Dyadic Green's Function

As mentioned earlier the applications of dyadic analysis facilitates simple manipulation of field vector calculations. The source of electromagnetic fields is the electric current which is a vector quantity. On the other hand small-signal electromagnetic fields satisfy the linearity conditions and therefore the behavior of the fields can be described in terms of the system impulse response. Since both the input (excitation current) and the output (field quantities) of the system are vector quantities, the impulse response of the system must be a dyadic quantity. In what follows the derivation of dyadic Green's function (impulse response for free space) is presented. Then the Fourier representation of the Green's function is derived which expresses the fields of an infinitesimal current source in terms of a continuous spectrum of plane waves. This form of the dyadic Green's function is useful for further development of dyadic Green's functions for more complicated media such as a dielectric half-space medium or a stratified (multi-layer) dielectric medium.

Consider an arbitrary time-harmonic electric current distribution $\overline{J}_e$ in an unbounded homogeneous medium with permittivity ϵ and permeability μ. Starting from the Maxwell's equation, the vector wave equation for the electric field can be obtained and is given by:

$$\nabla \times \nabla \times \overline{E}(\overline{r}) - k^2 \overline{E}(\overline{r}) = i\omega\mu \overline{J}_e(\overline{r}) \tag{1}$$

where $\overline{J}_e(\overline{r})$ is the impressed volumetric current distribution. As shown previously the electric field is usually calculated indirectly from the electric Hertz potential and is given by:

$$\overline{E}(\overline{r}) = \left(k^2 + \nabla\nabla\cdot\right) \overline{\Pi}_e(\overline{r}) \tag{2}$$

where

$$\overline{\Pi}_e(\overline{r}) = \frac{iZ}{k} \int_v \overline{J}_e(\overline{r}')g(\overline{r},\overline{r}')dv' \tag{3}$$

The electric field expression given by (2) is valid for all $\overline{r}$ in this medium including source points. Here

$$g(\overline{r},\overline{r}') = \frac{1}{4\pi}\frac{e^{ik|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|}$$

is the scalar Green's function satisfying the scalar wave equation

$$\left(\nabla^2 + k^2\right) g(\overline{r}, \overline{r}') = -\delta(\overline{r} - \overline{r}') \tag{4}$$

Let us now consider an infinitesimal current source along $\hat{x}$ direction given by

$$\overline{J}_e(\overline{r}) = \frac{\hat{x}}{i\omega\mu}\delta(\overline{r} - \overline{r}') = \frac{\hat{x}}{ikZ}\delta(\overline{r} - \overline{r}') \tag{5}$$

According to (3) and (2) the resulting electric field can be obtained from

$$\overline{G}^x(\overline{r}, \overline{r}') = \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right) g(\overline{r}, \overline{r}')\hat{x}$$

where $\overline{G}^x(\overline{r}, \overline{r}')$ denotes the impulse response to an x-directed excitation. In a similar manner the electric field in response to infinitesimal y-directed and z-directed currents are given by

$$\overline{G}^y(\overline{r}, \overline{r}') = \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right) g(\overline{r}, \overline{r}')\hat{y}$$
$$\overline{G}^z(\overline{r}, \overline{r}') = \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right) g(\overline{r}, \overline{r}')\hat{z}$$

Using the compact dyadic notation, the electric field due to an arbitrary oriented (along $\hat{p}$) infinitesimal current $\frac{\hat{p}}{ikZ}\delta(\overline{r} - \overline{r}')$ can be obtained from:

$$\overline{E}^p(\overline{r}, \overline{r}') = \overline{\overline{G}}(\overline{r}, \overline{r}') \cdot \hat{p}$$

where $\overline{\overline{G}}(\overline{r}, \overline{r}') = \overline{G}^x(\overline{r}, \overline{r}')\hat{x} + \overline{G}^y(\overline{r}, \overline{r}')\hat{y} + \overline{G}^z(\overline{r}, \overline{r}')\hat{z}$ is referred to as the dyadic Green's function of free-space. The explicit expression for $\overline{\overline{G}}(\overline{r}, \overline{r}')$ is given by

$$\begin{aligned}\overline{\overline{G}}(\overline{r}, \overline{r}') &= \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right) g(\overline{r}, \overline{r}')(\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}) \\ &= \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right) g(\overline{r}, \overline{r}')\,\overline{\overline{I}}\end{aligned}$$

where $\overline{\overline{I}}$ is the unit dyad (idemfactor). Noting that $\nabla \cdot \left(\psi\,\overline{\overline{I}}\right) = \nabla\psi \cdot \overline{\overline{I}} = \nabla\psi$ for any differentiable scalar function ψ, the expression for the dyadic Green's function is given by

$$\bar{\bar{G}}(\overline{r},\overline{r}') = \left(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right) g(\overline{r},\overline{r}') \tag{6}$$

Referring to (1) each vector component of $\bar{\bar{G}}(\overline{r},\overline{r}')$ $\left(\overline{G}^q(\overline{r},\overline{r}'); q = x, y, z\right)$ satisfies

$$\nabla \times \nabla \times \overline{G}^q(\overline{r},\overline{r}') - k^2\overline{G}^q(\overline{r},\overline{r}') = \hat{q}\delta(\overline{r} - \overline{r}') \tag{7}$$

By juxtaposing a unit vector $\hat{x}$, $\hat{y}$, or $\hat{z}$ at the posterior position of the three vector equations given by (7) and summing these equations, we obtain

$$\nabla \times \nabla \times \bar{\bar{G}}(\overline{r},\overline{r}') - k^2 \bar{\bar{G}}(\overline{r},\overline{r}') = \bar{\bar{I}}\,\delta(\overline{r} - \overline{r}') \tag{8}$$

1 Derivation of Field Quantities From the Dyadic Green's Function

Consider a homogeneous medium bounded by a closed surface S which includes an arbitrary electric current distribution $\overline{J}_e(\overline{r})$. Using the vector wave equation (1) and (8) in conjunction with the vector-dyadic Green's theorem given by

$$\begin{aligned}\iiint_v \left[\overline{P} \cdot \nabla \times \nabla \times \bar{\bar{Q}} - (\nabla \times \nabla \times \overline{P}) \cdot \bar{\bar{Q}}\right] dv = \\ - \oiint_s \left[(\hat{n} \times \nabla \times \overline{P}) \cdot \bar{\bar{Q}} + (\hat{n} \times \overline{P}) \cdot \nabla \times \bar{\bar{Q}}\right] ds\ .\end{aligned} \tag{9}$$

an explicit expression for the electric field due to the impressed electric current can be obtained. By letting $\overline{P} = \overline{E}(\overline{r})$ and $\bar{\bar{Q}} = \bar{\bar{G}}(\overline{r},\overline{r}')$ it can easily be shown that

$$\begin{aligned}\overline{E}(\overline{r}') &= ikZ \iiint_v \overline{J}_e(\overline{r}) \cdot \bar{\bar{G}}(\overline{r},\overline{r}')dv \\ &- \oiint_s \left[(\hat{n} \times \nabla \times \overline{E}(\overline{r})) \cdot \bar{\bar{G}}(\overline{r},\overline{r}') + (\hat{n} \times \overline{E}(\overline{r})) \cdot \nabla \times \bar{\bar{G}}(\overline{r},\overline{r}')\right] ds\ .\end{aligned} \tag{10}$$

Noting that $\nabla \times \overline{E}(\overline{r}) = ikZ\overline{H}(\overline{r})$, (10) can be written as

$$\begin{aligned}\overline{E}(\overline{r}') &= ikZ \iiint_v \overline{J}_e(\overline{r}) \cdot \bar{\bar{G}}(\overline{r},\overline{r}')dv \\ &- \oiint_s \left[ikZ(n \times \overline{H}(\overline{r})) \cdot \bar{\bar{G}}(\overline{r},\overline{r}') + (\hat{n} \times \overline{E}(\overline{r})) \cdot \nabla \times \bar{\bar{G}}(\overline{r},\overline{r}')\right] ds\end{aligned} \tag{11}$$

To find an expression for the magnetic field, we start with the vector wave equation for the magnetic field given by

$$\nabla \times \nabla \times \overline{H}(\overline{r}) - k^2\overline{H}(\overline{r}) = \nabla \times \overline{J}_e(\overline{r}) \tag{12}$$

Again by letting $\overline{P} = \overline{H}(\overline{r})$ and $\overline{\overline{Q}}=\overline{\overline{G}}\,(\overline{r},\overline{r}')$ in (9), we obtain

$$\begin{aligned}\overline{H}(\overline{r}') &= \iiint_v \left[\nabla \times \overline{J}_e(\overline{r})\right] \cdot \overline{\overline{G}}\,(\overline{r},\overline{r}')dv \\ &- \oiint_s \left[(\hat{n} \times \nabla \times \overline{H}(\overline{r}))\cdot \overline{\overline{G}}\,(\overline{r},\overline{r}') + (\hat{n} \times \overline{H}(\overline{r})) \cdot \nabla\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right] ds\end{aligned} \tag{13}$$

Applying the dyadic identity

$$\nabla \cdot (\overline{a}\times \overline{\overline{b}}) = \nabla \times \overline{a}\cdot \overline{\overline{b}} - \overline{a} \cdot \nabla\times \overline{\overline{b}} \;,$$

The volume integral in (13) can be written as

$$\iiint_v \left[\nabla \times \overline{J_e}(\overline{r})\right] \cdot \overline{\overline{G}}\,(\overline{r},\overline{r}')dv = \iiint_v \left\{\nabla \cdot \left[\overline{J_e}(\overline{r})\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right] + \overline{J_e}(\overline{r}) \cdot \nabla\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right\} dv$$

Using the divergence theorem

$$\begin{aligned}\iiint_v \nabla \cdot \left[\overline{J_e}(\overline{r})\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right] dv &= \oiint_s \hat{n} \cdot \left[\overline{J_e}(\overline{r})\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right] ds \\ &= \oiint_s \left[\hat{n} \times \overline{J_e}(\overline{r})\right] \cdot \overline{\overline{G}}\,(\overline{r},\overline{r}')\, ds\end{aligned}$$

and Maxwell's equation

$$\nabla \times \overline{H}(\overline{r}) = -ikY\overline{E}(\overline{r}) + \overline{J_e}(\overline{r})$$

in (13), the expression for the magnetic field reduces to

$$\begin{aligned}\overline{H}(\overline{r}') &= \iiint_v \overline{J}_e(\overline{r}) \cdot \nabla\times \overline{\overline{G}}\,(\overline{r},\overline{r}')dv \\ &+ \oiint_s \left\{ikY\left[n \times \overline{E}(\overline{r})\right] \cdot \overline{\overline{G}}\,(\overline{r},\overline{r}') - (\hat{n} \times \overline{H}(\overline{r})) \cdot \nabla\times \overline{\overline{G}}\,(\overline{r},\overline{r}')\right\} ds\end{aligned} \tag{14}$$

2 Field Quantities Generated from Magnetic and Electric Currents

Equations (11) and (14) provide the electric and magnetic field quantities in a bounded region originated from an electric current distribution and a certain surface field quantities at the surface of this bounded region. In this section these results are extended to allow for the existence of both electric and magnetic currents. This can easily be done by first obtaining the field expressions using a magnetic current distribution as the excitation. The duality relations can be employed to find the field quantities for a magnetic current excitation from those given by (11) and (14). We first point out that the magnetic dyadic Green's function for an unbounded homogeneous medium is the same as the electric one. Apply the duality relations to (11) and (14) the following expressions are obtained

$$\begin{aligned}\overline{H}_m(\overline{r}') &= ikY \iiint_v \overline{J}_m(\overline{r})\cdot \overline{\overline{G}}(\overline{r},\overline{r}')dv \\ &- \oint\!\!\!\oint_s \left[-ikY(\hat{n}\times\overline{E}_m(\overline{r}))\cdot \overline{\overline{G}}(\overline{r},\overline{r}') + (\hat{n}\times\overline{H}_m(\overline{r}))\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\right]ds \end{aligned} \tag{15}$$

$$\begin{aligned}\overline{E}_m(\overline{r}') &= -\iiint_v \overline{J}_m(\overline{r})\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\,dv \\ &- \oint\!\!\!\oint_s \left[ikZ(\hat{n}\times\overline{H}_m(\overline{r}))\cdot \overline{\overline{G}}(\overline{r},\overline{r}') + (\hat{n}\times\overline{E}_m(\overline{r}))\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\right]ds \end{aligned} \tag{16}$$

Superposition of (11) and (16) and (14) and (15) provides the total fields within S and are given by

$$\begin{aligned}\overline{E}(\overline{r}') &= \iiint_v \left[ikZ\overline{J}_e(\overline{r})\cdot \overline{\overline{G}}(\overline{r},\overline{r}') - \overline{J}_m(\overline{r})\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\right]dv \\ &- \oint\!\!\!\oint_s \left[ikZ(\hat{n}\times\overline{H}(\overline{r}'))\cdot \overline{\overline{G}}(\overline{r},\overline{r}') + (\hat{n}\times E(\overline{r}'))\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\right]ds \end{aligned} \tag{17}$$

$$\begin{aligned}\overline{H}(\overline{r}') &= \iiint_v \left[\overline{J}_e(\overline{r})\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}') + ikY\overline{J}_m(\overline{r})\cdot \overline{\overline{G}}(\overline{r},\overline{r}')\right]dv \\ &+ \oint\!\!\!\oint_s \left[ikY(\hat{n}\times\overline{E}(\overline{r}))\cdot \overline{\overline{G}}(\overline{r},\overline{r}') - (\hat{n}\times\overline{H}(\overline{r}))\cdot\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\right]ds \end{aligned} \tag{18}$$

3 Radiation Condition For Dyadic Green's Function

The contribution from the surface integrals of (13) and (14) should vanish as the surface approaches infinity according to the radiation condition first postulated by Sommerfeld.

The electric field far away from the source and observation points satisfies

$$\lim_{r\to\infty} r\left\{\nabla\times\overline{E}(\overline{r}) - ik\hat{r}\times\overline{E}(\overline{r})\right\} = 0 \tag{19}$$

The magnetic field also satisfies an identical equation. Using (19) for $\overline{G}^x(\overline{r},\overline{r}'), \overline{G}^y(\overline{r},\overline{r}')$, and $\overline{G}^z(\overline{r},\overline{r}')$ and by juxtaposing unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ at the posterior position of each equation respectively and then adding the three resulting equations we get

$$\lim_{r\to\infty} r\left\{\nabla\times \overline{\overline{G}}(\overline{r},\overline{r}') - ik\hat{r}\times \overline{\overline{G}}(\overline{r},\overline{r}')\right\} = 0 \tag{20}$$

which is known as the radiation condition for the free-space dyadic Green's function.

4 Explicit Forms of The Dyadic Green's Function

The compact form of the dyadic Green's function which is given by

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \left[\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right]\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}$$

can be expressed in any desired coordinate system. For example, in Cartesian coordinate system, where

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\ ,$$

the dyadic Green's function can be represented, in matrix form, in the following manner

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \begin{pmatrix} k^2 + \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z} \\ \frac{\partial^2}{\partial y\partial x} & k^2 + \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y\partial z} \\ \frac{\partial^2}{\partial z\partial x} & \frac{\partial^2}{\partial z\partial y} & k^2 + \frac{\partial^2}{\partial z^2} \end{pmatrix} \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi k^2|\overline{r}-\overline{r}'|} \tag{21}$$

It is quite obvious from (21) that $\overline{\overline{G}}(\overline{r},\overline{r}')$ is a symmetric dyad, i.e.

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \left[\overline{\overline{G}}(\overline{r},\overline{r}')\right]^T \tag{22}$$

Therefore, for any vector $\overline{V}$, we have:

$$\overline{V}\cdot \overline{\overline{G}}(\overline{r},\overline{r}') = \overline{\overline{G}}(\overline{r},\overline{r}')\cdot\overline{V}$$

Also noting that $\nabla \times \nabla = 0, \nabla\times \overline{\overline{G}} (\overline{r}, \overline{r}')$ can easily be evaluated as follows

$$\begin{aligned} \nabla\times \overline{\overline{G}} (\overline{r}, \overline{r}') &= \nabla \times \left[\left(\overline{\overline{I}} + \frac{1}{k^2} \nabla\nabla \right) g(\overline{r}, \overline{r}') \right] = \nabla \times \left[\overline{\overline{I}}\, g(\overline{r}, \overline{r}') \right] \\ &= \nabla g(\overline{r}, \overline{r}') \times \overline{\overline{I}} \end{aligned}$$

which in Cartesian coordinate system takes the following form

$$\nabla\times \overline{\overline{G}} (r, \overline{r}') = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \tag{23}$$

which is obviously anti-symmetric (any dyad of the form $\overline{C}\times \overline{\overline{I}}$ is anti-symmetric). Another expanded form of $\overline{\overline{G}} (\overline{r}, \overline{r}')$ can be obtained by noting that

$$\begin{aligned} \nabla g(R) = \frac{d}{dR} g(R) \nabla R &= (ik - \frac{1}{R}) g(R) \nabla R \\ &= \left(ik - \frac{1}{R} \right) g(R) \hat{R} \ , \end{aligned}$$

where $R = |\overline{r} - \overline{r}'|$ and

$$\hat{R} = \frac{\overline{r} - \overline{r}'}{|\overline{r} - \overline{r}'|} \ .$$

Hence,

$$\nabla\nabla g(R) = \nabla \left[\left(ik - \frac{1}{R} \right) g(R) \right] \hat{R} + \left(ik - \frac{1}{R} \right) g(R) \nabla \hat{R} \ , \tag{24}$$

$\nabla\hat{R}$ can be calculated easily noting that,

$$\nabla(\hat{R}) = \nabla \left(\frac{\overline{R}}{R} \right) = \frac{\nabla(\overline{R})}{R} + \overline{R} \nabla \left(\frac{1}{R} \right)$$

But $\nabla(\overline{R}) = \overline{\overline{I}}$ and therefore

$$\nabla(\hat{R}) = \left(\overline{\overline{I}} - \hat{R}\hat{R} \right) \frac{1}{R}$$

After some algebraic manipulations it can be shown that

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \left\{ \left(\frac{3}{k^2R^2} - \frac{3i}{kR} - 1 \right) \hat{R}\hat{R} + \left(1 + \frac{i}{kR} - \frac{1}{k^2R^2} \right) \overline{\overline{I}} \right\} g(R) \tag{25}$$

5 Far Field Expression of Dyadic Green's Function

In the evaluation of fields away from the sources where $|\overline{r}-\overline{r}'|$ is much larger than typical dimension of the source, simple expressions for the field quantities are usually obtained. Keeping only the terms of the order of $\frac{1}{R} \simeq \frac{1}{r}$, the far-field expression for the free-space dyadic Green's function can be obtained from (25) and is given by

$$\begin{aligned} \overline{\overline{G}}(\overline{r},\overline{r}') &\simeq \left[\overline{\overline{I}} - \hat{r}\hat{r} \right] \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi r} \\ &\simeq \left[\overline{\overline{I}} - \hat{r}\hat{r} \right] \frac{e^{ikr}}{4\pi r} e^{-ik\hat{r}\cdot\overline{r}'} \end{aligned} \tag{26}$$

Equation (26) indicates that the field quantities do not possess a radial component.

6 Fourier Representation of The Free-Space Dyadic Green's Function

Another useful representation of the dyadic Green's function is its Fourier Transform where the field response to an impulse excitation is expressed in terms of a continuous spectrum (angular) of plane waves. This expansion in terms of plane waves is useful since the scattering solution of many problems to plane wave excitation is known. Using the plane wave solution together with the superposition principle, the solution to any arbitrary source can be obtained.

The starting point is equation (4). Let us assume, without loss of generality, that the source point is at the origin. Then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2 \right) g(\overline{r}) = -\delta(\overline{r}) \tag{27}$$

The Fourier transform of $g(\overline{r})$, represented by $\tilde{g}(\overline{k})$, is given by

$$\tilde{g}(\overline{k}) = \iiint_{-\infty}^{+\infty} g(\overline{r}) e^{-i(k_x x + k_y y + k_z z)} dx\, dy\, dz$$

Conversely $g(\overline{r})$ in terms of its Fourier transform is obtained from

$$g(\overline{r}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \tilde{g}(\overline{k}) e^{i(k_x x + k_y y + k_z z)} dk_x\, dk_y\, dk_z \tag{28}$$

Substituting (28) in (27) and noting that

$$\delta(\overline{r}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} e^{i(k_x x + k_y y + k_z z)} dk_x\, dk_y\, dk_z$$

$\tilde{g}(\overline{k})$ can be evaluated and is given by

$$\tilde{g}(\overline{k}) = \frac{1}{k_x^2 + k_y^2 + k_z^2 - k_o^2} \tag{29}$$

Although the 3-dimensional Fourier transform can be used to express the field quantities in terms of plane waves $\left(e^{i\overline{k}\cdot\overline{r}}\right)$, it is not usually used because all three components of the propagation vector are independent, that is, the frequencies of these plane waves are not necessarily the same. To constrain the propagation vector $\overline{k}$ the integration with respect to one of the variables must be carried out. We consider integration of (28) over k_z, that is

$$I(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{k_z^2 - h^2} e^{ik_z z} dk_z\,; \qquad h^2 = k_o^2 - k_x^2 - k_y^2 \tag{30}$$

Considering the behavior of $g(\overline{r})$, we expect that $I(z)$ approaches zero at $z = \pm\infty$. This is justifiable if we let h to be complex with $Im[h] > 0$. Such assumption is common and corresponds to a slightly lossy media. After evaluation of the integral, the lossless condition is restored by allowing $Im[h] \to 0$. With this assumption the locations of the poles of the integrand (30) are shown in Figure 1. The contour of integration is assumed to be along the real axis. For $z \geq 0$ the contour can be closed in the upper half-plane with a semi-circle of a large radius ($R \to \infty$) noting that $Im[k_z] > 0$ (a radiation condition requirement). In this case the integrand along the semi-circle contour is zero.

For $z \leq 0$, the contour can be closed in the lower half-plane. Using Cauchy's residue theorem to the contour integrals, $I(z)$ can easily be evaluated and is given by

$$I(z) = \frac{i}{2h} \begin{cases} e^{ihz} & z \geq 0 \\ e^{-ihz} & z \leq 0 \end{cases} = \frac{i}{2h} e^{ih|z|}$$

where $h = \sqrt{k_o^2 - k_x^2 - k_y^2}$. Replacing h with k_z and keeping in mind that k_z is no longer an independent parameter, (28) takes the following form

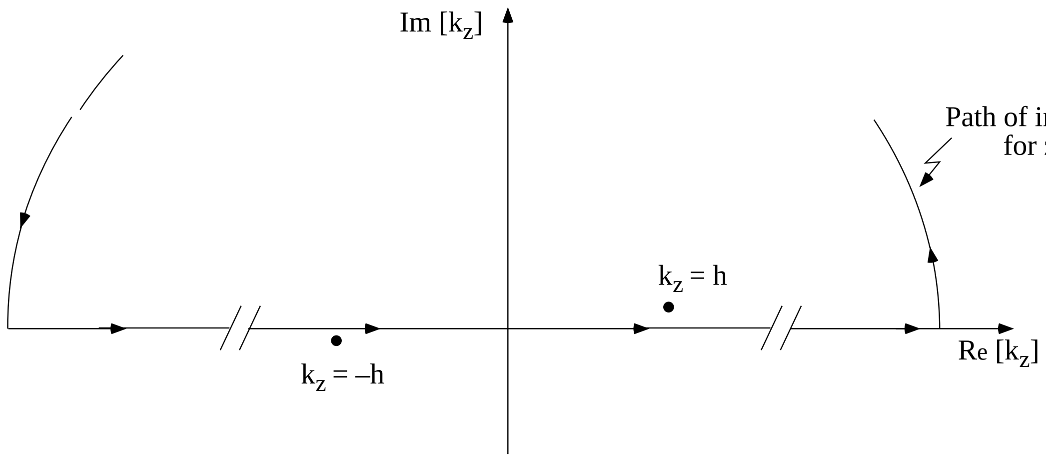

Figure 1: k_z-plane and the location of the poles of integrand (26).

$$g(\overline{r}) = \frac{i}{(2\pi)^2} \iint_{-\infty}^{+\infty} \frac{e^{+i\overline{k}_\perp \cdot \overline{r}_\perp + ik_z|z|} d\overline{k}_\perp}{2k_z} \tag{31}$$

where

$$\begin{aligned} \overline{k}_\perp &= k_x\hat{x} + k_y\hat{y} \\ \overline{r}_\perp &= x\hat{x} + y\hat{y} \\ k_z &= \sqrt{k^2 - k_\rho^2} \ , \quad Im[k_z] > 0 \\ k_\rho^2 &= k_x^2 + k_y^2 \end{aligned}$$

To find $\overline{\overline{G}}\ (\overline{r})$, derivatives of $g(\overline{r})$ must be evaluated, however, it should be noted that the derivative of $g(\overline{r})$ with respect to z is discontinuous, that is,

$$\frac{\partial}{\partial z} g(\overline{r}) = \left\{ \frac{-1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \frac{1}{2} e^{i\overline{k}_\perp \cdot \overline{r}_\perp + ik_z|z|} d\overline{k}_\perp \right\} f(z)$$

where $f(z)$ is a step function

$$f(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \end{cases}$$

Further differentiation with respect to z will give a dirac delta function

$$\frac{\partial^2}{\partial z^2} g(\overline{r}) \ = \ \left\{ -\frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} e^{i\overline{k}_\perp \cdot \overline{r}_\perp} dk_\perp \right\} \delta(z) -$$

$$\left\{ \frac{i}{(2\pi)^2} \iint_{-\infty}^{+\infty} e^{i\overline{k}_\perp \cdot \overline{r}_\perp + ik_z|z|} \frac{k_z}{2} dk_\perp \right\} f^2(z)$$

But $f^2(z) = 1$ and $\frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} e^{i\overline{k}_\perp \cdot \overline{r}_\perp} dk_\perp = \delta(x)\delta(y)$ and therefore

$$\frac{\partial^2}{\partial z^2} g(\overline{r}) = -\delta(\overline{r}) - \frac{i}{(2\pi)^2} \iint_{-\infty}^{+\infty} \frac{k_z}{2} e^{i(\overline{k}_\perp \cdot \overline{r}_\perp + k_z|z|)} dk_x \, dk_y \tag{32}$$

Substituting (31) in (6) and using (32) a simple expression for $\overline{\overline{G}}(\overline{r})$ can be obtained. Interchanging the order of differentiation and integration and noting that $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ can be replaced with ik_z, ik_y and $\pm ik_z$ ($+$ sign for $z > 0$ and $-$ sign for $z < 0$) respectively it can easily be shown that

$$\overline{\overline{G}}(\overline{r}) = \frac{-\hat{z}\hat{z}}{k^2} \delta(\overline{r}) + \begin{cases} \frac{i}{8\pi^2} \iint_{-\infty}^{+\infty} \frac{1}{k_z} \left[\overline{\overline{I}} - \frac{\overline{k}\overline{k}}{k^2} \right] e^{i\overline{k}\cdot\overline{r}} d\overline{k}_\perp & z > 0 \\ \frac{i}{8\pi^2} \iint_{-\infty}^{+\infty} \frac{1}{k_z} \left[\overline{\overline{I}} - \frac{\overline{K}\overline{K}}{k^2} \right] e^{i\overline{K}\cdot\overline{r}} d\overline{k}_\perp & z < 0 \end{cases} \tag{33}$$

where

$$\begin{aligned} \overline{k} &= k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ \overline{K} &= k_x\hat{x} + k_y\hat{y} - k_z\hat{z} \end{aligned}$$

Equation (33) is the expansion of the free-space dyadic Green's function in terms of a continuous spectrum of plane waves (monochromatic) propagating along the vectors $\overline{k}$ and $\overline{K}$ which are in general complex quantities. Propagation vector $\overline{K}$ is the mirror image of $\overline{k}$ in x-y plane $\left(\overline{K} = \overline{k} - 2(\overline{k} \cdot \hat{z})\hat{z}\right)$ and represents plane waves propagating along the $-\hat{z}$ direction. Another useful representation appropriate for planar boundaries can be obtained by decomposing the vectors in $\overline{\overline{G}}(\overline{r})$ into TE and TM components. Recognizing that $\overline{k}/k$ is a unit vector $(\hat{k})$ the horizontal (TE) and vertical (TM) unit vectors are, respectively, defined by

$$\begin{aligned} \hat{e}(k_z) &= \frac{\hat{k} \times \hat{z}}{|\hat{k} \times \hat{z}|} = \frac{k_y\,\hat{x} - k_x\,\hat{y}}{\sqrt{k_x^2 + k_y^2}} = \frac{1}{k_\rho}(k_y\hat{x} - k_x\hat{y}) \\ \hat{h}(k_z) &= \hat{e} \times \hat{k} = \frac{-k_z}{kk_\rho}(\hat{x}k_x + \hat{y}k_y) + \frac{k_\rho}{k}\hat{z} \end{aligned}$$

The triplet $\left(\hat{h}, \hat{e}, \hat{k}\right)$ form an orthonormal system, and therefore

$$\overline{\overline{I}} - \hat{k}\hat{k} = \hat{e}\hat{e} + \hat{h}\hat{h} \tag{34}$$

A similar orthonormal system can be formed with $\hat{K} = \frac{\overline{K}}{k}$ instead of $\hat{k}$. In this system the horizontal and vertical unit vectors are given by:

$$\begin{aligned} \hat{e}(-k_z) &= \frac{\hat{K} \times \hat{z}}{|\hat{K} \times \hat{z}|} = \hat{e}(k_z) \\ \hat{h}(-k_z) &= \hat{e} \times \hat{K} \end{aligned}$$

and as before

$$\overline{\overline{I}} - \hat{K}\hat{K} = \hat{e}(k_z)\hat{e}(k_z) + \hat{h}(-k_z)\hat{h}(-k_z) \tag{35}$$

Inserting (34) and (35) into (33) and translating the source from the origin to $\overline{r}'$, the free-space dyadic Green's function takes the following form:

$$\overline{\overline{G}}(\overline{r}, \overline{r}') = \frac{-\hat{z}\hat{z}}{k^2}\delta(\overline{r} - \overline{r}') + \begin{cases} \frac{i}{8\pi^2} \iint\limits_{-\infty}^{+\infty} \frac{1}{k_z} \left[\hat{e}\hat{e} + \hat{h}(k_z)\hat{h}(k_z)\right] e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d\,\overline{k}_\perp & z > z' \\ \frac{i}{8\pi^2} \iint\limits_{-\infty}^{+\infty} \frac{1}{k_z} \left[\hat{e}\hat{e} + \hat{h}(-k_z)\hat{h}(-k_z)\right] e^{i\overline{K}\cdot(\overline{r}-\overline{r}')} d\,\overline{k}_\perp & z < z' \end{cases} \tag{36}$$

This form of the dyadic Green's function is usually not appropriate for numerical evaluation, especially when $z - z' << \lambda$. In this case the convergence rate of the integral is very poor.

7 Dyadic Green's Function for Two-Dimensional Problems

In some electromagnetic scattering problems where the geometry of the problem is independent of one coordinate variable the formulation of the problem can be made somewhat simpler. Without loss of generality let us assume that the scatterer geometry is independent of z. In this case the scatterer is a cylinder of arbitrary cross-section whose axis is along $\hat{z}$. Since there is no variation with respect to z, all field quantities take the z dependence of the excitation. Suppose

$$\overline{J}(\overline{r}) = \overline{J}(\overline{\rho})e^{ik_{zi}z}$$

where $\overline{\rho} = x\hat{x} + y\hat{y}$. The differential operator can also be made explicit with respect to $\frac{\partial}{\partial z}$ which is replaced by ik_{zi}, that is

$$\nabla = \nabla_t + ik_{zi}\hat{z}$$

where $\nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$. Since the dependence of $\overline{J}(\overline{r})$ with respect to z is explicit, in field calculations the integral with respect to z can be carried out. Therefore the 2-D dyadic Green's function is given by

$$\overline{\overline{G}}(\overline{\rho}, \overline{\rho}') = \int_{-\infty}^{+\infty} \overline{\overline{G}}(\overline{r}, \overline{r}') e^{ik_{zi}z'} dz' = \frac{1}{4\pi}\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right) \int_{-\infty}^{+\infty} \frac{e^{ik|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} e^{ik_{zi}z'} dz'$$

Using the identity

$$\int_{-\infty}^{+\infty} \frac{e^{ik\sqrt{|\overline{\rho}-\overline{\rho}'|^2+(z-z')^2}}}{\sqrt{|\overline{\rho}-\overline{\rho}'|^2+(z-z')^2}} e^{ik_{zi}z'} dz' = i\pi H_0^{(1)}(k_\rho|\overline{\rho}-\overline{\rho}'|)e^{ik_{zi}z} \tag{37}$$

where $k_\rho = \sqrt{k^2 - k_{zi}^2}$, the 2-D dyadic Green's function takes the following form:

$$\overline{\overline{G}}(\overline{\rho}, \overline{\rho}') = \frac{i}{4}\left[\overline{\overline{I}} + \frac{1}{k^2}\left(\nabla_t\nabla_t + ik_{zi}\nabla_t\hat{z} + ik_{zi}\hat{z}\nabla_t - k_{zi}^2\hat{z}\hat{z}\right)\right] H_0^{(1)}(k_\rho|\overline{\rho}-\overline{\rho}'|)e^{ik_{zi}z} \ . \tag{38}$$

In matrix form (38) becomes

$$\overline{\overline{G}}(\overline{\rho}, \overline{\rho}') = \begin{pmatrix} 1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2} & \frac{1}{k^2}\frac{\partial^2}{\partial x \partial y} & \frac{ik_{zi}}{k^2}\frac{\partial}{\partial x} \\ \frac{1}{k^2}\frac{\partial^2}{\partial y \partial x} & 1 + \frac{1}{k^2}\frac{\partial^2}{\partial y^2} & \frac{ik_{zi}}{k^2}\frac{\partial}{\partial y} \\ \frac{ik_{zi}}{k^2}\frac{\partial}{\partial x} & \frac{ik_{zi}}{k^2}\frac{\partial}{\partial y} & \frac{k\rho^2}{k_0^2} \end{pmatrix} \frac{i}{4} H_0^{(1)}(k_\rho|\overline{\rho}-\overline{\rho}'|)e^{ik_{zi}z} \tag{39}$$

8 Fourier Representation of 2-D Dyadic Green's Function

A procedure similar to what was shown for 3-dimensional dyadic Green's function can be followed to obtain the Fourier representation of 2-D dyadic Green's function. The Fourier representation can also be obtained in a simpler way using the following identity

$$H_0^{(1)}(k_\rho\sqrt{(x-x')^2 + (y-y)^2}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{k_y} e^{ik_x(x-x')+ik_y|y-y'|} dk_x \tag{40}$$

where $k_y = \sqrt{k_\rho^2 - k_x^2}$. Substituting (40) in (39) and after some algebraic manipulations it can be shown that

$$\overline{\overline{G}}\,(\overline{\rho}, \overline{\rho}') = \frac{-\hat{y}\hat{y}}{k^2}\delta(\overline{\rho} - \overline{\rho}')e^{ik_{zi}z} + \begin{cases} \frac{ie^{ik_{zi}z}}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{k_y} \left[\overline{\overline{I}} - \frac{\overline{k}\overline{k}}{k^2}\right] e^{i[k_x(x-x')+k_y(y-y')}dk_x \;\; y > y' \\ \frac{ie^{ik_{zi}z}}{4\pi} \int \frac{1}{k_y} \left[\overline{\overline{I}} - \frac{\overline{K}\overline{K}}{k^2}\right] e^{i[k_x(x-x')-k_y(y-y')]}dk_x \;\; y < y' \end{cases} \tag{41}$$

where

$$\begin{aligned} \overline{k} &= k_x\hat{x} + k_y\hat{y} + k_{zi}\hat{z} \\ \overline{K} &= k_x\hat{x} - k_y\hat{y} + k_{zi}\hat{z} \;\; . \end{aligned}$$

As before $\overline{k}$ can be considered as the propagation vector of a plane wave going along positive y direction and $\overline{K}$ is that of a wave going along $-y$ direction. k_{zi} is a fixed known quantity.

9 Symmetrical Property of Dyadic Green's Function

Symmetrical property of dyadic Green's function allows for simple evaluation of the dyadic Green's function when the locations of source and observation points are interchanged. To demonstrate this property the dyadic-dyadic Green's second identity given by

$$\begin{aligned} \iiint_v \left[\nabla \times \nabla \times \overline{\overline{Q}}\right]^T \cdot \overline{\overline{P}} \;\; &- \;\; \left[\overline{\overline{Q}}\right]^T \cdot \nabla \times \nabla \times \overline{\overline{P}} \; dv = \\ &- \;\; \oiint_s \left\{\left[\overline{\overline{Q}}\right]^T \cdot \left(\hat{n} \times \nabla \times \overline{\overline{P}}\right) + \left[\nabla \times \overline{\overline{Q}}\right]^T \cdot \left(\hat{n} \times \overline{\overline{P}}\right)\right\} ds \end{aligned} \tag{42}$$

will be employed. Let us consider two situations where in each case the source location is at $\overline{r}_a$ and $\overline{r}_b$ respectively. The dyadic Green's functions for each case must satisfy

$$\nabla \times \nabla \times \overline{\overline{G}}\,(\overline{r}, \overline{r}_a) - k^2\, \overline{\overline{G}}\,(\overline{r}, \overline{r}_a) \;\; = \;\; \delta(\overline{r} - \overline{r}_a)\, \overline{\overline{I}} \tag{43}$$

$$\nabla \times \nabla \times \overline{\overline{G}}\,(\overline{r}, \overline{r}_b) - k^2\, \overline{\overline{G}}\,(\overline{r}, \overline{r}_b) \;\; = \;\; \delta(\overline{r} - \overline{r}_b)\, \overline{\overline{I}} \tag{44}$$

Substituting $\overline{\overline{G}}\,(r, \overline{r}_a)$ for $\overline{\overline{Q}}$ and $\overline{\overline{G}}\,(\overline{r}, \overline{r}_b)$ for $\overline{\overline{P}}$ in (42) and using the radiation condition at infinity and equations (43) and (44) it can readily be shown that

$$\overline{\overline{G}}(\overline{r}_a, \overline{r}_b) = \left[\overline{\overline{G}}(\overline{r}_b, \overline{r}_a)\right]^T \tag{45}$$

That is, the dyadic Green's function when the source is at $\overline{r}_b$ and the observation point is at $\overline{r}_a$ is the transpose of the Green's function when the source point is at $\overline{r}_a$ and the observation point is at $\overline{r}_b$. Although the proof is given for the free space Green's function, (40) is a general result. Equations (21) and (25) show that the free space Green's function is symmetric, i.e.,

$$\overline{\overline{G}}(\overline{r}, \overline{r}') = \left[\overline{\overline{G}}(\overline{r}, \overline{r}')\right]^T$$

In view of (45) for free space Green's function we have

$$\overline{\overline{G}}(\overline{r}, \overline{r}') = \overline{\overline{G}}(\overline{r}', \overline{r}) \tag{46}$$

However, it should be noted that (46) <u>is not</u> a general result.

10 Dyadic Green's Function for Piece-Wise Homogenous Media

In the previous sections we considered the properties of dyadic Green's function for homogeneous media. In practice however, the medium of interest is often complex which may be composed of many homogeneous media such as the one shown in Figure 2.

For these problems it is usually desired to derive the expression for a dyadic Green's function which satisfies the necessary field boundary conditions at the interfaces.

The field quantities in response to a volumetric electric current distribution $\overline{J}_e(\overline{r})$ in the unbounded inhomogeneous medium are simply given by

$$\overline{E}(\overline{r}') = ik_n Z_n \iiint_v \overline{J}_e(\overline{r}) \cdot \overline{\overline{G}}(\overline{r}, \overline{r}')\, dv \tag{47}$$

$$\overline{H}(\overline{r}') = \iiint_v \nabla' \times [J_e(r) \cdot \overline{\overline{G}}(\overline{r}, \overline{r}')] dv \tag{48}$$

The simple form of (47) is obtained by imposing certain boundary conditions on $\overline{\overline{G}}(\overline{r}, \overline{r}')$. To derive these boundary conditions consider a simple medium composed of two homogeneous media. Suppose the source exists only in medium 1 where we have

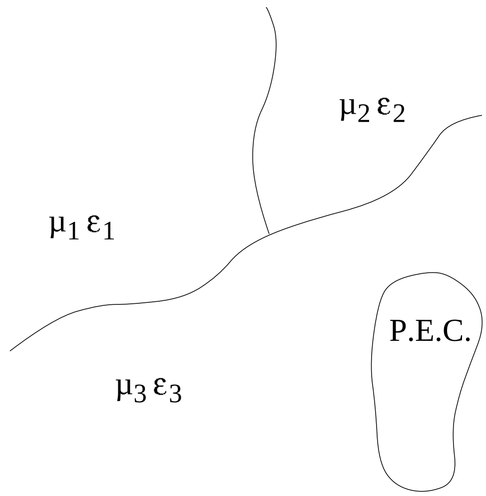

Figure 2: A complex medium composed of a number of homogeneous media and perfect electric conductors.

$$\nabla \times \nabla \times \overline{E}_1(\overline{r}) - k_1^2 E_1(\overline{r}) = i\omega\mu_1 \overline{J}_1(\overline{r}) \tag{49}$$

and in the second medium

$$\nabla \times \nabla \times \overline{E}_2(\overline{r}) - k_2^2 E_2(\overline{r}) = 0 \tag{50}$$

Let us denote the expression for the dyadic Green's function in medium 1 by $\overline{\overline{G}}^{(11)}(r, r')$ which satisfies

$$\nabla \times \nabla \times \overline{\overline{G}}^{(11)}(\overline{r}, \overline{r}') - k_1^2 \overline{\overline{G}}^{(11)}(\overline{r}, \overline{r}') = \overline{\overline{I}}\,\delta(\overline{r} - \overline{r}') \tag{51}$$

and the dyadic Green's function in medium 2 by $\overline{\overline{G}}^{(21)}(r, r')$ which satisfies

$$\nabla \times \nabla \times \overline{\overline{G}}^{(21)}(\overline{r}, \overline{r}') - k_2^2 \overline{\overline{G}}^{(21)}(\overline{r}, \overline{r}') = 0\ . \tag{52}$$

The application of the vector-dyadic Green's second identity to (49) and (51) gives:

$$\begin{aligned} \overline{E}_1(\overline{r}') &= i\omega\mu_1 \iiint_v \overline{J}_1(\overline{r}) \cdot \overline{\overline{G}}^{(11)}(r, r') dv' \qquad (53) \\ &+ \iint_s \Big[i\omega\mu_1 (\hat{n}_1 \times \overline{H}_1(\overline{r}_1)) \cdot \overline{\overline{G}}^{(11)}(r, r') + (\hat{n}_1 \times \overline{E}_1(\overline{r})) \cdot \nabla \times \overline{\overline{G}}^{(11)}(r, r') \Big] ds \end{aligned}$$

where s is the boundary between the two media.

The application of the vector-dyadic Green's second identity to (50) and (52) provides

$$\iint_s \Big[i\omega\mu_2(\hat{n}_1 \times \overline{H}_2(\overline{r}))\cdot \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}') + (\hat{n}_1 \times \overline{E}_2(\overline{r}))\cdot \nabla\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}')\Big] ds = 0 \qquad (54)$$

Noting that

$$\begin{aligned} \hat{n}_1 \times \overline{H}_1(\overline{r}) &= \hat{n}_1 \times \overline{H}_2(\overline{r}) \\ \hat{n}_1 \times \overline{E}_1(\overline{r}) &= \hat{n}_1 \times \overline{E}_2(\overline{r}) \end{aligned}$$

and using the following identities

$$\begin{aligned} \hat{n}_1 \times \overline{H}_1(\overline{r})\cdot \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}') &= -\overline{H}_1(\overline{r})\cdot(\hat{n}_1\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}')) \\ \hat{n}_1 \times \overline{E}_1(\overline{r})\cdot\nabla\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}') &= -\overline{E}_1(\overline{r})\cdot\hat{n}_1\times(\nabla\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}')) \end{aligned}$$

the contribution from the surface integral of (53) can be shown to vanish if

$$\mu_1\hat{n}\times \overline{\overline{G}}^{(11)}(\overline{r},\overline{r}') = \mu_2\hat{n}\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}') \qquad (55)$$

and

$$\hat{n}\times\nabla\times \overline{\overline{G}}^{(11)}(\overline{r},\overline{r}') = \hat{n}\times\nabla\times \overline{\overline{G}}^{(21)}(\overline{r},\overline{r}') \qquad (56)$$

Equations (55) and (56) are the necessary boundary conditions for the dyadic Green's function.

On the surface of perfect electric conductors

$$\hat{n}(\overline{r})\times\overline{E}(\overline{r}) = 0$$

which mandates $\hat{n}(\overline{r})\times \overline{\overline{G}}(\overline{r}',\overline{r}) = 0$.

The symmetry property of the dyadic Green's function can be shown easily by following the same procedure outlined for the free space dyadic Green's function. Let us consider two experiments where in one experiment the source is placed at an arbitrary point $\overline{r}_a$ and the observation point is at $\overline{r}$ in the nth region. In the second experiment we place the source point at an arbitrary point $\overline{r}_b$ while keeping the observation point at the same location $\overline{r}$ as shown in Figure 3. In both experiments the dyadic Green's functions satisfy (51) (without the superscripts). Applying the dyadic-dyadic Green's second identity to

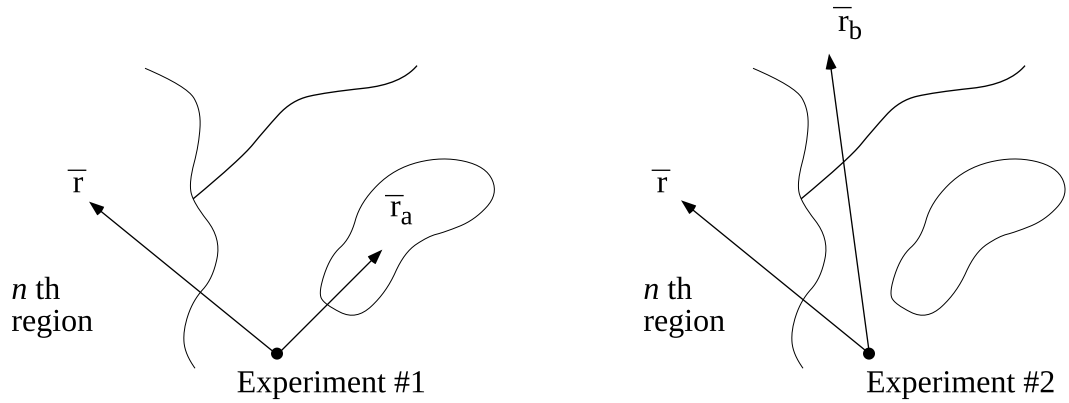

Figure 3: Locations of sources and observation points in a complex dielectric medium for demonstrating the symmetry property of dyadic Green's function.

these dyadic Green's functions which satisfy the radiation condition, it can readily be shown that

$$\bar{\bar{G}}\,(\overline{r}_a, \overline{r}_b) = \left[\bar{\bar{G}}\,(\overline{r}_b, \overline{r}_a)\right]^T . \tag{57}$$

It should be noted that it is not very easy to find a dyadic Green's function for a general piece-wise homogeneous media with arbitrary boundaries. In such cases it is more convenient to use the free space dyadic Green's function with surface integrals given by (11) and (14).

In view of (57), equation (47) can be written as

$$\overline{E}(\overline{r}') = i\,k_n\,Z_n \iiint_v \bar{\bar{G}}\,(\overline{r}', \overline{r}) \cdot \overline{J}_e(\overline{r})\,dv$$

where we used the fact that

$$\overline{J}_e(r)\cdot \bar{\bar{G}}\,(r, r') = \left[\bar{\bar{G}}\,(\overline{r}, \overline{r}')\right]^T \cdot \overline{J}_e(\overline{r})$$

and

$$\left[\bar{\bar{G}}\,(\overline{r}, \overline{r}')\right]^T = \bar{\bar{G}}\,(\overline{r}', \overline{r}).$$

Now interchanging $\overline{r}'$ with r and vice versa we get

$$\overline{E}(\overline{r}) = ik_n Z_n \iiint_v \bar{\bar{G}}\,(\overline{r}, \overline{r}') \cdot \overline{J}_e(r')dv' \tag{58}$$

11 Dyadic Green's Function For Inhomogeneous Media

Consider an inhomogeneous isotropic medium whose permittivity and permeability are functions of position and are, respectively, denoted by $\epsilon(\overline{r})$ and $\mu(\overline{r})$. As before we are seeking simple expressions for electric and magnetic fields for an arbitrary source using the impulse response of the medium. It should be emphasized that the evaluation of the dyadic Green's function for this type of problem, in general, is very complex; however, here a formal analysis is provided[1]. Taking the curl of the modified Amper's law, it can be shown that the vector wave equation takes the following form

$$\nabla \times \left[\frac{1}{\mu(\overline{r})} \nabla \times \overline{E}(\overline{r}) \right] - \omega^2 \, \epsilon(\overline{r}) \, \overline{E}(\overline{r}) = i\omega \overline{J}_e(\overline{r}) \tag{59}$$

Comparing (1) and (59)for this problem one can define a Green's function such that it would satisfy

$$\nabla \times \left[\frac{1}{\mu(\overline{r})} \nabla \times \overline{\overline{G}} \, (\overline{r}, \overline{r}') \right] - \omega^2 \epsilon(\overline{r}) \, \overline{\overline{G}} \, (\overline{r}) = \frac{1}{\mu(\overline{r})} \, \overline{\overline{I}} \, \delta(\overline{r} - \overline{r}') \tag{60}$$

Noting that

$$\nabla \cdot \left(\frac{1}{\mu(\overline{r})} \overline{P} \times \nabla \times \overline{Q} \right) = \frac{1}{\mu(\overline{r})} \nabla \times \overline{Q} \cdot \nabla \times \overline{P} - \overline{P} \cdot \nabla \times \left(\frac{1}{\mu(\overline{r})} \nabla \times \overline{Q} \right)$$

The Green's second vector identity can be written as

$$\begin{aligned} \iiint_v \left\{ \overline{P} \cdot \nabla \times \left[\frac{1}{\mu} \nabla \times \overline{Q} \right] - \overline{Q} \cdot \nabla \times \left[\frac{1}{\mu} \nabla \times \overline{P} \right] \right\} dv = \\ = \oiint_s \frac{1}{\mu} \left\{ \overline{Q} \times \nabla \times \overline{P} - \overline{P} \times \nabla \times \overline{Q} \right\} \cdot \hat{n} \, ds \end{aligned} \tag{61}$$

from which the vector-dyadic Green's second identity can be obtained and is given by

$$\begin{aligned} \iiint_v \left\{ \overline{P} \cdot \nabla \times \left[\frac{1}{\mu(\overline{r})} \nabla \times \overline{\overline{Q}} \right] - \nabla \times \left[\frac{1}{\mu(\overline{r})} \nabla \times \overline{P} \right] \cdot \overline{\overline{Q}} \right\} dv \\ = \oiint_s \frac{1}{\mu(\overline{r})} \left\{ \overline{P} \cdot \left(\hat{n} \times \nabla \times \overline{\overline{Q}} \right) - \left(\hat{n} \times \nabla \times \overline{P} \right) \cdot \overline{\overline{Q}} \right\} ds \end{aligned} \tag{62}$$

[1] W.C. Chew, " ".

Substituting $\overline{E}$ for $\overline{P}$ and $\overline{\overline{G}}$ for $\overline{\overline{Q}}$ in (62), applying (59) and (60), and using the radiation condition it can easily be shown that

$$\overline{E}(\overline{r}) = i\omega\mu(\overline{r}) \iiint_v \overline{J}_e(\overline{r}')\cdot \overline{\overline{G}}(\overline{r}',\overline{r})dv' \tag{63}$$

The magnetic field can be obtained from the application of Faraday's law ($\overline{H}(\overline{r}) = \frac{1}{i\omega\mu(\overline{r})}\nabla\times\overline{E}(\overline{r})$).

$$\overline{H}(\overline{r}) = \frac{1}{\mu(\overline{r})} \iiint_v \overline{J}(\overline{r})\cdot\nabla\times\left[\mu(\overline{r})\,\overline{\overline{G}}(\overline{r}',\overline{r})\right] dv \tag{64}$$

Using the dyadic-dyadic Green's second identity

$$\iiint_v \left\{\left[\nabla\times\left(\frac{1}{\mu(\overline{r})}\nabla\times\overline{\overline{Q}}\right)\right]^T\cdot\overline{\overline{P}} - [\overline{\overline{Q}}]^T\cdot\nabla\times\left(\frac{1}{\mu(\overline{r})}\nabla\times\overline{\overline{P}}\right)\right\} dv =$$
$$-\oiint_s \frac{1}{\mu(r)}\left\{\left[\overline{\overline{Q}}\right]^T\cdot\left(\hat{n}\times\nabla\times\overline{\overline{P}}\right) + \left[\nabla\times\overline{\overline{Q}}\right]^T\cdot\left(\hat{n}\times\overline{\overline{P}}\right)\right\} ds$$

it can easily be shown that

$$\mu(\overline{r}')\,\overline{\overline{G}}(\overline{r},\overline{r}') = \mu(\overline{r})\left[\overline{\overline{G}}(\overline{r}',\overline{r})\right]^T. \tag{65}$$