1 Dielectric Mixing Formula

One of the classical problems in electromagnetic is the characterization of effective dielectric constant of a mixture of two or more constituents with different permittivity. Strictly speaking, the effective dielectric constant is a macroscopic parameter which relates the "average" electric field in the medium to the average flux density. In this sense we are implying that the physical size of the mixture constituents are small compared to the wavelength, hence quasi-static solutions are sufficient to analyze the fields locally. For simplicity let us consider a mixture of two constituents: 1) inclusion particles with permittivity ϵ_i (inclusion particle), and 2) a homogeneous dielectric medium as a background ϵ_h (host medium) as shown in Figure 1.

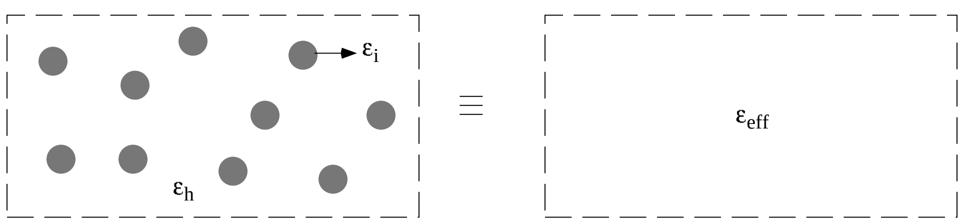

Figure 1: An inhomogeneous mixture of two constituents and its macroscopic equivalent.

The incident field on each particle is the sum of the scattered fields from all other particles in the medium. However, it would be difficult to characterize the sum of the scattered fields since we also have to consider the effect of multiple scattering. To circumvent this difficulty we assume the sum of the scattered fields is simply the field generated from the effective medium in a spherical hole with dielectric constant ϵ_h. This is shown in Figure 2.

To find the field in the spherical cavity, let us consider a uniform field in the equivalent medium denoted by E. Then we can carve out a sphere off of the equivalent medium, but before doing that we somehow freeze the polarization (internal field) inside the sphere. Therefore we end up with a spherical hole and a uniformly polarized dielectric sphere as shown in Figure 3.

By superposition, the field inside the cavity in addition to the field on the uniformly polarized sphere in the background of ϵ_h is the field inside the equivalent medium, i.e.,

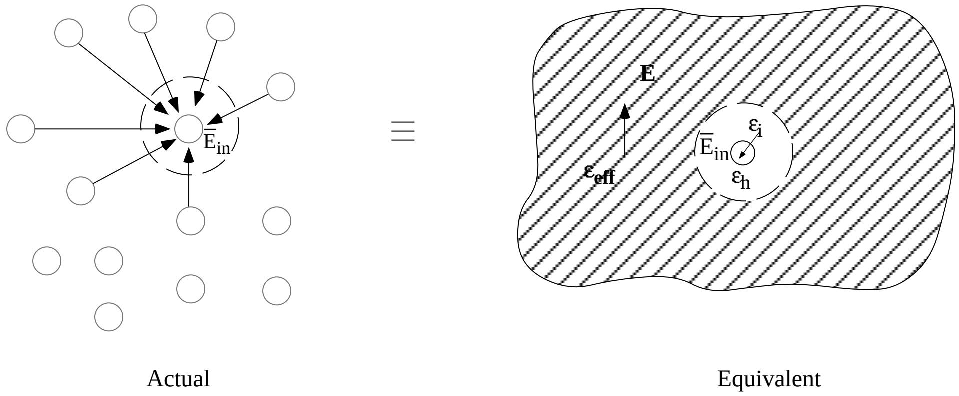

Figure 2: An equivalent method of calculating the incident field on a particle. In the equivalent method the incident field is an “averaged” quantity.

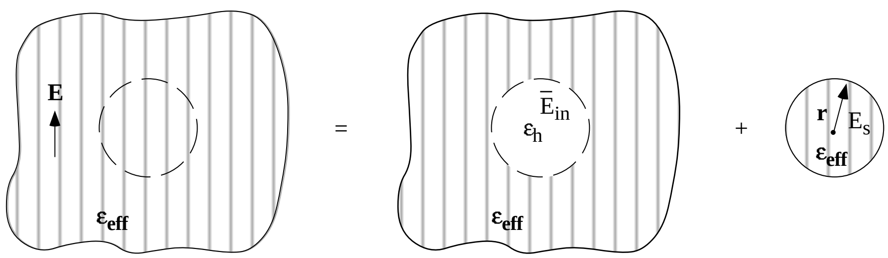

Figure 3: The field in the equivalent medium is the superposition of the field in the cavity ($\overline{E}_{in}$) and the field inside the sphere of uniform polarization ($\overline{E}_s$).

$$\overline{E} = \overline{E}_{in} + \overline{E}_s \tag{1}$$

Hence, once we obtain the internal field of uniformly polarized sphere the field inside the hole can be obtained. Noting that

$$\epsilon_{eff}\overline{E} = \epsilon_h\overline{E} + \overline{P} \tag{2}$$

the frozen uniform polarization in the sphere is given by

$$\overline{P} = (\epsilon_{eff} - \epsilon_h)\overline{E} \tag{3}$$

Now we try to establish a relation between the polarization $\overline{P}$ and the field inside the sphere $\overline{E}_s$. Since polarization is uniform, the internal field must be uniform, suppose

$$\overline{E}_s = c\hat{z} = c\left[\cos\theta\hat{r} - \sin\theta\ \hat{\theta}\right]$$

for some constant c. Therefore the potential inside is

$$\Psi^- = -cZ = -cr\cos\theta \tag{4}$$

The potential outside the sphere $\Psi^+(r,\theta,\phi)$ must satisfy Laplace's equation

$$\nabla^2\Psi^+ = 0 \tag{5}$$

Since Ψ^- is not a function of ϕ, Ψ^+ is independent of ϕ as well. Using the method of separation of variables solution (5) must be of the form

$$\Psi^+(r,\theta) = \sum_{n=0}^{\infty} b_n \frac{P_n(\cos\theta)}{r^{n+1}} \tag{6}$$

where $P_n(\cos\theta)$ is the Legendre polynominal of order n. The boundary condition mandates the continuity of the potential function at $r = r_1$, that is,

$$-cr_1\cos\theta = \sum_{n=0}^{\infty} b_n \frac{P_n(\cos\theta)}{r_1^{n+1}} \tag{7}$$

But since $P_1(\cos\theta) = \cos\theta$ and the fact that P_n and $P_m (m \neq n)$ are orthogonal functions

$$\begin{aligned} b_n &= 0 \quad n \neq 1 \\ b_1 &= -cr_1^3 \quad , \end{aligned}$$

Therefore

$$\Psi^+(r,\theta) = -c\frac{r_1^3}{r^2}\cos\theta \tag{8}$$

The electric field outside the sphere is then computed from

$$\begin{aligned} E^+(r,\theta) &= -\nabla\Psi^+ = -\frac{\partial}{\partial r}\Psi^+\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}\Psi^+\hat{\theta} \\ &= -\frac{cr_1^3}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\ \hat{\theta}\right] \end{aligned} \tag{9}$$

Since there are no free charges on the boundary

$$\hat{n}\cdot(\overline{D}^+ - \overline{D}_s)|_{r=r_1} = 0$$

or

$$\hat{r}\cdot\left[\epsilon_h\overline{E}^{+}-\epsilon_h\overline{E}_s-\overline{P}\right]|_{r=r_1}=0$$

Therefore

$$\hat{r}\cdot\left[\overline{E}^{+}-\overline{E}_s\right]|_{r=r_1}=\frac{\hat{r}\cdot\overline{P}}{\epsilon_h}=\frac{|\overline{P}|\cos\theta}{\epsilon_h} \tag{10}$$

but $\hat{r}\cdot\left[\overline{E}^{+}-\overline{E}_s\right]=-3c\cos\theta$ which provides

$$c=-\frac{|\overline{P}|}{3\epsilon_h} \tag{11}$$

Equation (11) indicates that the field inside a uniformly polarized dielectric sphere is simply given by

$$\overline{E}_s=-\frac{\overline{P}}{3\epsilon_h} \quad , \tag{12}$$

Hence according to (1), the field inside the spherical cavity is

$$\overline{E}_{in}=\overline{E}+\frac{\overline{P}}{3\epsilon_h} \tag{13}$$

At low frequencies the scattered electrostatic field from a particle resembles that of a dipole whose dipole moment $\overline{p}$ is proportional to the incident field. Suppose a particle is in the spherical cavity where the incident field is given by (13), thus

$$\overline{p}=\alpha\overline{E}_{in} \tag{14}$$

where α is the polarizability of the particle which depends on the particle size, shape, and permittivity. If there are N particles per unit volume, the polarization $\overline{P}$ is given by

$$\overline{P}=N\overline{p}=N\alpha\overline{E}_{in} \tag{15}$$

Substituting (13) in (15) and solving for $\overline{P}$

$$\overline{P}=\frac{N\alpha\overline{E}}{1-\frac{N\alpha}{3\epsilon_h}} \tag{16}$$

and finally substituting (16) in (2)

$$\epsilon_{eff} = \epsilon_h \frac{1 + \frac{2N\alpha}{3\epsilon_h}}{1 - \frac{N\alpha}{3\epsilon_h}} \tag{17}$$

which is the well-known Clausius-Mossotti or Lorentz-Lorenz formula.

1.1 Polarization of Spherical Particles

It is a well-known result that at low frequencies the field excited inside a dielectric sphere when placed in a uniform field has a uniform distribution and is linearly proportional to the exciting field. And the field outside the sphere resembles that of an infinitesimal dipole whose electrostatic potential is given by

$$\Psi = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_h r^2} \tag{18}$$

where $\vec{p}$ is the dipole moment $(q\vec{\ell})$. To find the equivalent dipole moment of a dielectric sphere placed in a uniform field, Laplace's equation subject to the boundary conditions is used. Assuming a uniform incident field polarized along $\hat{z}$, the incident potential is given by

$$\Psi_o = -E_o r \cos\theta \tag{19}$$

Due to the symmetry with respect to ϕ the potential inside and outside the sphere are independent of ϕ. The scattered potential outside the sphere Ψ^+ must be finite, hence

$$\Psi^+ = \sum_{n=0}^{\infty} b_n \frac{P_n(\cos\theta)}{r^{n+1}} \tag{20}$$

The total potential inside the sphere may also be written as

$$\Psi^- = \sum_{n=0}^{\infty} c_n P_n(\cos\theta) r^n \tag{21}$$

The boundary condition mandates

$$(\Psi_o + \Psi^+)|_{r=a} = \Psi^-|_{r=a} \tag{22}$$

$$\epsilon_h \frac{\partial(\Psi_o + \psi^+)}{\partial r}|_{r=a} = \epsilon_i \frac{\partial \Psi^-}{\partial r}|_{r=a} \tag{23}$$

where ϵ_h is the permittivity of the background and ϵ_i is the permittivity of the dielectric sphere. Using (22) we can easily show

$$-E_o a + \frac{b_1}{a^2} = c_1 a \tag{24}$$

and using (23)

$$-\left[E_o + \frac{2b_1}{a^3}\right]\epsilon_h = c_1\epsilon_i \tag{25}$$

Solving (24) and (25) simultaneously

$$\begin{aligned} b_1 &= \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} a^3 E_o \\ c_1 &= \frac{-3\epsilon_h}{\epsilon_i + 2\epsilon_h} E_o \end{aligned}$$

Also for $n > 1$

$$b_n = c_n = 0$$

Thus the scattered potential outside the sphere is

$$\Psi^+ = \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} a^3 E_o \frac{\cos\theta}{r^2} \tag{26}$$

Comparing (26) with (18), the equivalent dipole moment of the sphere is given by

$$p = 4\pi\epsilon_h \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} a^3 E_o \tag{27}$$

The potential inside the sphere is

$$\Psi^- = -\frac{3\epsilon_h}{\epsilon_i + 2\epsilon_h} E_o r\cos\theta$$

and the electric field inside is

$$\overline{E}^- = -\frac{\partial\Psi^-}{\partial z} = \frac{3\epsilon_h}{\epsilon_i + 2\epsilon_h} E_o \hat{z} \tag{28}$$

which indicates that the field inside the sphere is constant and proportional to the incident field.

1.2 Mixing Formula for Spherical Particles

For spherical particles the polarizability α given by (14) can be obtained from (27). In this case

$$\alpha = 4\pi\epsilon_h \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} a^3 \tag{29}$$

Substituting (29) in (17) and noting that $\frac{4\pi}{3}a^3 N = f$ is the volume fraction of spherical particles, it can be shown that

$$\epsilon_{eff} = \epsilon_h \frac{1 + 2f\frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h}}{1 - f\frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h}} \tag{30}$$

Or equivalently

$$\frac{\epsilon_{eff} - \epsilon_h}{\epsilon_{eff} + 2\epsilon_h} = f\frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} \tag{31}$$

which is known as Maxwell-Garnet mixing formula. This formula is accurate for low volume fractions. However, it has the property that matches the limiting conditions, that is, for $f = 0$ we have $\epsilon_{eff} = \epsilon_h$ and for $f = 1\,, \epsilon_{eff} = \epsilon_i$ as expected. But it does not satisfy the symmetry property which requires an identical ϵ_{eff} once $\epsilon_i \rightleftharpoons \epsilon_h$ and $f \to (1 - f)$, that is, different results are obtained when the roles of the inclusion and host are interchanged. Noting that for low volume fractions ϵ_{eff} is approximately equal to ϵ_h then (31) can be written as

$$\frac{\epsilon_{eff} - \epsilon_h}{3\epsilon_h} = f\frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} \tag{32}$$

which is the Polder van Santen mixing formula obtained previously using the strong permittivity fluctuation theory.

1.3 Mixing Formula for Ellipsoidal Particles

If the inclusion particles are non-spherical, then the effective permittivity of the mixture may become a tensor instead of a scalar. This happens when the particles have a preferred orientation. To demonstrate this let us consider an ellipsoidal particle whose geometry is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{33}$$

with $a \geq b \geq c$. Suppose this particle (with permittivity ϵ_i) is placed in a parallel field

$$\overline{E}_o = E_{ox}\hat{x} + E_{oy}\hat{y} + E_{oz}\hat{z} \tag{34}$$

Applying the technique of separation of variables to Laplace's equation it can be shown that the potential inside and outside the ellipsoid can be expressed in terms of elliptic integrals[1]. After some algebraic manipulation it can be shown that the internal field is a constant field which in general is not parallel to the applied field E_o. The internal field is given by

$$\overline{E}^- = \left[\frac{E_{ox}\hat{x}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_1} + \frac{E_{oy}\hat{y}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_2} + \frac{E_{oz}\hat{z}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_3} \right] \tag{35}$$

where $v_i = \frac{4}{3}\pi abc$ is the volume of the particle and

$$\begin{aligned} A_1 &= \frac{abc}{2} \int_0^\infty \frac{ds}{(s + a^2)R_s} \\ A_2 &= \frac{abc}{2} \int_0^\infty \frac{ds}{(s + b^2)R_s} \\ A_3 &= \frac{abc}{2} \int_0^\infty \frac{ds}{(s + c^2)R_s} \end{aligned} \tag{36}$$

with $R_s = \sqrt{(s + a^2)(s + b^2)(s + c^2)}$. It can be shown that

$$A_1 + A_2 + A_3 = 1 \tag{37}$$

The scattered potential outside the ellipsoid is given by

$$\Psi = -\frac{abc\frac{\epsilon_i - \epsilon_h}{\epsilon_h}}{2} \left[\frac{\int_\zeta^\infty \frac{ds}{(s+a^2)R_s}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_1} \Psi_{ox} + \frac{\int_\zeta^\infty \frac{ds}{(s+b^2)R_s}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_2} \Psi_{oy} + \frac{\int_\zeta^\infty \frac{ds}{(s+c^2)R_s}}{1 + \frac{\epsilon_i - \epsilon_h}{\epsilon_h} A_3} \Psi_{oz} \right] \tag{38}$$

where $\Psi_{ox} = -xE_{ox}, \Psi_{oy} = -yE_{oy}$, and $\Psi_{oz} = -zE_{oz}$. At observation points away from the surface of the ellipsoid the constant ζ surface given by

$$\frac{x^2}{a^2 + \zeta} + \frac{y^2}{b^2 + \zeta} + \frac{z^2}{c^2 + \zeta} = 1$$

can be written as

[1] Straton, J.A., Electromagnetic Theory, pp. 207-213, McGraw Hill, 1941.

$$\zeta = \frac{x^2}{1+\frac{a^2}{\zeta}} + \frac{y^2}{1+\frac{b^2}{\zeta}} + \frac{z^2}{1+\frac{c^2}{\zeta}} \simeq x^2 + y^2 + z^2 = r^2$$

hence

$$\int_\zeta^\infty \frac{ds}{(s+c^2)R_s} \simeq \int_\zeta^\infty \frac{ds}{s^{5/2}} = \frac{2}{3}\zeta^{-3/2} \simeq \frac{2}{3}r^{-3} \quad (39)$$

Using approximation (39) in (38), it is obvious that the potential outside the ellipsoid can be approximated by the superposition of potentials from three dipoles along $\hat{x}, \hat{y}$, and $\hat{z}$

$$\Psi^+ = P_x \frac{\hat{x}\cdot\hat{r}}{4\pi\epsilon_h r^2} + P_y \frac{\hat{y}\cdot\hat{r}}{4\pi\epsilon_h r^2} + P_z \frac{\hat{z}\cdot\hat{r}}{4\pi\epsilon_h r^2}$$

where

$$P_x = \epsilon_h \frac{v_i \frac{\epsilon_i-\epsilon_h}{\epsilon_h}}{1+\frac{\epsilon_i-\epsilon_h}{\epsilon_h}A_1} E_{ox} \quad (40)$$

and P_y and P_z can be obtained from (40) by replacing A_1 with A_2 and A_3 and E_{ox} with E_{oy} and E_{oz} respectively. It should be noted that when the ellipsoid degenerates into a sphere $(a = b = c) A = \frac{1}{3}$ and (40) becomes

$$P = 4\pi\epsilon_h \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} a^3 E_{ox} \quad (41)$$

which is the same as (27). The polarizability ($\bar{\bar{\alpha}}$) in a general case can be defined as (see (14))

$$\overline{P} = \bar{\bar{\alpha}} \cdot \overline{E}_o$$

where for an ellipsoid particle we can write

$$\bar{\bar{\alpha}} = v_i(\epsilon_i - \epsilon_h) \begin{bmatrix} \frac{1}{1+\frac{\epsilon_i-\epsilon_h}{\epsilon_h}A_1} & 0 & 0 \\ 0 & \frac{1}{1+\frac{\epsilon_i-\epsilon_h}{\epsilon_h}A_2} & 0 \\ 0 & 0 & \frac{1}{1+\frac{\epsilon_i-\epsilon_h}{\epsilon_h}A_3} \end{bmatrix} \quad (42)$$

of course if the axis of the ellipsoid does not coincide with the global coordinate system a transformation is required to find the particle polarizability in the global coordinate system ($\bar{\bar{\alpha}}_g$)

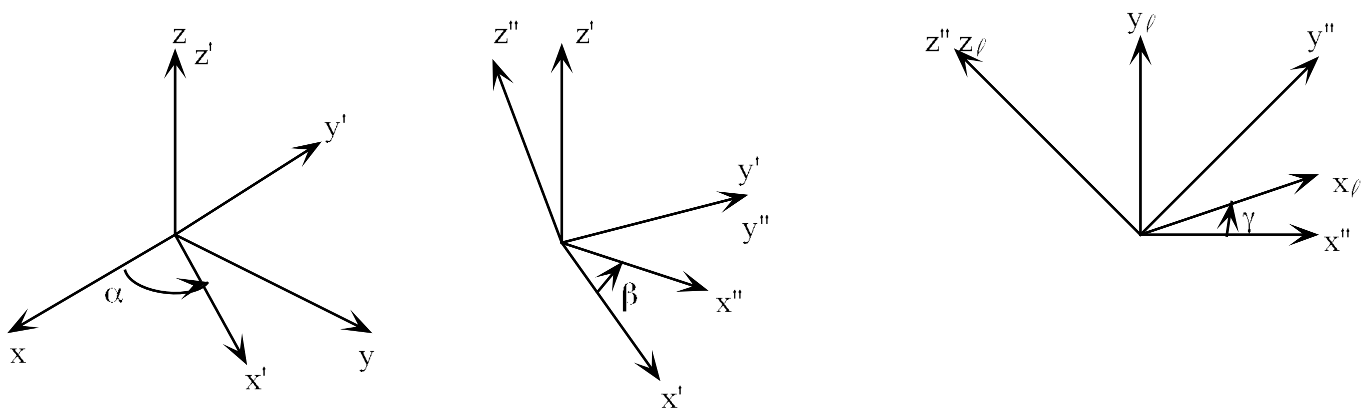

Figure 4: Coordinate transformation using Eulerian angles α, β, and γ

$$\bar{\bar{\alpha}}_g = \bar{\bar{T}}^{-1} \bar{\bar{\alpha}} \bar{\bar{T}} \tag{43}$$

where $\bar{\bar{T}}$ is a transformation from the global to local coordinate transformation. Suppose the global coordinate goes through the flowing rotations to arrive at the local coordinate system.

First a rotation $\alpha (0 \leq \alpha < 2\pi)$ of $x - y$ plane about the z-axis to form a new coordinate system (x', y', z'). Then a rotation of $x' - z'$ plane by angle $\beta (0 \leq \beta < \pi)$ about y'-axis to form (x'', y'', z'') coordinate system. Finally a rotation of $x'' - y''$ plane about z''-axis by an angle $(0 \leq \gamma < 2\pi)$ to arrive at the local coordinate system. These transformations are depicted in Fig. 4. The product of these three simple rotation transformations produces the desired transformation. Angles α, β, and γ are known as the Eulerian angles of rotation in terms of which T is given by [1]

$$\bar{\bar{T}} = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

This transformation is unitary and therefore

$$\bar{\bar{T}}^{-1} = \bar{\bar{T}}^{t}$$

In two special cases analytical solutions for A_j's can be obtained. When the ellipsoid reduces to an spheroid that is when $a = b > c$ known as oblate spheroid (disk shape) or $a = b =< c$, known as Prolate spheroid (rod shape). For oblate spheroid we have

$$A_3 = \frac{1}{e_0^2} - \frac{e}{ac_0^3} \tan^{-1}(\frac{a}{c} e_0) \tag{44}$$

where $e_0 = \sqrt{1 - \frac{c^2}{a^2}}$. For prolate spheroid

$$A_3 = -\frac{a^2}{2c^2 e_p^3} \left[2e_p + \ln\left(\frac{1 - e_p}{1 + e_p} \right) \right] \tag{45}$$

where $e_p = \sqrt{1 - \frac{a^2}{c^2}}$ and $A_1 = A_2 = (1 - A_3)$.

In limit as $c \to 0 (e_0 \to 1)$ the oblate spheroid becomes a thin disk for which $A_3 = 1$ and $A_1 = A_2 = 0$. Hence for a thin dielectric disk the polarizability tensor takes the following form

$$\overline{\overline{\alpha}}_{\text{disk}} = v_i(\epsilon_i - \epsilon_h) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\epsilon_h}{\epsilon_i} \end{bmatrix}$$

For the prolate spheroid when $a \to 0$ the spheroid approaches a thin cylinder for which $A_3 = 0$ and $A_1 = A_2 = \frac{1}{2}$. In this case the polarizability tensor is

$$\overline{\overline{\alpha}}_{\text{rod}} = v_i(\epsilon_i - \epsilon_u) \begin{bmatrix} \frac{2\epsilon_h}{\epsilon_i + \epsilon_h} & 0 & 0 \\ 0 & \frac{2\epsilon_h}{\epsilon_i + \epsilon_h} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the mixing formula an ellipsoidal or spherical cavity can be considered. If we expect some orientation distribution a spherical cavity is a good approximation where the field inside in terms of the polarization and the field in the equivalent medium is given by (13). A modified form of (15) is given by

$$\overline{P} = N \left\langle \overline{\overline{\alpha}} \right\rangle \cdot \overline{E}_{in} \tag{46}$$

where $\left\langle \overline{\overline{\alpha}} \right\rangle$ is the ensemble average of $\overline{\overline{\alpha}}$ over size and orientation angles. Using (13)

$$\overline{P} = N \left\langle \overline{\overline{\alpha}} \right\rangle \cdot \overline{E} + \frac{N \left\langle \overline{\overline{\alpha}} \right\rangle \cdot \overline{P}}{3\epsilon_h} \tag{47}$$

From the definition of $\overline{\overline{\epsilon}}_{eff}$ we have

$$\overline{\overline{\epsilon}}_{eff} \cdot \overline{E} = \epsilon_h \overline{\overline{I}} \cdot \overline{E} + \overline{P} \tag{48}$$

Using (47) in (48)

$$\overline{\overline{\epsilon}}_{eff} = \epsilon_h \left[\overline{\overline{I}} + \left(\overline{\overline{I}} - \frac{N\langle\overline{\overline{\alpha}}\rangle}{3\epsilon_h} \right)^{-1} \cdot \frac{N\langle\overline{\overline{\alpha}}\rangle}{\epsilon_h} \right] \tag{49}$$

2 Improved Mixing Formula

Maxwell Garnet mixing formula given by

$$\frac{\epsilon_{eff} - \epsilon_h}{\epsilon_{eff} + 2\epsilon_h} = f \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h} \tag{50}$$

is valid only if the volume fraction of inclusions is small. This fact has been proven using numerical techniques (see for example [1]). To improve the region of validity of this formulation let us consider an iterative procedure. Basically to get to a desired volume fraction f, one may proceed by incrementally adding inclusions with permittivity ϵ_i to the host and each time substituting ϵ_{eff} of the resulting mixture of the previous step as the host dielectric constant (ϵ_h) for the next step. In this procedure ϵ_h can be represented by $\epsilon(f)$ and (50) may be written in terms of a first-order differential equation.

$$\begin{aligned} \frac{\Delta\epsilon}{3\epsilon} &= \Delta f \frac{\epsilon_i - \epsilon}{\epsilon_i + 2\epsilon} \\ \frac{d\epsilon}{df} &= \frac{3\epsilon(\epsilon_i - \epsilon)}{\epsilon_i + 2\epsilon} \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{df}{d\epsilon} &= \frac{1}{3\epsilon} + \frac{1}{\epsilon_i - \epsilon} \\ f &= \int_{\epsilon_h}^{\epsilon_{eff}} (\frac{1}{3\epsilon} + \frac{1}{\epsilon_i - \epsilon}) d\epsilon = ln(\frac{\epsilon^{1/3}}{\epsilon_i - \epsilon}) \mid_{\epsilon_h}^{\epsilon} \\ f &= ln[(\frac{\epsilon^{1/3}}{\epsilon_h}) \frac{\epsilon_i - \epsilon_h}{\epsilon_i - \epsilon}] \end{aligned} \tag{51}$$

This satisfies one limiting condition, that is, when $f = 0$, $\epsilon = \epsilon_h$ as expected; however, for $f = 1$, $\epsilon < \epsilon_i$ equation (51) may be solved for ϵ in terms of f

$$\frac{\epsilon^{1/3}}{\epsilon_i - \epsilon} = \frac{(\epsilon_h)^{1/3}}{\epsilon_i - \epsilon_h} \; e^f \tag{52}$$

or

$$\epsilon = 3\frac{\epsilon_h}{(\epsilon_i - \epsilon_h)^3}(\epsilon_i - \epsilon)^3 \tag{53}$$

References

[1] Tsang, L., J.A. Kong, and R.T. Shin, Theory of Microwave Remote Sensing, New York, NY: John Wiley and Sons, Wiley-Interscience, 1985.