EECS 730

Due Thursday, March 26, 2009

Homework Set No. 7

Problems from "Theory of Microwave Remote Sensing," by L. Tsang, A. J. Kong, and R.T. Shin. Problems 5.18 and 5.19, pages 419-422.

5.18

Assume a mixture of two constituents. One constituent is background medium with permittivity ϵ_b and fractional volume f_b. The other constituent is scatterer with permittivity ϵ_s and fractional volume f_s. The scatterers are assumed to be spherical with a radius a. We have $f_b + f_s = 1$. Next use simple physical arguments to construct correlation functions $\xi(\bar{r})$ for the mixture. First look at the one-dimensional problem. Consider pulses of height ξ_s and width ι (which corresponds with to 2a in the three dimensional case) distributed on a line. For the rest of the line, the amplitude is ξ_b (Figure 1a). The problem is to determine the correlation function. To simplify the problem, the line is partitioned into segments of length ι, with the amplitude in each partition assuming either the value ξ_s or ξ_b (Figure 2b).

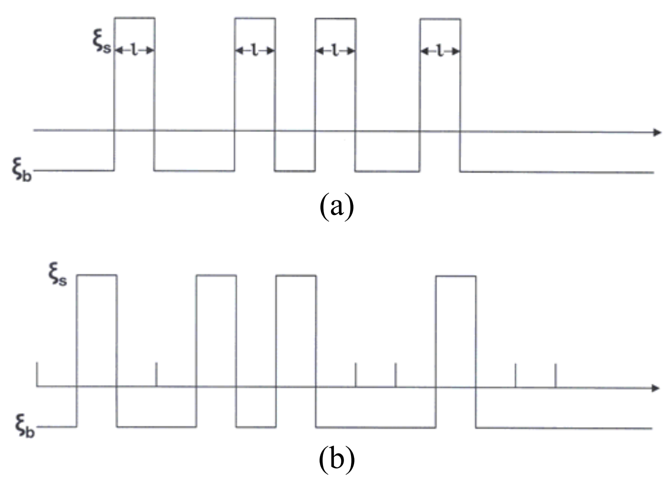

Fig. 1 (a): One–dimensional random process of $\xi(r)$ consisting of Pulses of width ι and height ξ_s.
Fig. 2 (b): Same as (a) except the line is partitioned into segments of length ι.

Let $\xi(\boldsymbol{k})$ be the value of amplitude of partition $\boldsymbol{k}$. Then, Pr $(\xi(\boldsymbol{k}) = \xi_s) = f_s$, Pr $(\xi(\boldsymbol{k}) = \xi_b) = f_b$ and $< \xi(\boldsymbol{k}) > = f_s \xi_s + f_b \xi_b = 0$. To calculate the correlation function, verify that:

$$\langle \xi(\kappa)\xi(\kappa+m)\rangle = f_s \xi_s^2 + f_b \xi_b^2 \qquad \text{for } \boldsymbol{m} = 0$$

For $\boldsymbol{m} \neq 0$

$$\begin{aligned}\langle \xi(\kappa)\xi(\kappa+m)\rangle = \quad & \Pr[\xi(\kappa)=\xi_s, \xi(\kappa+m)=\xi_s]\xi_s^2 \\ & + \Pr[\xi(\kappa)=\xi_s, \xi(\kappa+m)=\xi_b]\xi_s\xi_b \\ & + \Pr[\xi(\kappa)=\xi_b, \xi(\kappa+m)=\xi_s]\xi_b\xi_s \\ & + \Pr[\xi(\kappa)=\xi_b, \xi(\kappa+m)=\xi_b]\xi_b^2 \\ & = f_s^2\xi_s^2 + 2 f_b f_s \xi_s \xi_b + f_b^2 \xi_b^2 = 0\end{aligned}$$

Using the results from the 3-D continuum case, show :

$$C_\xi(r) = \begin{cases} \langle \xi^2 \rangle & 0 < r \le a \\ 0 & Otherwise \end{cases}$$

Where $\quad C_\xi(|\bar{r}_1 - \bar{r}_2|) = \langle \xi(\bar{r}_1)\xi(\bar{r}_2)\rangle$

5.19

Dyson's equation for the mean scalar Green's function was derived in the text form:

$$\begin{aligned}&\left(\nabla^2 + \kappa_m^2\right) \langle G(\bar{r}, \bar{r}_0)\rangle + \int d^3 r_1 Q(\bar{r}, \bar{r}_1) \, \langle G(\bar{r}_1, \bar{r}_o)\rangle \\ &= -\delta(\bar{r} - \bar{r}_0)\end{aligned}$$

where the mass operator, $Q(\bar{r}, \bar{r}_1)$ is defined as the sum of all strongly connected diagrams.

1. Show that the mass operator is translationally invariant. In other words show that $Q(\bar{r}_1, \bar{r}_2) = Q(\bar{r}_1 - \bar{r}_2)$.

2. Taking the source point at the origin (i.e. $\bar{r}_o = 0$), show that the solution to the above Dyson's equation may be written as:

$$\langle G(\bar{r})\rangle = \frac{1}{(2\pi)^3}\int d^3\kappa \frac{e^{-i\bar{k}.\bar{r}}}{\kappa^2 - \kappa_m^2 - \int d^3RQ(\bar{R})e^{-i\bar{k}.\bar{R}}}$$

3. In the bilocal approximation, the mass operator is approximated as:

$$Q(R) \approx C(\bar{R})\frac{e^{ik_m R}}{4\pi R}$$

Taking the correlation function to be $C(R) = \delta\kappa_m^{'4}\exp(-R/a)$ show that in the limit $\kappa_m a \langle\langle 1$, the mean Green's function obtained in part (b) is a spherical wave, with an effective propagation constant given by:

$$K^2 = k_m^2 + \frac{\delta k_m^{'4} a^2}{(1 - ik_m a)^2}$$

which is similar to the result obtained in the text using the Taylor expansion method.