Ewald-Oseen Extinction Theorem[1]

Integral equation methods are often used to formulate the electromagnetic scattering problems for which the standard method of separation of variables is not applicable. In general for non-metallic scatterers the integral equation formulation can be cast in terms of either a volumetric integral equation or a coupled surface field integral equation. The volume integral equation is usually used for inhomogeneous scatterers whereas the surface integral equation is commonly applied to scattering problems with homogeneous scatterers.[2]

Our objective in this section is to present a different method for deriving a surface-field integral equation for homogeneous isotropic scatterers known as "extended boundary condition" (EBC) method based on the extinction theorem. Before we proceed with the formal analysis of EBC method, it is worth mentioning that a similar result may be obtained by invoking the "Field Equivalence Principle" as outlined by Harrington.[3] However, the proof of extinction theorem provides some useful insight to the physics of the scattering problems.

Consider an electrically and magnetically permeable object with constitutive parameters ϵ_2 and μ_2 whose boundary is specified by a closed surface S. Suppose this object is illuminated by an incident wave $\overline{E}^i$ and is embedded in a homogeneous medium with constitutive parameters ϵ_1 and μ_1 as shown in Figure 1.

The nature of the scattered-field can be attributed, physically, to the induced electric and magnetic dipole moments within the scatterer. Mathematically the scattered field can be interpreted by noting that

$$\nabla \times \overline{E} = i\omega\mu(r)\overline{H} = i\omega\mu_1\overline{H} + i\omega(\mu(r) - \mu_1)\overline{H} \tag{1}$$

where $\mu(r) = \mu_2$ over the region occupied by the scatterer and $\mu(r) = \mu_1$ elsewhere. Comparing (1) with

[1]

[2] Poggio, A.J., and E.K. Miller, "Integral equation solutions of three-dimensional scattering problem," in *Computer Techniques for Electromagnetics*, R. Mittra (Editor), New York: Hemisphere Publishing Corp., 1987.

[3] Harrington, R.F., "Time-Harmonic Electromagnetic Fields," pp. 106-116, New York: McGraw Hill, 1961.

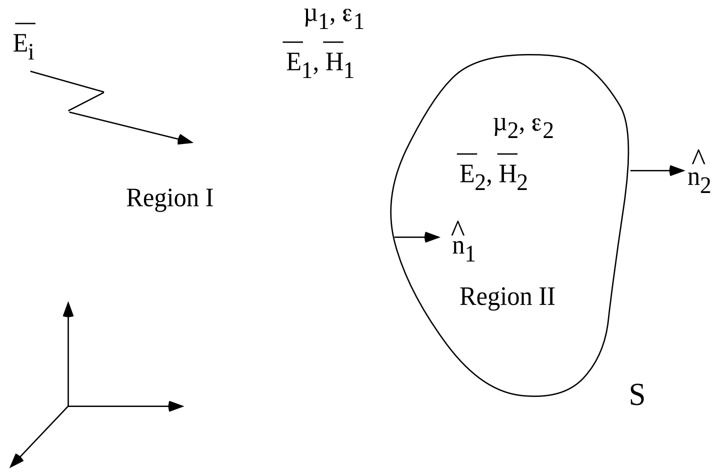

Figure 1: Geometry of the Scattering Problem.

$$\nabla \times \overline{E} = i\omega\mu_1\overline{H} - \overline{J}_m$$

the second term in the right-hand of (1) can be interpreted as an induced magnetic current

$$\overline{J}_m = -i\omega(\mu_2 - \mu_1)\overline{H} \tag{2}$$

radiating in a homogeneous medium. Similarly

$$\nabla \times \overline{H} = -i\omega\epsilon(r)\overline{E} = -i\omega\epsilon_1\overline{E} - i\omega(\epsilon(r) - \epsilon_1)\overline{E} \tag{3}$$

which can be compared with

$$\nabla \times \overline{H} = -i\omega\epsilon_1\overline{E} + \overline{J}_e$$

to find the induced electric polarization current

$$\overline{J}_e = -i\omega(\epsilon_2 - \epsilon_1)\overline{E} \tag{4}$$

over the volume of the scatterer. Hence $\overline{J}_e$ and $\overline{J}_m$ are the sources of the scattered field in a homogeneous medium. In (2) and (4), $\overline{E}$ and $\overline{H}$ are the total internal fields within Region II which are in general unknown quantities.

It should be noted that these unknown currents are not independent. In fact these are related by the curl operations, and are $\overline{E}$ and $\overline{H}$ in Region II.

The extinction theorem, which was first established for crystalline media by Ewald[4] and then for isotropic media by Oseen[5], states that the expression for the scattered field within the scatterer can be decomposed into two terms: 1) a field which satisfies the wave equation in Region I and cancels out exactly the incident wave, and 2) a field which satisfies the wave equation in Region II. This is an expected result since the propagation constant, or equivalently the phase velocity, and the field impedance in Region II are different, in general, from those of the incident field and therefore the scattered field must contain an expression that would cancel the incident field in Region II. In other words the incident field is extinguished at any point within the scatterer by the scattered field and a new field which satisfies the wave equation within the scatterer is established.

As mentioned before the fields in Region I ($\overline{E}_1, \overline{H}_1$) can be considered as the sum of the incident waves and the scattered waves, that is,

$$\overline{E}_1 = \overline{E}_i + \overline{E}_s \tag{5a}$$

$$\overline{H}_1 = \overline{H}_i + \overline{H}_s \tag{5b}$$

The electric fields outside and inside of the scatterer satisfy the source free vector wave equations

$$\begin{aligned} \nabla \times \nabla \times \overline{E}_1 - k_1^2 \overline{E}_1 &= 0 && (6) \\ \nabla \times \nabla \times \overline{E}_2 - k_2^2 \overline{E}_2 &= 0 && (7) \end{aligned}$$

where $k_1^2 = \omega^2 \mu_1 \epsilon_1$ and $k_2^2 = \omega^2 \mu_2 \epsilon_2$.

These equations can be integrated together with appropriate dyadic Green's function using the vector-dyadic Green's second identity given by

$$\begin{aligned} &\iiint_v \left[\overline{P} \cdot \nabla \times \nabla \times \overline{\overline{Q}} - (\nabla \times \nabla \times \overline{P}) \cdot \overline{\overline{Q}} \right] dv = \\ &- \oiint_s \left[(\hat{n} \times \nabla \times \overline{P}) \cdot \overline{\overline{Q}} + (\hat{n} \times \overline{P}) \cdot \nabla \times \overline{\overline{Q}} \right] ds \end{aligned} \tag{8}$$

First we consider the dyadic Green's function $\overline{\overline{G}}_1$ for a homogenous medium with constituitive parameters μ_1 and ϵ_1 (Region I) which satisfies the equation

$$\nabla \times \nabla \times \overline{\overline{G}}_1 - k_1^2 \overline{\overline{G}}_1 = \overline{\overline{I}} \, \delta(\overline{r} - \overline{r}') \tag{9}$$

[4]Ewald, P.P., (Dissertation, Munchen 1912); Ann. d. Physik, 49 (1916).
[5]Oseen, C.W., Ann. d. Physik, 48 (1915), 1.

and is given by

$$\overline{\overline{G}}_1(\overline{r},\overline{r}') = (\overline{\overline{I}} + \frac{1}{k_1^2}\nabla\nabla)\frac{e^{ik_1|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}$$

Using $\overline{\overline{G}}_1$ for $\overline{\overline{Q}}$ and $\overline{E}_1$ for $\overline{P}$ in equation (8) and noting that Region I is bounded by S and S_∞, we have:

$$\iiint_{Region\ I} \left[\overline{E}_1(\overline{r})\cdot\nabla\times\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}') - \nabla\times\nabla\times\overline{E}_1(\overline{r})\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}')\right]dv = \tag{10}$$
$$-\oint\!\!\!\oint_{S+S_\infty}\left[(\hat{n}_1\times\nabla\times\overline{E}_1(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_1(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r})\right]ds$$

where as before S_∞ is the surface of a sphere with a very large radius ($R\to\infty$).

Substituting (6) and (9) in the left-hand side of (10), with $\overline{r}'$ once in Region I and once in Region II, renders $\overline{E}_1(\overline{r}')$ when $\overline{r}' \in$ Region I and zero when $\overline{r}' \in$ Region II. Also noting that in the absence of the scatterer for all values of $\overline{r}(\overline{r}\in\mathcal{R}^3)$

$$\overline{E}_i(\overline{r}') = -\oint\!\!\!\oint_{S_\infty}\left[(\hat{n}_1\times\nabla\times\overline{E}_i(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_i(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\right]ds \tag{11}$$

and using (5a), equation (10) can be written in the following form:

$$\begin{aligned}\overline{E}_i(\overline{r}') &- \oint\!\!\!\oint_{S_\infty}\left[(\hat{n}_1\times\nabla\times\overline{E}_s(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_s(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\right]ds \\ &- \oint\!\!\!\oint_{S}\left[(\hat{n}_1\times\nabla\times\overline{E}_1(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_1(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\right]ds = \\ &= \begin{cases}\overline{E}_1(\overline{r}') & \overline{r}'\in \text{ Region I}\\ 0 & \overline{r}'\in \text{ Region II}\end{cases}\end{aligned} \tag{12}$$

It can easily be shown that the integral over S_∞ vanishes since at infinity $\overline{E}_s$ and $\overline{\overline{G}}_1$ satisfy the radiation and far-field conditions. Basically $\overline{E}_s(\overline{r}')$ is a spherical wave satisfying

$$\hat{r}\times\overline{E}_s = Z_1\overline{H}_s \quad ; \quad Z_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \tag{13}$$

and the Green's function and its curl $(\nabla\times)$ can be approximated by

$$\overline{\overline{G}}_1(\overline{r},\overline{r}') \simeq \left(\overline{\overline{I}} - \hat{r}\hat{r}\right)\frac{e^{ik_1 r}}{4\pi r}e^{-ik_1\overline{r}'\cdot\hat{r}} \tag{14a}$$

$$\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}') \simeq ik_1\hat{r}\times\overline{\overline{G}}_1(\overline{r},\overline{r}') \simeq ik_1\hat{r}\times\left(\overline{\overline{I}} - \hat{r}\hat{r}\right)\frac{e^{ik_1 r - ik_1\overline{r}'\cdot\hat{r}}}{4\pi r}\ . \tag{14b}$$

Using Faraday's law

$$\nabla\times\overline{E}_s(\overline{r}) = ik_1 Z_1 \overline{H}_s(\overline{r})\ ,$$

and noting that

$$\begin{aligned}(\hat{r}\times\overline{E}_s(\overline{r}))\cdot\hat{r}\times(\overline{\overline{I}} - \hat{r}\hat{r}) &= \left[(\hat{r}\times\overline{E}_s(\overline{r}))\times\hat{r}\right]\cdot(\overline{\overline{I}} - \hat{r}\hat{r}) \\ &= -\left[\hat{r}\times Z_1\overline{H}_s(\overline{r})\right]\cdot(\overline{\overline{I}} - \hat{r}\hat{r})\end{aligned}$$

the two terms in the integrand of the integral over S_∞ cancel out each other. Hence,

$$\begin{aligned}\overline{E}_i(\overline{r}') &- \iint_S \left[ik_1 Z_1(\hat{n}_1\times\overline{H}_1(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_1(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\right]ds \\ &= \begin{cases}\overline{E}_1(\overline{r}') & \overline{r}'\in \ \text{Region I} \\ 0 & \overline{r}'\in \ \text{Region II}\end{cases}\end{aligned} \tag{15}$$

where it should be noted here that $\hat{n}_1$ is the inward unit normal to the surface S. Equation (15) is the representation of the extinction theorem which states that the surface fields are formed in such manner to completely cancel out the incident field within Region II. It also provides a tool for evaluating the scattered field in Region I, once the surface fields are obtained, i.e.,

$$\overline{E}^s(\overline{r}') = -\iint_S \left[ik_1 Z_1(\hat{n}_1\times\overline{H}_1(\overline{r}))\cdot\overline{\overline{G}}_1(\overline{r},\overline{r}') + (\hat{n}_1\times\overline{E}_1(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\right] \tag{16}$$

In a similar manner using the vector-dyadic Green's second identity in conjunction with the dyadic Green's function of a homogeneous unbounded medium with parameters μ_2, ϵ_2 (Region II), the following equation is obtained:

$$\begin{aligned}&- \oiint_S \left[ik_2 Z_2(\hat{n}_2\times\overline{H}_2(\overline{r}))\cdot\overline{\overline{G}}_2(\overline{r},\overline{r}') + (\hat{n}_2\times\overline{E}_2(\overline{r}))\cdot\nabla\times\overline{\overline{G}}_2(\overline{r},\overline{r}')\right]ds \\ &= \begin{cases}0 & \overline{r}'\in \ \text{Region I} \\ E_2(\overline{r}') & \overline{r}'\in \ \text{Region II}\end{cases}\end{aligned} \tag{17}$$

where $\hat{n}_2$ is the outward unit normal to surface S. The boundary conditions mandate that

$$\begin{aligned} \hat{n}_1 \times \overline{H}_1(\overline{r}) &= -\hat{n}_2 \times \overline{H}_2(\overline{r}) \\ \hat{n}_1 \times \overline{E}_1 &= -\hat{n}_2 \times \overline{E}_2 \end{aligned}$$

Therefore once the surface fields are known, (17) provides a formulation for evaluating the field inside the scatterer. It also shows that the scattered fields are formed such that the surface fields acted on the dyadic Green's function of Region II produce no field in Region I as expected (the phase velocity embedded in $\overline{\overline{G}}_2(\overline{r}, \overline{r}')$ is different from that of Region I).

Similar results can be obtained for the magnetic fields in each region by applying the duality principal. Basically by changing:

$$\begin{aligned} \overline{E}_{1,2} &\rightarrow \overline{H}_{1,2} \\ \overline{H}_{1,2} &\rightarrow -\overline{E}_{1,2} \\ \mu &\rightarrow \epsilon \\ \epsilon &\rightarrow \mu \\ Z_{1,2} &\rightarrow Y_{1,2} \qquad \text{where } Y_{1,2} = \frac{1}{Z_{1,2}}\ , \end{aligned}$$

the desired expressions for the magnetic fields can be obtained.

It is common to express the observation point with unprimed coordinates. This can be done by changing $\overline{r}' \rightarrow \overline{r}, \overline{r} \rightarrow \overline{r}'$ and $\nabla \rightarrow \nabla'$ in (15) and (17) and noting that

$$(\hat{n} \times \overline{H}(\overline{r}')) \cdot \overline{\overline{G}}(\overline{r}', \overline{r}) = \overline{\overline{G}}(\overline{r}, \overline{r}') \cdot (\hat{n} \times \overline{H}(\overline{r}')) \tag{18}$$

since $\nabla' \times \overline{\overline{G}}(\overline{r}', \overline{r})$ is antisymmetric and $\overline{\overline{G}}(\overline{r}', \overline{r}) = \overline{\overline{G}}(\overline{r}, \overline{r}')$

$$(\hat{n} \times \overline{E}(\overline{r}')) \cdot \nabla' \times \overline{\overline{G}}(\overline{r}', \overline{r}) = -\nabla' \times \overline{\overline{G}}(\overline{r}, \overline{r}') \cdot (\hat{n} \times \overline{E}(\overline{r}')) \tag{19}$$

and $\nabla' \times \overline{\overline{G}}(\overline{r}, \overline{r}') = -\nabla \times \overline{\overline{G}}(\overline{r}, \overline{r}')$ it can be shown that

$$(\hat{n} \times E(\overline{r}')) \cdot \nabla' \times \overline{\overline{G}}(\overline{r}, \overline{r}') = \nabla \times \overline{\overline{G}}(\overline{r}, \overline{r}') \cdot (\hat{n} \times \overline{E}(\overline{r}'))$$

Hence (15) may be written as

$$\begin{aligned}\overline{E}_i(\overline{r}) &- \oiint_S \left[ik_1 Z_1 \overline{\overline{G}}_1(\overline{r},\overline{r}')\cdot(\hat{n}_1\times\overline{H}(\overline{r}')) + \nabla\times\overline{\overline{G}}_1(\overline{r},\overline{r}')\cdot(\hat{n}_1\times E(\overline{r}'))\right] ds' \\ &= \begin{cases}\overline{E}_1(\overline{r}) & \overline{r}\in \quad \text{Region I}\\ 0 & \overline{r}\in \quad \text{Region II}\end{cases}\end{aligned} \tag{20}$$

and (17) takes the following form:

$$\begin{aligned}&- \oiint_S \left[ik_2 Z_2 \overline{\overline{G}}_2(\overline{r},\overline{r}')\cdot(\hat{n}_2\times\overline{H}(\overline{r}')) + \nabla\times\overline{\overline{G}}_2(\overline{r},\overline{r}')\cdot(\hat{n}_2\times \overline{E}(\overline{r}'))\right] ds' \\ &= \begin{cases}0 & \overline{r}\in \quad \text{Region I}\\ \overline{E}_2(\overline{r}) & \overline{r}\in \quad \text{Region II}\end{cases}\end{aligned} \tag{21}$$

where again $\hat{n}_1$ is pointing towards Region II and $\hat{n}_2$ points towards Region I.

The integrals in (20) and (21) can be made simpler by noting that

$$\begin{aligned}\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}') &= \nabla\times\left[(\overline{\overline{I}} - \frac{1}{k_o^2}\nabla\nabla)g(\overline{r},\overline{r}')\right] \\ &= \nabla g(\overline{r},\overline{r}')\times\overline{\overline{I}}\end{aligned} \tag{22}$$

and

$$\nabla\times\overline{\overline{G}}(\overline{r},\overline{r}')\cdot(\hat{n}\times\overline{E}(\overline{r}')) = -(\hat{n}\times\overline{E})\times\nabla g(\overline{r},\overline{r}')$$