Evaluation of Attenuation Rate For a Layer of Randomly Distributed Particles[1]

Evaluation of the mean-field in a random medium was studied using the perturbation method applied to the wave equation. As discussed Dyson's equation for the mean-field is valid for media with relatively low permittivity fluctuations. In random media with high dielectric contrasts, such as vegetation canopies, the methods based on the perturbation analysis would fail. For these types of problems other techniques may be used. In situations where the particle number density is low, multiple scattering among the particles is low and an approximate solution for estimating the attenuation rate of the mean-field in the random medium can be obtained. This methodology is referred to as the single-scattering theory.

In this section two heuristic methods are considered for evaluating the attenuation rate of the mean-field for a sparse random medium. The first method is an incoherent approach developed based on the optical theorem and the second approach is a coherent one which provides an equivalent index of refraction (or effective dielectric constant) for the medium.

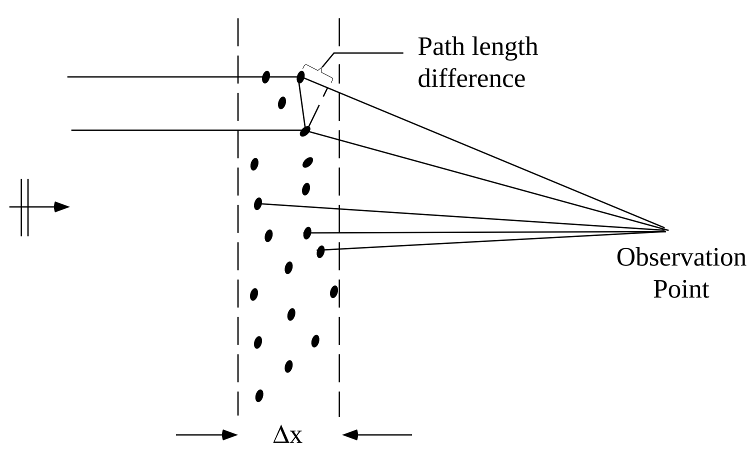

Figure 1: A layer of randomly distributed particles illuminated by a plane wave.

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Geometry of the problem is shown in Figure 1 where a layer of randomly distributed particles confined over a relatively thin layer with thickness Δx is illuminated by a plane wave. The orientation, size, position, and permittivity joint probability distribution of the particles is denoted by a prescribed function $P(\theta_p, \phi_p; \bar{a}; \bar{r}; \epsilon)$, where (θ_p, ϕ_p) denotes the particle orientation, $\bar{a}$ is a vector parameter denoting particle size, $\bar{r}$ is the position vector, and ϵ is the permittivity of the particles. Assuming that the observation point is far enough away from the scatterers, only the scatterers near the incident ray which goes through the observation point contribute to the mean-field. This is due to the fact that the phase differences of the scattered fields of the scatterers outside the first Fresnel zone are uniformly distributed. Hence it is expected that the field attenuation rate caused by scattering and absorption be related to the forward scattering properties of individual particles. To relate the energy loss due to scattering and absorption to the particles' parameters and their statistics, the "optical theorem" can be used which will be discussed next.

0.1 Optical Theorem

Optical theorem provides a relationship between the total power scattered in all directions and the power dissipated inside a particle, due to ohmic loss, to the scattering amplitude of the particle in the forward direction. Consider a particle with an arbitrary geometry and permittivity illuminated by a plane wave as shown in Figure 2. The incident, scattered, and the total field in the lossless background medium are, respectively, denoted by $\overline{E}_i, \overline{E}_s$, and $\overline{E}$. For a monochromatic plane wave the expression for the incident field may be expressed by

$$\overline{E}_i = \hat{p} E_0 e^{i\bar{k}_i \cdot \bar{r}} \tag{1}$$

The scattered field in the far-field region of the scatterer can be expressed in terms of a spherical wave given by

$$\overline{E}_s = \frac{e^{ik_0 r}}{r} \,\overline{\overline{S}}\,(\hat{k}_s, \hat{k}_i) \cdot \hat{p} E_0$$

where $\overline{\overline{S}}\,(\hat{k}_s, \hat{k}_i)$ is the bistatic scattering matrix of the scatterer and k_0 is the propagation constant in the background medium. In order to evaluate the total power carried by the scattered field, consider a fictitious surface S_1 enclosing the scatterer. The only restriction on the arbitrary surface S_1 is that every point on S_1 is in the far-field of the scatterer. The time-average Poynting vector is

$$\overline{S} = \frac{1}{2} Re \left\{ \overline{E} \times \overline{H}^* \right\} \tag{2}$$

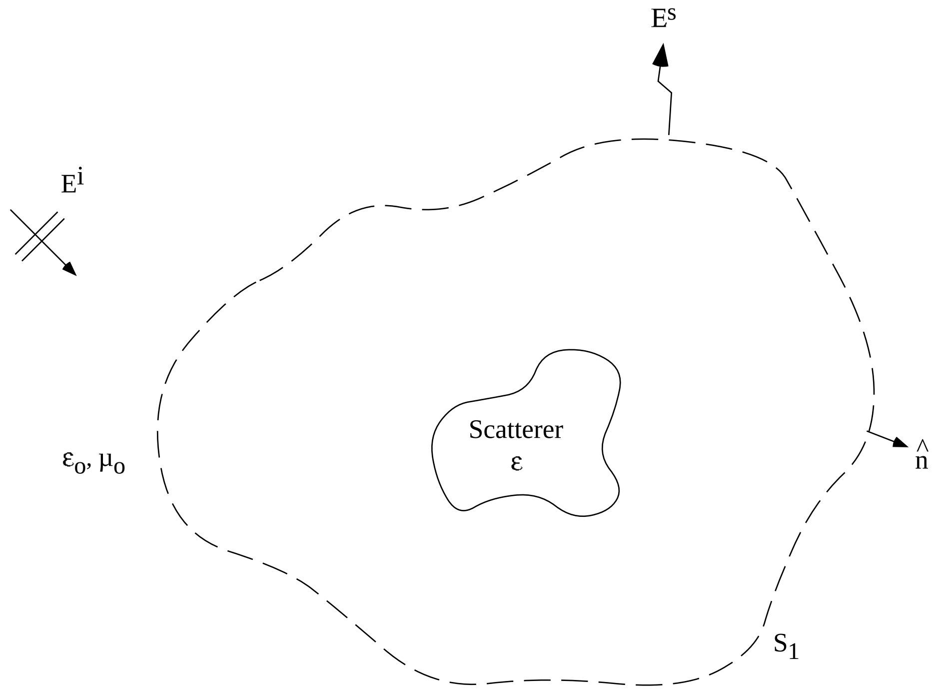

Figure 2: Geometry of a scatterer illuminated by a plane wave and a fictitious surface S_1 for evaluating the total scattered power.

where

$$\overline{E} = \overline{E}_i + \overline{E}_s \quad , \quad \overline{H} = \overline{H}_i + \overline{H}_s \tag{3}$$

The integration of $\overline{S}$ over the closed surface provides the total (incident plus scattered) power leaving the surface. If the scatterer is lossless and passive, the net flow of power out of the close surface must be zero according to the law of conservation of energy. If the scatterer is lossy, however, the total dissipated power by the scatterer (P_a) can be calculated from

$$-P_a = \oint_{S_1} \overline{S} \cdot \hat{n} ds \tag{4}$$

Substituting (3) in (2), the total Poynting vector is given by

$$\overline{S} = \frac{1}{2} Re \left\{ \overline{E}_i \times \overline{H}_i^* \right\} + \frac{1}{2} Re \left\{ \overline{E}_s \times \overline{H}_s^* \right\} + \frac{1}{2} Re \left\{ \overline{E}_i \times \overline{H}_s^* + \overline{E}_s \times \overline{H}_i^* \right\} \tag{5}$$

Integration of the first and second terms gives the net outflow of the incident and scattered power. Of course we expect that the net outflow of the incident power from the close surface be zero. This can be shown mathematically by noting that

$\nabla \cdot S_i = \frac{1}{2} Re\left[\nabla \cdot (\overline{E}_i \times \overline{H}_i^*)\right] = 0$, and this is true since $\overline{E}_i \times \overline{H}_i^* = \frac{|E_o|^2}{Z_0}\hat{k}_i$ is independent of position. Representing the scattered power loss by

$$P_s = \frac{1}{2} Re\left\{\oint_{S_1} \left(\overline{E}_s \times \overline{H}_s^*\right) \cdot \hat{n} ds\right\}, \tag{6}$$

according to (4) the total power loss due to absorption and scattering is given by

$$-P_t = -\frac{1}{2} Re\left\{\oint_{S_1} \left(\overline{E}_i \times \overline{H}_s^* + \overline{E}_s \times \overline{H}_i^*\right) \cdot \hat{n} ds\right\} = P_a + P_s \tag{7}$$

Conjugating the first term inside the bracket and substituting the appropriate expressions for $\overline{E}_i$ and $\overline{H}_i$, we get

$$-P_t = \frac{1}{2} Re\left\{\oint_{S_1} E_0^* e^{-i\bar{k}_i \cdot \bar{r}} \left[\hat{p} \cdot \left(\hat{n} \times \overline{H}_s\right) - \frac{1}{Z_0}\left(\hat{k}_i \times \hat{p}\right) \cdot \left(\hat{n} \times \overline{E}_s\right)\right] ds\right\} \tag{8}$$

where we have made the use of the following vector identifies

$$\begin{aligned} \left(\overline{E}_i^* \times \overline{H}_s\right) \cdot \hat{n} &= -\overline{E}_i^* \cdot \left(\hat{n} \times \overline{H}_s\right) && (9) \\ \left(\overline{E}_s \times \overline{H}_i^*\right) \cdot \hat{n} &= \overline{H}_i^* \cdot \left(\hat{n} \times \overline{E}_s\right) && (10) \end{aligned}$$

An expression for the scattered field in terms of surface fields on S_1 can be obtained from

$$\overline{E}_s(\bar{r}) = \oint_{S_1} \left\{ik_0 Z_0 \,\overline{\overline{G}}\,(\bar{r}, \bar{r}') \cdot (\hat{n} \times \overline{H}) + \nabla \times \overline{\overline{G}}\,(\bar{r}, \bar{r}') \cdot (\hat{n} \times \overline{E})\right\} ds' \tag{11}$$

Suppose the observation point is in the far-field region of S_1, then

$$\begin{aligned} \overline{\overline{G}}\,(\bar{r}, \bar{r}') &\simeq (\hat{h}_s\hat{h}_s + \hat{v}_s\hat{v}_s)\frac{e^{ik_0 r}}{4\pi r} e^{-i\bar{k}_s \cdot \bar{r}'} && (12) \\ \overline{E}_s(\bar{r}) &= \frac{e^{ik_0 r}}{r}\,\overline{\overline{S}}\,(\hat{k}_s, \hat{k}_i) \cdot \hat{p} E_o && (13) \end{aligned}$$

Where as before $\hat{v}_s$ and $\hat{h}_s$ are vertical and horizontal unit vectors perpendicular to $\hat{k}_s$. Substituting (12) and (13) in (11), we obtain

$$\begin{aligned} \overline{\overline{S}}\,(\hat{k}_s, \hat{k}_i) \cdot \hat{p} E_0 &= \frac{1}{4\pi} ik_0 Z_0 \oint_{S_1} e^{-i\bar{k}_s \cdot \bar{r}'} \left\{(\hat{h}_s\hat{h}_s + \hat{v}_s\hat{v}_s) \cdot (\hat{n} \times \overline{H}_s)\right. && (14) \\ &+ \left.\frac{1}{Z_0}\hat{k}_s \times (\hat{h}_s\hat{h}_s + \hat{v}_s\hat{v}_s) \cdot (\hat{n} \times \overline{E}_s)\right\} ds' \end{aligned}$$

Evaluating the co-polarized component of the scattered field in the forward direction, we obtain,

$$\hat{p}\cdot \overline{\overline{S}}(\hat{k}_i,\hat{k}_i)\cdot \hat{p}E_0 = \frac{1}{4\pi} ik_0 Z_0 \oint_{S_1} e^{-i\bar{k}_i\cdot\bar{r}'}\left\{(\hat{p}\cdot(\hat{n}\times\overline{H}_s)) - \frac{1}{Z_0}(\hat{k}_s\times\hat{p}))\cdot(\hat{n}\times\overline{E}_s)\right\} ds' \tag{15}$$

where we used the fact that

$$\hat{p}\cdot(\hat{h}_s\hat{h}_s+\hat{v}_s\hat{v}_s) = \hat{p}\cdot(\overline{\overline{I}} - \hat{k}_s\hat{k}_s) = \hat{p}\cdot \overline{\overline{I}} = \hat{p}$$

and

$$\hat{p}\cdot\hat{k}_s\times(\hat{h}_s\hat{h}_s+\hat{v}_s\hat{v}_s) = \hat{p}\cdot\hat{k}_s\times(\overline{\overline{I}} - \hat{k}_s\hat{k}_s) = \hat{p}\cdot(\hat{k}_s\times \overline{\overline{I}}) = -\hat{k}_s\times\hat{p}$$

Comparing (15) with the right-hand side of (8), we get

$$-P_t = Im\left\{\frac{2\pi}{k_0 Z_0}E_0^*\hat{p}\cdot \overline{\overline{S}}(\hat{k}_i,\hat{k}_i)\cdot\hat{p}E_0\right\} \tag{16}$$

The extinction cross section is defined by

$$\sigma_{ext} = \frac{-P_t}{\frac{1}{2Z_0}|E_0|^2} \tag{17}$$

which is an effective area of the target capable of intercepting the incident power density and losing it in absorption and scattering and has units of m^2. Using (16) in (17) we get

$$\sigma_{ext} = \frac{4\pi}{k_0} Im\left\{\hat{p}\cdot \overline{\overline{S}}(\hat{k}_i,\hat{k}_i)\cdot\hat{p}\right\} \tag{18}$$

which is the statement of the optical theorem. Using the standard vertical and horizontal polarizations the vertical and horizontal extinction cross sections of a target can be obtained from

$$\begin{aligned}\sigma_{ext}^{v} &= \frac{4\pi}{k_0} Im\left\{S_{vv}(\hat{k}_i,\hat{k}_i)\right\}\\ \sigma_{ext}^{h} &= \frac{4\pi}{k_0} Im\left\{S_{hh}(\hat{k}_i,\hat{k}_i)\right\}\end{aligned}$$

0.2 Attenuation Rate Calculation

Optical theorem can be used for evaluating the attenuation rate in a sparse random medium. Consider a segment of the thin slab shown in Figure 1. Suppose the area of this segment is A and the number density of particles is n. The average power loss due to absorption and scattering from such a segment is given by

$$\Delta P = nA\Delta x\langle\sigma_{ext}\rangle P_i \tag{19}$$

Hence the rate of change of power density ($\langle S\rangle$) or power attenuation rate is

$$\alpha = \lim_{\Delta x\to 0}\frac{\Delta P/A}{\Delta x P_i} = n\langle\sigma_{ext}\rangle$$

or

$$\alpha = \frac{4\pi n}{k_0} Im\left\{\langle\hat{p}\cdot \overline{\overline{S}}\,(\hat{k}_i,\hat{k}_i)\cdot\hat{p}\rangle\right\} \tag{20}$$

As discussed earlier the scattered field from individual scatterers in the forward direction are in phase and therefore the field attenuation rate is simply $\alpha/2$. It should also be emphasized that (20) is valid only for sparse media as we have assumed that the incident field is the dominant field illuminating individual scatterers.

In order to calculate α from (20), one should be careful about the accuracy with which $\overline{\overline{S}}\,(\hat{k}_i,\hat{k}_i)$ is calculated. For example using low frequency approximation (to the zeroth order in k_0) for evaluation $\overline{\overline{S}}\,(\hat{k}_i,\hat{k}_i)$ provides erroneous results for lossless particles. To demonstrate this fact, let us consider a spherical particle whose radius is much smaller than a wavelength. It was shown previously that the equivalent dipole moment of a dielectric sphere in a uniform field is given by

$$\bar{P} = 3v_0\epsilon_0\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\overline{E}_0 \tag{21}$$

where v_0 is the volume of the sphere and ϵ is the permittivity of the spherical particle. The far-field expression for the field from a dipole is given by

$$\overline{E}^s = -\frac{k_0 Z_0\omega}{4\pi}\frac{e^{ikr}}{r}\hat{k}_s\times(\hat{k}_s\times\bar{P}) \tag{22}$$

Substituting (21) in (22) we have

$$\overline{E}^s = -\frac{3v_0k_0^2}{4\pi}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{e^{ikr}}{r}\hat{k}_s\times(\hat{k}_s\times\overline{E}_0)$$

Therefore

$$\overline{\overline{S}}\,(\hat{k}_s,\hat{k}_i) = \frac{3v_0k_0^2}{4\pi}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\begin{bmatrix} \hat{v}_s\cdot\hat{v}_i & \hat{v}_s\cdot\hat{h}_i \\ \hat{h}_s\cdot\hat{v}_i & \hat{h}_s\cdot\hat{h}_i \end{bmatrix} \tag{23}$$

where $\hat{v}_s, \hat{h}_s, \hat{v}_i, \hat{h}_i$ are the vertical and horizontal unit vectors for the scattered and incident fields respectively. In the forward direction the matrix in (23) becomes diagonal and

$$\alpha = 3v_0nk_0Im\left\{\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right\} \qquad (Np/m) \tag{24}$$

It is obvious that when the particles are lossless the attenuation constant α predicted by (24) would be zero. In fact (24) predicts only the absorption loss. In order to get the scattering loss using the optical theorem the scattering solution to a higher order in frequency is needed. An alternative method is the calculation of the scattering loss directly from (23), i.e.

$$P_s = \frac{1}{2}Re\left\{\int_0^{2\pi}\int_0^{\pi}\frac{|\overline{E}_s|^2}{Z_0}\sin\theta_s d\theta_s d\phi_s\right\} \tag{25}$$

For a z-directed incident wave ($\hat{v}_i = \hat{z}$) and v-polarized scattered wave ($\hat{v}$ polarized field is the only component of the scattered wave) we have

$$\begin{aligned} P_s &= \frac{1}{2Z_0}\left(\frac{3v_0k_0^2}{4\pi}\right)^2\left|\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right|^2|\overline{E}_0|^2\int_0^{2\pi}\int_0^{\pi}\sin^3\theta_s d\theta_s d\phi_s \\ &= \frac{3(v_0k_0^2)^2}{4\pi Z_0}\left|\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right|^2|\overline{E}_0|^2 \end{aligned} \tag{26}$$

Therefore the attenuation rate due to scattering loss is

$$\begin{aligned} \alpha &= \frac{nP_s}{\frac{1}{2}\frac{|E_0|^2}{Z_0}} = \frac{3n(v_0k_0^2)^2}{2\pi}\left|\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right|^2 \quad (Np/m) \\ &= 2fa^3k_0^4\left|\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right|^2 \end{aligned}$$

1 Calculation of Propagation Constant for a Sparse Random Medium

In the previous section a formulation for the evaluation of attenuation constant of a sparse random medium was developed using the optical theorem. This result may be extended by introducing an equivalent homogeneous medium in which the propagation constant k is the same as that for the mean-field in the random medium. In this problem the objective is to relate k to the geometrical and dielectric parameters of the constituent particles of the random medium. Unlike dielectric mixing formulas where the effect of scattering loss was ignored this formulation provides a term for the scattering loss. The only limitation here is that the particle number density has to be small. Practically speaking, this formulation is suitable for the prediction of wave propagation in vegetation canopies where usually the volume fraction is less than 1%.

In this approach, we first consider a thin slab of the random medium in free-space illuminated by a plane wave at normal incidence. Then by calculating the ensemble average of the total field on the other side of the dielectric slab an expression for the effective propagation constant is obtained.[2,3] To formulate the problem let us consider a narrow slab of the medium which consists of scatterers whose dimensions are comparable or larger than a wavelength as shown in Figure 3. It is assumed that the scatterers are sparse enough so that multiple scattering among the constituent particles is negligible. It is also assumed that the incident field travel along the x-direction and can be represented as

$$\overline{E}^i = \hat{p} e^{ik_0 x}. \tag{27}$$

The total field at an observation point located on the x-axis and denoted by $P(x_0, 0, 0)$ is composed of two components: 1) the direct incident field, 2) the scattered field from the three dimensional scatterers in the slab.

For a sparse medium, the incident field on the n^{th} scatterer located at (x_n, y_n, z_n) is represented by:

$$\overline{E}^i = \hat{p} e^{ik_0 x_n} \tag{28}$$

Assuming the observation point is far away from the slab, the scattered field due to this scatterer observed at $P(x_0, 0, 0)$ is given by

$$\overline{E}_n^s = e^{ik_0 x_n} \frac{e^{ik_0 r_n}}{r_n} \overline{S}_n(\hat{r}_n, \hat{x}) \tag{29}$$

where r_n is the distance between the n^{th} scatterer and the observation point and $\overline{S}_n(\hat{r}_n, \hat{x})$ is the bistatic scattering amplitude of the scatterer for scattering in direction $\hat{r}_n$. Since the medium is sparely populated, multiple scattering between scatterers may be ignored.

[2]Tavakoli, A., K. Sarabandi, and F.T. Ulaby, "Microwave propagation constant for a vegetation canopy at x-band," *Radio Sci.*,Vol. 28, No. 4, pp. 549-588, July-Aug. 1993

[3]Ulaby, F.T., A. Tavakoli, and J.B.A. Senior, "Microwave propagation constant for a vegetation canopy with vertical stalks," *IEEE Trans. Geosci. Remote Sensing*, Vol. 25, No. 6, pp. 714-725, Nov. 1987.

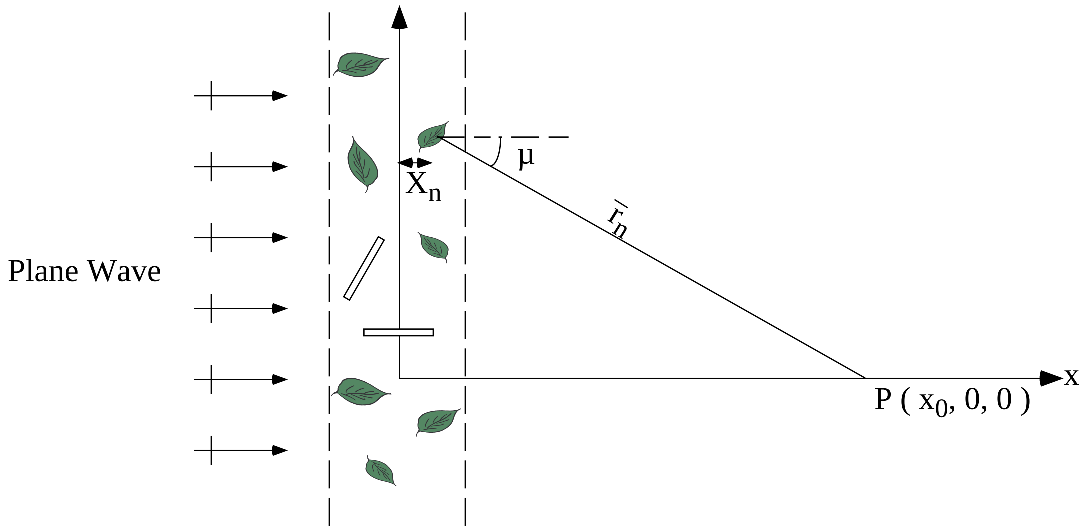

Figure 3: Geometry of a thin layer of a sparse random medium illuminated by a plane wave.

Thus the total scattered field at the observation point is the superposition of contributions of all individual scatterers in the slab, i.e.

$$\overline{E}^s = \sum_n e^{ik_0 x_n} \frac{e^{ik_0 r_n}}{r_n} \overline{S}_n(\hat{r}_n, \hat{x}) \tag{30}$$

If the extent of the medium is large and the observation point is sufficiently far away, then the summation in (30) can be approximated by an integral. In (30), S_n is a random variable function of orientation, size, and dielectric properties of the particles. Assuming that there are N particles per unit volume of the medium then

$$\langle \overline{E}^s \rangle = N \iiint e^{ik_0 x'} \frac{e^{ik_0 r'}}{r'} < \vec{S}(\hat{r}'; \hat{x}) > dx'dy'dz' \tag{31}$$

where $< \vec{S}(\hat{x}, \hat{r}') >$ is the scattering amplitude averaged over the prescribed size and orientation distributions. To evaluate (31), two changes of variables are in order: 1) the integration in $y' - z'$ plane can be performed by cylindrical variables ρ' and ϕ' and 2) the integration with respect to ρ' and x' can be performed by changing to variables ν and μ defined by:

$$\begin{aligned} \nu &= x' \\ \mu &= tan^{-1} \frac{\rho'}{x_0 - x'} \end{aligned} \tag{32}$$

In (32), μ is the angle between forward scattering direction (x-axis) and the observation point P as shown in Figure 3. Thus (31) becomes

$$\langle \overline{E}^s \rangle = N \int_0^{2\pi} \int_{-d/2}^{d/2} e^{ik_0\nu} \left[\int_0^{2\pi} < \overline{S}(\pi - \mu) > \frac{e^{ik_0 \frac{x_0 - \nu}{\cos\mu}}}{\cos\mu} (x_0 - \nu) \tan\mu d\mu \right] d\nu d\phi' \qquad (33)$$

Since the observation point is far away from the slab, that is $k_0 x_0 >> 1$, then the integrand of (33) rapidly changes with small changes in μ for values of μ away from the stationary phase point ($\mu = 0$). Therefore, the integral is dominated by the contribution of the integrand around the stationary phase point where the integrand can be approximated by its Taylor series expansion, i.e.,

$$\begin{aligned} \langle \overline{E}^s \rangle \cong\; & N \int_0^{2\pi} \int_{-d/2}^{d/2} e^{ik_0\nu} \\ & \left[\int_0 < \overline{S}(\pi) > e^{ik_0(x_0-\nu)(1+\frac{\mu^2}{2})} (x_0 - \nu) \mu d\mu \right] d\nu d\phi' \end{aligned} \qquad (34)$$

Direct evaluation of (34) leads to

$$\langle \overline{E}^s \rangle \cong e^{ik_0 x_0} \frac{i2\pi N d}{k_0} < \overline{S}(\pi) > \qquad (35)$$

The following observations are in order:

1. Since the number density of particles is small the effective dielectric constant of the medium is very close to that of the background medium (ϵ_0). This implies that the boundary between the medium and the background may be considered diffused, that is, no significant reflection at the boundaries take place.

2. Assuming that the particles have no preferred orientation in $x-y$ plane, the equivalent slab may be considered as a uniaxial anisotropic medium with the optical axis along the $\hat{z}$ direction. In this case the medium has two distinct propagation constants $k_x = k_y = k_0 n_y = k_0\sqrt{\epsilon_y}$ and $k_z = k_0 n_z = k_0\sqrt{\epsilon_z}$. The magnitude of each mode is excited depends on the polarization of the incoming wave.

The total field at the observation point P is

$$\begin{aligned} \overline{E}^t &= \overline{E}^i + < \overline{E}^s > \\ &= e^{ik_0 x_0} \left[\hat{p} + \frac{i2\pi N d}{k_0} < \overline{S}(\pi) > \right] \end{aligned} \qquad (36)$$

Now, if we represent the thin layer of scatterers by an equivalent homogeneous and anisotropic dielectric, with diffused boundaries, the total field at the observation point may be written as

$$\begin{aligned}\overline{E}^t &= (\hat{p}\cdot\hat{y})e^{ik_0x_0}e^{i(n_y-1)k_0d}\hat{y} + (\hat{p}\cdot\hat{z})e^{ik_0x_0}e^{i(n_z-1)k_0d}\hat{z} \\ &\cong e^{ik_0x_0}\left\{\hat{p} + ik_0d[(n_y-1)(\hat{p}\cdot\hat{y})\hat{y} + (n_z-1)(\hat{p}\cdot\hat{z})\hat{z}]\right\} \qquad (37)\end{aligned}$$

Taylor series expansion used in the above approximation is justified by the fact that d is small, $n' = Re[n] \approx 1$ and $n'' = Im[n] << 1$. Upon equating (36) and (37), we obtain the following expressions for the index of refraction of the equivalent medium by setting once $\hat{p} = \hat{y}$ and once $\hat{p} = \hat{z}$

$$\begin{aligned}n_y &= n'_y + n''_y \\ n_y &= 1 + \frac{2\pi N}{k_0^2} < S_{hh}(\pi) > \qquad (38)\end{aligned}$$

and

$$n_z = 1 + \frac{2\pi N}{k_0^2} < S_{vv}(\pi) >$$

where $< S_{hh}(\pi) >$ and $< S_{vv}(\pi) >$ are the elements of the scattering matrix in forward direction for horizontal and vertical polarizaions. The phase constant of the equivalent medium β is given by:

$$\begin{aligned}\beta_v &= k_0(1 + \frac{2\pi N}{k_0^2} < Re[S_{vv}(\pi)] >) \\ \beta_h &= k_0(1 + \frac{2\pi N}{k_0^2} < Re[S_{hh}(\pi)] >) \qquad (39)\end{aligned}$$

The field attenuation constant of the medium can also be obtained from

$$\begin{aligned}\alpha_v &= \frac{2\pi N}{k_0} < Im[S_{vv}(\pi)] > \\ \alpha_h &= \frac{2\pi N}{k_0} < Im[S_{hh}(\pi)] > \qquad (40)\end{aligned}$$

The result for the attenuation rate is in agreement with the result obtained in the previous section.

For a medium composed of many species with different number densities and scattering properties the results given by (39) and (40) can easily be extended to find the equivalent index of refraction of this medium

$$n_y = 1 + \sum_{i=1}^{m} \frac{2\pi N_i}{k_0} < S^i_{hh}(\pi) > \tag{41}$$

so long as the total number of particles per unit volume $\sum_{i=1}^{m} N_i$ is small enough to ensure a small volume fraction of scatterers.

As an example let us consider a vegetation canopy consisting of leaves. At microwave frequencies the leaf thickness is much smaller than a wavelength. Assuming that the other dimensions of the vegetation leaves are comparable or larger than a wavelength, a physical optics model can be used to develop the expression for the scattering amplitude. Under these approximations a leaf can be considered a resistive sheet whose resistivity, R, in ohms per unit area is given by[4]

$$R = \frac{iZ_0}{k_0 \tau(\epsilon - 1)} \tag{42}$$

where Z_0 is the free space impedance, τ is the leaf thickness and ϵ is the relative dielectric constant of the leaf. When $R = 0$ the sheet appears perfectly conducting and when $R = \infty$ it ceases to exist. The sheet is an electric current sheet whose strength is proportional to the tangential electric field and is related to R. As mentioned before the scattering properties of a leaf can be derived by using the physical optics approximation where for a finite size sheet the same electric current supported by an infinite resistive sheet is used. The reflection coefficients of an infinite resistive sheet for perpendicular and parallel polarizations are:

$$\begin{aligned} \Gamma_E &= \left(1 + \frac{2R\cos\psi}{Z_0}\right)^{-1} \\ \Gamma_H &= \left(1 + \frac{2R\sec\psi}{Z_0}\right)^{-1} \end{aligned} \tag{43}$$

where ψ is the incidence angle with respect to the normal to the sheet surface. For a plane wave traveling in $\hat{k}_i$ direction and for a leaf whose normal is denoted by $\hat{n}$ the bistatic scattering amplitude is given by [3]

$$\overline{S} \;=\; \Big\{(\hat{n}\cdot\hat{p})\Gamma_E \hat{k}_s \times \hat{k}_s \times [(\hat{n}\cdot\hat{k}_i)\hat{k}_i - \hat{n}] + [\hat{n}\cdot(\hat{k}_i \times \hat{p})]\Gamma_H$$

[4]Sarabandi, K., "Electromagnetic scattering from vegetation canopies," pp. 43, Ph.D. Dissertation, The University of Michigan, 1989.

$$\cdot \left[(\hat{k}_s \cdot \hat{k}_i)(\hat{n} \times \hat{k}_s) + (\hat{n} \cdot \hat{k}_s)(\hat{k}_s \times \hat{k}_i)\right]\Big\} \frac{\hat{n} \cdot \hat{k}_i}{|\hat{n} \times k_i|^2} K \tag{44}$$

where $\hat{k}_s$ denotes the direction of the observation point and

$$K = -\frac{i}{\lambda_0} \int_s e^{ik_0(\hat{k}_i - \hat{k}_s)\cdot\overline{r}'} ds'. \tag{45}$$

In the forward direction $\hat{k}_s = \hat{k}_i = \hat{x}$ and

$$K = -\frac{iA}{\lambda_0} \tag{46}$$

where A is the area of a single leaf. Suppose the angle between $\hat{n}$ and $\hat{x}$ is represented by ψ and the angle between projection of $\hat{n}$ on $y-z$ plane and y axis is denoted by ξ as shown in Figure 4. Then

$$\hat{n} = \cos\psi\hat{x} + \sin\psi\cos\xi\hat{y} + \sin\psi\sin\xi\hat{z}. \tag{47}$$

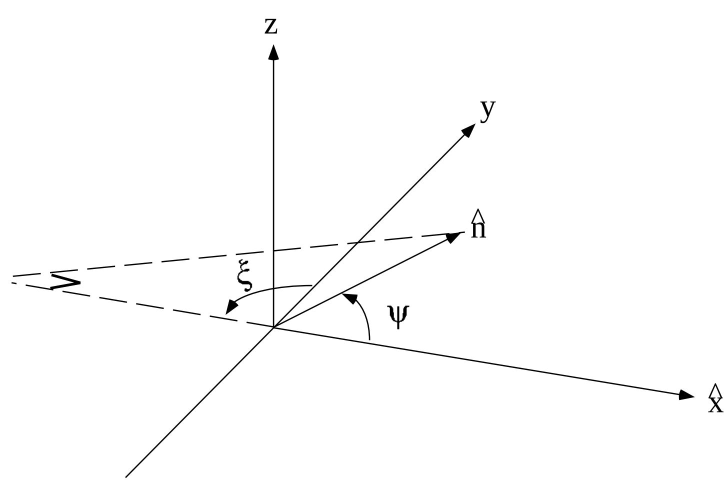

Figure 4: Angles ψ, and ξ define the unit normal $\hat{n}$ of a flat leaf in the coordinate system.

Substituting (46) and (47) in (44) the forward scattering amplitude simplifies to

$$\begin{aligned} \overline{S}(\pi) \quad = \quad & -\frac{iA\cos\psi k_0}{2\pi} \Big\{ \hat{y}[p_y(\Gamma_H \sin^2\xi + \Gamma_E \cos^2\xi) + p_z \sin\xi\cos\xi(\Gamma_E - \Gamma_H)] \\ & + \hat{z}[p_y \sin\xi\cos\xi(\Gamma_E - \Gamma_H) + p_z(\Gamma_H \cos^2\xi + \Gamma_E \sin^2\xi)] \Big\} \end{aligned} \tag{48}$$

In the propagation model, we need an ensemble average of the forward scattering amplitude of the leaves. The average forward scattering amplitude for randomly oriented leaves can be obtained by averaging $\overline{S}(\pi)$ over all possible orientation angles. Assuming a spherical distribution function of $\hat{n}$, we have

$$< \overline{S}(\pi) >= \frac{1}{4\pi} \int_0^{2\pi} d\xi \int_0^{\pi} \overline{S}(\pi) \sin\psi d\psi \tag{49}$$

Since Γ_E and Γ_H are independent of ξ, the integration over ξ can be easily performed to give:

$$< \overline{S}(\pi) >= -\frac{iAk_0}{8\pi} \hat{p} \int_0^{\pi} (\Gamma_V + \Gamma_H) \cos\psi \sin\psi d\psi \tag{50}$$

Since there is no preferred orientation in this example $< \overline{S}(\pi) >$ is parallel to the polarization of incident field and the effective medium is isotropic. Upon substitution of the values of Γ_E and Γ_H in (48), the following relation is obtained.

$$\begin{aligned} < \overline{S}(\pi) > \quad = \quad & -\frac{iAk_0}{8\pi} \hat{p} \left\{ \int_0^{\pi} (1 + \frac{2R\cos\psi}{Z_0})^{-1} \cos\psi \sin\psi d\psi + \right. \\ & \left. \int_0^{\pi} (1 + \frac{2R}{Z_0 \cos\psi})^{-1} \cos\psi \sin\psi d\psi \right\} \end{aligned} \tag{51}$$

By setting $\gamma = \cos\psi$, $< \overline{S}(\pi) >$ assumes the form

$$< \overline{S}(\pi) >= -\frac{iAk_0}{8\pi} \hat{p} \left\{ \int_{-1}^{1} \frac{\alpha}{1 + \frac{2R}{Z_0}\alpha} d\alpha + \int_{-1}^{1} \frac{\alpha^2}{\alpha + \frac{2R}{Z_0}} d\alpha \right\} \tag{52}$$

which leads to the result

$$< \overline{S}(\pi) >= -\frac{iAk_0}{8\pi} \hat{p} \left\{ \frac{Z_0}{R} - \frac{4R}{Z_0} + i4\pi \frac{R^2}{Z_0^2} + \left[\frac{4R^2}{Z_0^2} - \frac{Z_0^2}{4R^2} \right] \ln\left(\frac{Z_0 + 2R}{Z_0 - 2R} \right) \right\} \tag{53}$$

Because the physical-optics approximation was used in the derivation of $< \overline{S}(\pi) >$, equation (51) is valid for leaves with surface dimensions larger than a wavelength. Upon substituting (51) in (38), we obtain the following expressions for the equivalent index of refraction of the canopy:

$$n = 1 - \frac{i\zeta}{4k_0} \left\{ \frac{Z_0}{R} - \frac{4R}{Z_0} + i4\pi \frac{R^2}{Z_0^2} + \left[\frac{4R^2}{Z_0^2} - \frac{Z_0^2}{4R^2} \right] \ln\left(\frac{Z_0 + 2R}{Z_0 - 2R} \right) \right\} \tag{54}$$

where $\zeta = NA$ is the leaf area per unit volume. It can be seen that as long as the leaves are large compared with λ, the exact sizes and numbers of leaves are not needed in the computations, and only the knowledge of ζ suffices.