Lemma 2. Let \( a_n, N = 1, 2, \ldots \) be a bounded sequence of numbers and let
\[
d = \sup_{N} a_N \quad \text{and} \quad \varepsilon = \inf_{N} a_N,
\]
(By a bounded sequence, we mean that \( a < d \) and \( \varepsilon > -\infty \).) Assume that, for all \( n \geq 1 \), and all \( N > n \),
\[
a_N \leq \frac{n}{N} a_n + \frac{N-n}{N} a_{n-1} \tag{4A.4}
\]
Then
\[
\lim_{N \to \infty} a_N = d \tag{4A.5}
\]
Conversely, if for all \( n \geq 1 \) and \( N > n \),
\[
a_N \leq \frac{n}{N} a_n + \frac{N-n}{N} a_{n-1} \tag{4A.6}
\]
we have
\[
\lim_{N \to \infty} a_N = \varepsilon \tag{4A.7}
\]
Proof. Assume (4A.4) is valid and, for any \( \varepsilon > 0 \), choose \( n \) to satisfy
\[
a_n \geq d - \varepsilon \tag{4A.9}
\]
Choosing \( N = 2n \), (4A.4) becomes
\[
a_{2n} \geq \frac{a_n}{2} + \frac{a_n}{2} \geq d - \varepsilon \tag{4A.9}
\]
Similarly, choosing \( N = mn \) for any integer \( m \geq 2 \),
\[
a_{mn} \geq \frac{a_m}{m} + \frac{(m-1)a_{m-1}}{m} \tag{4A.10}
\]
Using induction, we assume that \( a_{mn+i} \geq d - \varepsilon \), and (4A.10) then implies that
\[
a_{mn+i} \geq d - \varepsilon \quad \text{for all } m \geq 1 \tag{4A.11}
\]
Now, for any \( N > n \), we can represent \( N \) as \( mn + j \) where \( 0 \leq j \leq n - 1 \). Using \( j \) in place of \( n \) in (4A.4), we have
\[
a_N \geq \frac{n}{N} a_n + \frac{N-n}{N} a_{n-1} = a_{mn+j} = (j/N)(a_{n+i} - a_{n+i-1}) \geq d - \varepsilon + (j/N)(d - a_{n+i-1}) \tag{4A.12}
\]
It follows that, for all sufficiently large \( N \), \( a_N \geq d - 2\varepsilon \), since \( a_N \leq d \), and \( \varepsilon \) is arbitrary. (4A.5) follows from (4A.9) by observing that (4A.4) applies to the sequence \(-a_n\) and thus
\[
\lim \frac{d}{a_n} = -\inf_{N} \frac{a_n}{a_N} \tag{4A.13}
\]
}\[111,112]