1. In this problem you will show that the MSE of a noisy channel quantizer can be decomposed into the sum of a source distortion and a channel distortion.

Consider a k-dimensional noisy-channel quantizer with partition \( S = \{ S_1, \ldots, S_M \} \), binary codebook \( C_b = \{ c_1, \ldots, c_M \} \) whose codewords have length \( L \), and decoder codebook \( C = \{ w_1, \ldots, w_{2^L} \} \). Let \( p(d|c) \) denote the probability that the channel outputs the length \( L \) vector \( d \) given that the channel input is the length \( L \) vector \( c \). Let \( f_X(x) \) denote the density of the k-dimensional source random vector \( X \).

Show that the mean squared error of this noisy channel quantizer can be decomposed into a source distortion term \( D_s \) plus a channel distortion terms \( D_c \). Specifically,

\[
D = D_s + D_c
\]

where the source distortion \( D_s \) is the distortion of a quantizer consisting of the partition \( S \) and codebook \( C_c = \{ u_1, \ldots, u_M \} \) that consists of the centroids of the cells of \( S \) (not the codebook \( C \) given above), and where the channel distortion \( D_c \) is

\[
D_c = \frac{1}{k} \sum_{i=1}^{M} \sum_{d} \| u_i - w_d \|^2 p(d|c_i) P_i
\]

where \( P_i = \Pr(X \in S_i) \) is the probability of the ith cell of the partition.

Suggested approach: Start with

\[
D = \frac{1}{k} \mathbb{E} \| X - Y \|^2 = \frac{1}{k} \sum_{i=1}^{M} \int_{S_i} \int_d \| x - w_d \|^2 p(d|c_i) f(x) \, dx
\]

Substitute \( \| x - w_d \|^2 = \| (x - u_i) + (u_i - w_d) \|^2 \). Expand the square into three terms, and write the sum-integral-sum as the sum of three such terms. Use the fact that \( u_i \)'s are centroids to show that one of the three terms is zero. Show that one of the remaining terms is \( D_s \) and the other is \( D_c \).