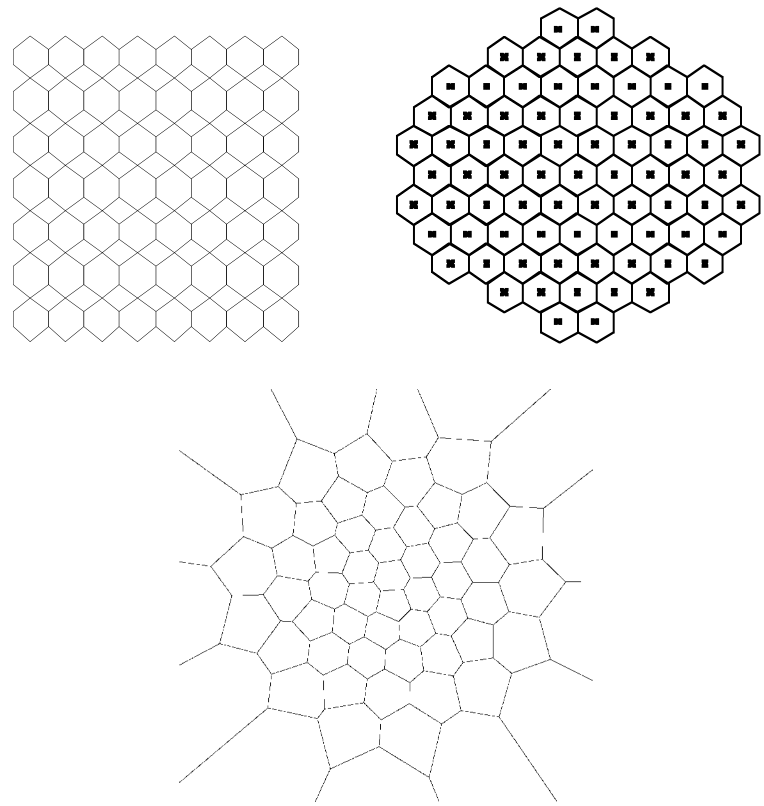
k=2



Bennett-3



Bennett-4



The Key Gross Characteristics of a High-Resolution Quantizer:

- Dimension k
- Size M
- The distribution or density of points/cells over R^k
- Some way of characterizing the shapes of the cells as a function of location in R^k.

Case 1: Bennett's Integral for Quantizers with Congruent Cells

Theorem

The MSE distortion of a k-dimensional VQ with size M and "small" congruent, or at least approximately congruent, cells is approximately given by

$$D \cong \frac{1}{M^{2/k}}\, m \int \frac{1}{\lambda^{2/k}(\underline{x})}\, f_{\underline{X}}(\underline{x})\, d\underline{x}$$

where

m = quantity, to be defined later, that depends only on the shape of the cells, but not on their size, nor on the source density

$\lambda(\underline{x})$ = "quantization density function". This is a function that characterizes, approximately, the density of cells/points in the vicinity of $\underline{x}$. It is also called the "point" or "cell density" function.

The formula above is called "Bennett's integral". It was first derived by Bennett[1] for scalar quantizers, and then extended to VQ's with congruent cells by Gersho[2].

Our derivation comes later. Examples of VQ's with congruent cells can be found on pages 4 and 5. Later we also drop the congruent cell restriction.

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[1] W. Bennett, "Spectra of quantized signals," *Bell Syst. Tech. J.*, vol. 27, pp. 446-472, July 1948.
[2] A. Gersho, "Asymptotically optimal block quantizers," *IEEE Trans. Inform. Thy.*, vol. 25, pp. 373-380, 1979.

Properties of a Quantization Density $\lambda(\underline{x})$

The following are properties that you would expect of any function that deserves to be called a "density".

1. $\lambda(\underline{x}) \geq 0$, $\int \lambda(\underline{x})\, d\underline{x} = 1$
2. $\int_A \lambda(\underline{x})\, d\underline{x} \cong$ fraction of codevectors (or cells) in region A
3. if A is small, then $\lambda(\underline{x})\, |A| \cong \frac{\#\,\text{cells/points in } A}{M}$

   (assuming A is much larger than cells in the vicinity of $\underline{x}$)
4. Ordinarily $\lambda(\underline{x})$ is a smooth or piecewise smooth function.
5. $\lambda(\underline{x}) \cong \frac{1}{M|S_i|}$ when $\underline{x} \in S_i$ or equivalently, $|S_i| \cong \frac{1}{M\lambda(\underline{x})}$

   Why? In a small region A containing $\underline{x}$, one expects most cells to have approximately the same volume.

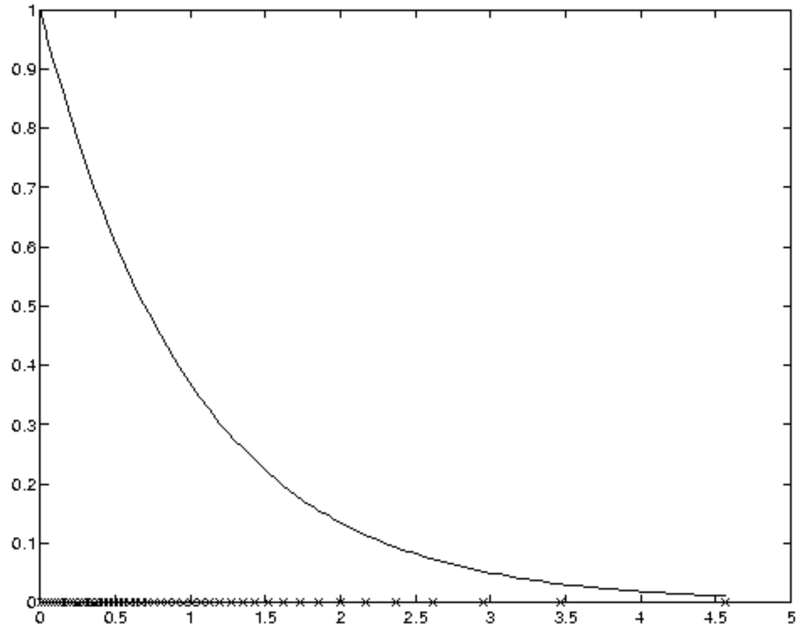
   Therefore, $\#\,\text{cells in } A \cong \frac{|A|}{\text{cell vol}}$.

   And by 3, $\lambda(\underline{x})\, |A| \cong \frac{\#\,\text{cells in } A}{M} = \frac{|A|/(\text{cell vol})}{M}$

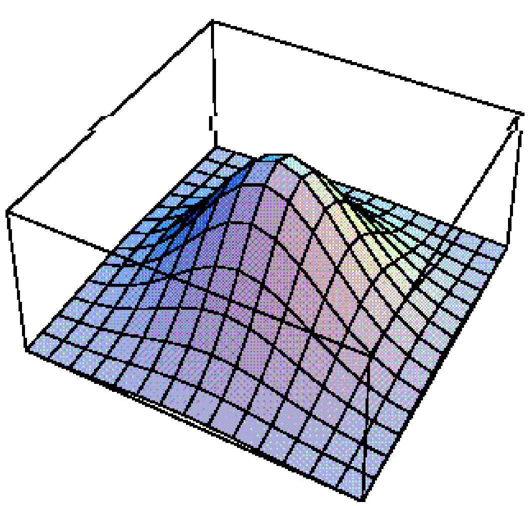
   $$\Rightarrow \lambda(\underline{x}) \cong \frac{1}{M|S_i|}$$

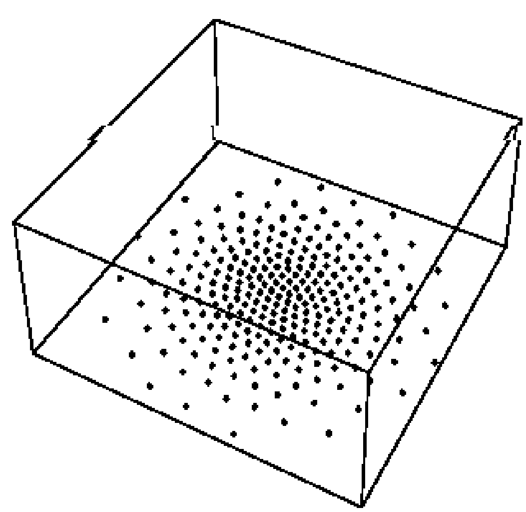
Examples of Point Densities

k=1



Optimal Quantizer for a 2-dimensional IID Gaussian random vector and its point density





Derivation of Bennett's Integral

$$D = \frac{1}{k} E\,||\underline{X}-Q(\underline{X})||^2 = \frac{1}{k}\sum_{i=1}^{M}\int_{S_i}||\underline{x}-\underline{w}_i||^2 f_{\underline{X}}(\underline{x})\,d\underline{x}$$

$$\cong \frac{1}{k}\sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\int_{S_i}||\underline{x}-\underline{w}_i||^2 d\underline{x} \quad \text{because } f_{\underline{X}}(\underline{x}) \cong f_{\underline{X}}(\underline{w}_i) \text{ when } \underline{x}\in S_i$$

$f_{\underline{X}}(\underline{x}) \cong f_{\underline{X}}(\underline{w}_i)$ when $\underline{x}\in S_i$ is a valid assumption because most S_i's are small, and because $f_{\underline{X}}(\underline{x})$ is ordinarily either continuous, in which case it changes little or over a small cell, or piecewise continuous, in which case it is continuous on most cells and the previous argument applies.

$$= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\,\frac{1}{k}\,M(S_i,\underline{w}_i)$$

where $M(S,\underline{w}) = \int_S ||\underline{x}-\underline{w}||^2 d\underline{x}$ = "moment of inertia" (mi) of S about $\underline{w}$

Normalized Moment of Inertia

Let us now separate the effect on $M(S,\underline{w})$ of the shape of S from its size.

$$\frac{1}{k} M(S,\underline{w}) = \frac{\int_S ||\underline{x}-\underline{w}||^2 d\underline{x}}{k|S|^{1+2/k}} \times |S|^{1+2/k} \quad , \text{ where } |S| = \text{vol of } S = \int_S 1\, d\underline{x}$$

$$= m(S,\underline{w}) \times |S|^{1+2/k}$$

where

$$m(S,\underline{w}) = \frac{\int_S ||\underline{x}-\underline{w}||^2 d\underline{x}}{k|S|^{1+2/k}}$$

= "normalized moment of inertia" (nmi) of S about $\underline{w}$.

Fact: $m(S,\underline{w})$ is not affected by a scaling nor a translation. Thus, it is determined only by its shape, but not its size nor position.

Proof: First, consider scaling by a factor $a>0$: $S \to aS = \{\underline{z} = a\underline{x} : \underline{x} \in S\}$; $\underline{w} \to a\underline{w}$

$$m(aS,a\underline{w}) = \frac{\int_{aS} ||\underline{x}-a\underline{w}||^2 d\underline{x}}{k|aS|^{1+2/k}} = \frac{\int_S ||a\underline{z}-a\underline{w}||^2 a^k d\underline{z}}{k|aS|^{1+2/k}} \quad \text{where } a\underline{z} = \underline{x},\ a^k d\underline{z} = d\underline{x}$$

$$= \frac{\int_S a^2 ||\underline{z}-\underline{w}||^2 a^k d\underline{z}}{k|S|^{1+2/k}(a^k)^{1+2/k}} \quad \text{because } |aS| = a^k |S|$$

$$= \frac{\int_S ||\underline{z}-\underline{w}||^2 d\underline{z}}{k|S|^{1+2/k}} = m(S,\underline{w})$$

Next, consider translating by a vector $\underline{v}$: $S \to S+\underline{v} = \{\underline{z} = \underline{x}+\underline{v} : \underline{x} \in S\}$; $\underline{w} \to \underline{w}+\underline{v}$

$$m(S+\underline{v},\underline{w}+\underline{v}) = \frac{\int_{S+\underline{v}} ||\underline{x}-\underline{w}-\underline{v}||^2 d\underline{x}}{k|S+\underline{v}|^{1+2/k}} = \frac{\int_S ||\underline{z}-\underline{w}||^2 d\underline{z}}{k|S|^{1+2/k}} \quad \text{where } \underline{z} = \underline{x}-\underline{v},\ d\underline{z} = d\underline{x}$$

$$= m(S,\underline{w}),$$

Completion of the Derivation of Bennett's Integral

$$D \cong \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{1}{k} M(S_i,\underline{w}_i) \qquad \text{derived earlier}$$

$$= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, m(S_i,\underline{w}_i)\, |S_i|^{1+2/k} \qquad \text{by the definition of nmi}$$

$$= m(S_o,\underline{w}_o) \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, |S_i|^{1+2/k} \qquad \text{since all } S_i\text{'s are congruent to } S_o$$

Note: For the least equality to hold each $\underline{w}_i$ to be in the same relative position within S_i that $\underline{w}_o$ is in S_o. We define "congruence" so as to imply this.
Note: Already we see how the cell sizes and shape separately affect distortion.

Now recall: $|S_i| \cong \dfrac{1}{M\lambda(\underline{w}_i)}$, which implies $|S_i|^{1+2/k} = \dfrac{1}{M^{2/k}} \dfrac{1}{\lambda^{2/k}(\underline{w}_i)} |S_i|$

Therefore,

$$D \cong \frac{1}{M^{2/k}}\, m(S_o,\underline{w}_o) \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{1}{\lambda^{2/k}(\underline{w}_i)} |S_i|$$

$$\cong \frac{1}{M^{2/k}}\, m(S_o,\underline{w}_o) \int f_{\underline{X}}(\underline{x}) \frac{1}{\lambda^{2/k}(\underline{x})} d\underline{x} \quad = \quad \text{Bennett's integral}$$

where the last "$\cong$" is by the definition of an integral and the fact that most $|S_i|$'s are small.

Later, we'll have more discussion of when it's appropriate to use Bennett's integral, i.e. of the conditions under which it leads to accurate approximations.

Special case: Scalar quantization ($k=1$)

Cells are intervals.

If codepoints (levels) are in the centers of the cells, then it is easy to show that

$$m(\text{interval}) = \frac{1}{12}$$

Then

$$D \cong \frac{1}{12M^2} \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)}\, dx$$

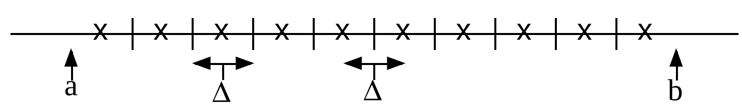
This is what Bennett originally derived.

Special case: Uniform scalar quantizer.

Consider a quantizer that is *uniform* over the interval $[a,b]$, in the sense that

(a) the partition divides $[a,b]$ into M cells of width $\Delta = \frac{b-a}{M}$

(c) the codepoints (levels) are in the centers of the cells (i.e. they are uniformly spaced Δ apart)



Δ is called the "quantizer stepsize".

Suppose Δ is small. Then

$$\lambda(x) \cong \frac{1}{M\Delta} = \frac{1}{b-a}$$

Suppose also that $\Pr(a \le X \le b) \cong 1$. Then

$$D \cong \frac{1}{12M^2} \int_a^b \frac{f_X(x)}{(1/a)^2}\, dx \cong \frac{a^2}{12M^2} \int_a^b f_X(x)\, dx \cong \frac{\Delta^2}{12}$$

This is a formula worth remembering!

Bennett's Integral for Vector Quantizers -- General Case

Theorem[3]: Under the *high-resolution conditions* stated below, the MSE distortion of a k-dimensional VQ with size M applied to random vector $\underline{X}$ can be approximated by

$$D \cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

High-resolution conditions:

+ Most cells are small enough that the prob. density can be approximated as constant on each. (Union of cells for which prob. density cannot be so approximated has very small probability. The *overload distortion* is negligible. M is large.)
+ Neighboring cells have similar sizes and shapes, i.e. cell size & shape change slowly, if at all, with $\underline{x}$. ("shape" also refers to placement of codevectors within cells.)
+ The quantization density is approximately $\lambda(\underline{x})$.
+ The *inertial profile* is approximately $m(\underline{x})$
+ $f_{\underline{X}}(\underline{x})$ = k-dimensional source probability density

Inertial profile: A function $m(\underline{x})$ is a valid inertial profile for a given VQ if

$$m(\underline{x}) \cong \text{NMI of cell containing } \underline{x} = m(S_i,\underline{w}_i) \quad \text{if } \underline{x} \in S_i$$

We also require $m(\underline{x}) \geq 0$, all $\underline{x}$

[3] S. Na and D. Neuhoff, "Bennett's integral for vector quantizers," *IEEE Trans. Inform. Thy.*, vol. 41, pp. 886-900, July 1995.

Derivation of Bennett's Integral -- General Case

$$\begin{aligned}
D &= \frac{1}{k} E\, ||\underline{X}-Q(\underline{X})||^2 \\
&= \frac{1}{k} \sum_{i=1}^{M} \int_{S_i} ||\underline{x}-\underline{w}_i||^2 f_{\underline{X}}(\underline{x})\, d\underline{x} \\
&\cong \frac{1}{k} \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \int_{S_i} ||\underline{x}-\underline{w}_i||^2\, d\underline{x} \\
&= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, \frac{1}{k} \mathcal{M}(S_i,\underline{w}_i) \qquad \text{MI of } S_i \text{ about } \underline{w}_i \\
&= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, m(S_i,\underline{w}_i)\, |S_i|^{1+2/k}, \qquad \text{recall: } \mathcal{M}(S_i,\underline{w}_i) = m(S_i,\underline{w}_i)\, k\, |S_i|^{1+2/k} \\
&\cong \frac{1}{M^{2/k}} \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{m(\underline{w}_i)}{\lambda^{2/k}(\underline{w}_i)} |S_i|, \qquad \text{recall: } |S_i| \cong \frac{1}{M\lambda(\underline{w}_i)},\ m(\underline{w}_i) \cong m(S_i,\underline{w}_i) \\
&\cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x} \qquad \text{by the definition of an integral}
\end{aligned}$$

Notes on Bennett's Integral

- Bennett's integral identifies point density and inertial profile as key characteristics of VQ's, in addition to k and M. For example, with quantizers with the same k, M and point density, distortion is proportional to the the NMI of the cells.
- When M is large, both left and righthand sides of the Bennett integral relation are approximately zero; so what really needs to be shown if one is doing a careful derivation is

$$\frac{D}{\frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}} \cong 1$$

- Bennett's integral shows that distortion decreases as $\frac{1}{M^{2/k}}$,

  assuming point density and inertial profile stay the same.

  To see that this makes sense, consider what happens when M doubles, while maintaining the same point density function and the same inertial profile. Doubling M cuts the volumes of cells in a given region in half. This decreases the linear dimensions of such cells by the factor $1/2^{1/k}$, and causes the average squared distance between points $\underline{x}$ and the codevector of the cell in which $\underline{x}$ lies to decrease on the average by $1/2^{2/k}$. This indicates that mean squared error decreases as $1/M^{2/k}$.
- Equivalently, SNR increases 6 dB for each one bit increase of rate.

$$\frac{1}{M^{2/k}} = 2^{-2R} \;\Rightarrow\; D \cong 2^{-2R} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$\Rightarrow \text{SNR} = 10 \log_{10} \frac{\sigma^2}{D} = 10 \log_{10} 2^{2R} + 10 \log_{10} \frac{\sigma^2}{\int m(\underline{x})\, \lambda^{-2/k}(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}}$$

$$= 6.02\, R + 10 \log_{10} \frac{\sigma^2}{\int m(\underline{x})\, \lambda^{-2/k}(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}}$$

This is called the "6 dB per bit rule".

- Usually, we don't employ a quantization density or inertial profile to describe a VQ unless most cells are small, where "small" means that the probability density changes little across the cell and "most" means that the probability of the cells that are small is large.
- Usually, λ and m are fairly smooth functions that do not convey the detailed locations of codevectors and cells. VQ's with the same quantization density can differ in the number of points, in the exact placement of codepoints and in the shapes of the cells. VQ's with the same inertial profile can differ in the number and placement of codepoints
- Quantization density and inertial profile are, generally, idealizations or models. Often we pick a target quantization density and/or target inertial profile and try to make our quantizer approximate it.

  Later we'll find what quantization densities and inertial profiles are desirable.

- We don't use the following as definitions because if we did, a quantizer would almost never have a specified quantization density or inertial profile:

$$\lambda(\underline{x}) = \frac{1}{M|S_i|} \text{ when } \underline{x} \in S_i \,, \quad m(\underline{x}) = m(S_i, \underline{w}_i) \text{ if } \underline{x} \in S_i$$

- Sketch of why Property 5 on p. 10 implies Property 2

$$\int_A \lambda(\underline{x})\, d\underline{x} = \sum_{i=1}^{M} \int_{S_i \cap A} \lambda(\underline{x})\, d\underline{x} \cong \sum_{i=1}^{M} \int_{S_i \cap A} \frac{1}{M\,\mathrm{vol}(S_i)}\, d\underline{x}$$

$$= \frac{1}{M} \sum_{i=1}^{k} \frac{\mathrm{vol}(S_i \cap A)}{\mathrm{vol}(S_i)} \cong \frac{\#\text{cells in } A}{M}$$

because $\frac{\mathrm{vol}(S_i \cap A)}{\mathrm{vol}(S_i)} = \begin{cases} 1 \,, & \text{if } S_i \subset A \\ 0 \,, & \text{if } S_i \cap A = \phi \\ <1 \,, & \text{otherwise} \end{cases}$, and because most S_i's are small

- When, as usual, $\lambda(x)$ is smooth, Prop. 5, p. 10 implies neighboring cells have similar sizes; e.g. it rules out quantizers with alternating large and small cells.
- When, as usual, $m(x)$ is smooth, the defining property of m implies neighboring cells mostly have similar NMI; e.g. it rules out quantizers whose cell shapes change rapidly.
- We see from Bennett's integral that to make D small, we want larger quantization density where $f_{\underline{X}}(\underline{x})$ is larger. Small inertial profile is desired, everywhere.

Properties and Examples of Normalized Moment of Inertia (NMI)

- **Definition:** NMI of S about point $\underline{w}$ is

$$m(S) = m(S,\underline{w}) = \frac{1}{k} \frac{\int_S ||\underline{x}-\underline{w}||^2 \, d\underline{x}}{\mathrm{vol}^{1+2/k}(S)}$$

- NMI is the same for cubes of all dimensions.

  This is why the definition of $m(S)$ includes $1/k$.

  Defn: k-dimen'l cube $= \left\{ \underline{x} : a \le x_i \le b, i = 1,\dots,k \right\}$ for some $a < b$

  Proof: Let $S = \{\underline{x} : -1/2 \le x_i \le 1/2, i = 1,\dots,k\}$ and $\underline{w} = (0,\dots,0)$. Then $\mathrm{vol}(S) = 1$ and

$$m(S) = \frac{1}{k} \frac{1}{\mathrm{vol}(S)^{1+2/k}} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \sum_{i=1}^{k} x_i^2 \, dx_1 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \left(\sum_{i=2}^{k} x_i^2 + \frac{x_1^3}{3}\Big|_{-1/2}^{1/2} \right) dx_2 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \left(\sum_{i=2}^{k} x_i^2 + \frac{1}{12} \right) dx_2 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \left(\sum_{i=3}^{k} x_i^2 + \frac{x_2^3}{3}\Big|_{-1/2}^{1/2} + \frac{1}{12} \right) dx_3 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \left(\sum_{i=4}^{k} x_i^2 + \frac{x_3^3}{3}\Big|_{-1/2}^{1/2} + \frac{1}{12} + \frac{1}{12} \right) dx_4 \dots dx_k$$

$$= \frac{1}{k} \sum_{i=1}^{k} \frac{1}{12} = \frac{1}{12}$$

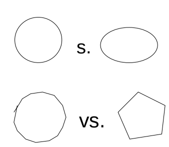
- The NMI of various cell shapes

cell shape	dimension	NMI
1×2 rectangle	2	.104
cube	any	$.0833 = \frac{1}{12}$
hexagon	2	$.0802 = \frac{5\sqrt{3}}{108}$
circle	2	$.0796 = \frac{1}{4\pi}$
sphere	3	$.0770 = \frac{(4\pi/3)^{-2/3}}{5}$
sphere	k	$\frac{1}{(k+2)(V_k)^{2/k}}$
sphere	∞	$.0585 = \frac{1}{2\pi e} \cong \frac{1}{17}$
$s_1 \times s_2 \times \ldots \times s_k$ rectangle	k	$\frac{1}{12} \frac{\frac{1}{k}\Sigma_{i=1}^{k}(s_i)^2}{\left(\Pi_{i=1}^{k}(s_i)^2\right)^{1/k}} = \frac{1}{12} \frac{\text{arith mean of sides}^2}{\text{geom mean of sides}^2}$

where V_k = volume of k-dimensional sphere with radius 1

Bennett-25

- Shapes that tend to make NMI smaller
  - \+ Spheroidal rather than oblong
  - \+ More finely faceted (many sides rather than few)
  - \+ Higher rather than lower dimension



- A spheres has the lowest NMI of any cell of a given dimension.
- NMI of a sphere decreases with dimension to the limit $1/2\pi e = .0585$

Bennett-26

Volume of k-dimensional sphere

(Wozencraft & Jacobs, p. 357)

V_k = vol. of k-dim. sphere with radius 1

$$= \begin{cases} \dfrac{\pi^{k/2}}{(k/2)!}, & k \text{ even} \\[2ex] \dfrac{2^k \pi^{(k-1)/2} \left(\frac{k-1}{2}\right)!}{k!}, & k \text{ odd} \end{cases}$$

From Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\varepsilon_N} \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{where } 0 < \varepsilon_n < \frac{1}{12n},$$

which somewhat underestimates n!, one can show

$$V_k \cong \frac{N}{\sqrt{\pi k}}\left(\frac{2\pi e}{k}\right)^{k/2}.$$

From the above,

$$V_k \cong \exp\left\{\frac{k}{2}\left(\ln(2\pi e) - \ln(k)\right)\right\}$$

$$\to 0 \text{ as } k \to \infty \quad \text{because } -k \ln k \to -\infty$$

Bennett-27

It follows that

$$m(\text{k-dim'l sphere}) = \frac{1}{(k+2)(V_k)^{2/k}} \cong \frac{1}{k+2}\left(\frac{1}{\sqrt{\pi k}}\left(\frac{2\pi e}{k}\right)^{k/2}\right)^{-2/k}$$

$$= (\pi k)^{1/k}\frac{k}{k+2}\frac{1}{2\pi e} = \exp\{\tfrac{1}{k}\ln(\pi k)\}\frac{k}{k+2}\frac{1}{2\pi e}$$

$$\to \frac{1}{2\pi e} \text{ as } k \to \infty.$$

Bennett-28

Formal Statement of the Validity of Bennett's Integral

Theorem[4]**:** $\lim_{M\to\infty} M^{2/k} D(Q_M) = \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$

Assuming

+ $Q_1, Q_2, \ldots$ is sequence of k-dim'l VQ's, with Q_M having size M, partition S_M
+ $\lambda_M(\underline{x}) \to \lambda(\underline{x})$ in probability as $M \to \infty$

  where $\lambda_M(\underline{x}) \stackrel{\Delta}{=} \frac{1}{M \text{ vol(cell containing } \underline{x})}$ = *specific point denstiy* of Q_M
+ $m_M(\underline{x}) \to m(\underline{x})$ as $M \to \infty$ in prob.

  where $m_M(\underline{x}) \stackrel{\Delta}{=} m(S_{\underline{x}}, \underline{y}_{\underline{x}})$ = *specific inertial profile* of Q_M
+ $\{M^{2/k} \|\underline{x} - Q_M(\underline{x})\|^2\}$ has *uniformly absolutely continous integrals*
+ diam(cell of S_M containing $\underline{X}$) $\to 0$ in prob.
+ $f_{\underline{X}}(\underline{x})$ is piecewise continuous
+ Bennett's integral is finite

"Sequence approach" first used by Bucklew & Wise (1982), for scalar quantizers.

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[4] S. Na and D. Neuhoff, "Bennett's integral for vector quantizers," *IEEE Trans. Inform. Thy.*, vol. 41, pp. 886-900, July 1995.