Homework submission policy: Same as before.

1. Show that the opta function $\delta(k, M)$ satisfies the following property, which is called “subadditivity”:
   For any positive integers $k, m$
   \[ \delta(k+m, R) \leq \frac{k}{k+m} \delta(k, R) + \frac{m}{k+m} \delta(m, R) \text{ for all } R \]
   (Comment: The subadditivity implies that $\delta(nk, R) \leq \delta(k, R)$, for any $k$ and $n$. This indicates that opta functions tend to decrease with dimension $k$. Unfortunately, it is not known if $\delta(k+1, R) \leq \delta(k, R)$ is always a true statement. In other words, to my knowledge, there is neither a proof of this, nor a counterexample.)
   (Hint: $\delta(k+m, R)$ is less than or equal to the distortion of any $(k+m)$-dimensional quantizer obtained by taking the product of two quantizers with rate $R$, one with dimension $k$ and the other with dimension $m$.)

2. Consider the scalar quantizer shown below, called a compander, that quantizes by preceding the encoder of an $M$ level uniform scalar quantizer with support $[0,1]$ with a memoryless nonlinear function $c(x)$. At the decoder, the output of the decoder for the uniform scalar quantizer is followed by the inverse of $c$. The levels and thresholds of the uniform scalar quantizer are distributed evenly over the interval $[0,1]$. The function $c$ is nonnegative and monotonically increasing, and it maps $(-\infty, \infty)$ into $[0,1]$. The plot below the block diagram may help you to visualize the operation of the compander.

   (a) Find formulas for the levels $w_1, \ldots, w_M$ and thresholds $t_0, \ldots, t_M$ of the compander in terms of the function $c$.
   (b) Assuming $M$ is large, find the point density and an approximate expression for the distortion of this quantizer in terms of $M$, the function $c$, and the probability density of $X$. Simplify as much as possible (Hint: It should be an integral expression.)
   (c) Show that any scalar quantizer can be implemented with a compander, provided its levels lie within its cells.

3. Suppose a scalar quantizer with many cells and point density $\lambda(x)$ has reconstruction levels located at the thresholds that separate the cells, rather than in the middle of the cells, as we usually presume. By what factor does this increase the distortion over a quantizer that has thresholds in the middle of the cells.
4. Suppose that to a zero-mean Gaussian random variable $X$ with variance $\sigma^2$ and pdf $f(x)$, we apply a scalar quantizer with many cells, reconstruction levels in the middle of the cells, and point density $\lambda(x) = \tilde{f}(x)$, where $\tau > 1$ is some constant greater than one.

(a) Find an approximate, though accurate, expression for the mean squared error of this quantizer. Your answer should be expressed in terms of $\sigma^2$, $\tau$, and the number of quantization levels $M$.

(b) Evaluate your expression for $\tau = 1$. What can be concluded from your answer?

5. Consider a two-dimensional VQ that is the product of two scalar quantizers, one with $M_1$ levels, point density $\lambda_1(x)$, and levels in the middle of its cells, and the other with $M_2$ levels, point density $\lambda_2(x)$, and levels in the middle of its cells. Assume $M_1$ and $M_2$ are large. The goal of this problem is to do a high resolution analysis of the product quantizer.

(a) Find the inertial profile of the product quantizer, in terms of $\lambda_1$, $\lambda_2$, $M_1$ and $M_2$. (Hint: Find the shape of the cell containing an arbitrary point $\tilde{x} = (x_1, x_2)$.)

(b) Find the point density of the product quantizer, in terms of $\lambda_1$, $\lambda_2$, $M_1$ and $M_2$. (Hint: Find an approximate expression for the volume of the cell containing an arbitrary point $\tilde{x} = (x_1, x_2)$.)

(c) Suppose $\lambda_1(x) = \lambda_2(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Describe the shape of the point density of the product quantizer.

(d) Find an approximate, though accurate, expression for the mean squared error of the product quantizer, in terms of $\lambda_1$, $\lambda_2$, $M_1$ and $M_2$.

6. Suppose a scalar quantizer with rate $R$, partition $S$, codebook $C$, quantization rule $Q$, and binary encoding and decoding rules $\alpha_b$ and $\beta_b$ is arranged into a new quantizer, as shown below, where $0 < a < 1$.

(a) Draw the block diagram of a decoder that produces $y_n$ from $z_n$. The blocks of your block diagram should involve $S$, $C$, $Q$, $\alpha_b$, $\beta_b$ and $a$, or a subset thereof.

(b) Find the rate of this quantizer.

(c) Suppose that for a given value of $\tilde{x}_n$, we consider the system from $x_n$ to $y_n$ to be a scalar quantizer. Find it’s partition $S'$, codebook $C'$ and quantization rule $Q'$ in terms of $S$, $C$, $Q$, $a$, and $\tilde{x}_n$. Can you recognize this scalar quantizer as being the result of some sort of transformation of the original quantizer?

(d) Show that $E(X_n - Y_n)^2 = E(U_n - V_n)^2$, i.e. the distortion for this quantizer applied to $X$ equals the distortion of the original quantizer applied to $U$.

(Comment: This kind of quantization is called Differential Pulse Code Modulation (DPCM).)