Review of Fundamental Equations[1]

Maxwell's equations together with constitutive relations are the fundamental equations of electromagnetic wave theory. These fundamental equations were put into differential form and were completed by Maxwell in 1865. In three-dimensional vector notation the electric and magnetic field quantities obey

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{1}$$

$$\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J}_e \tag{2}$$

$$\nabla \cdot \overline{B} = 0 \tag{3}$$

$$\nabla \cdot \overline{D} = \rho_e \tag{4}$$

where $\overline{E}$ and $\overline{H}$ are, respectively, the electric and magnetic field strength and $\overline{D}$ and $\overline{B}$ are the electric displacement and magnetic flux density respectively. $\overline{J}_e$ and ρ_e are the sources of electromagnetic waves and denote the electric current density and the electric charge density respectively. It should be remembered that (1) and (3) are not independent. This can be shown by applying the divergence operator on both sides of (1) and noting that $\nabla \cdot \nabla \times a = 0$ for all vector functions $\overline{A}$. Also by applying the divergence operator on both sides of (2) and using (4), the law of conservation of charges given by

$$\nabla \cdot \overline{J}_e = -\frac{\partial}{\partial t}\rho_e \tag{5}$$

can be derived.

The physics behind Maxwell's equations can be easily understood from the definition of curl ($\nabla\times$) and divergence ($\nabla\cdot$) operators. These operators are basically the point form of the integral representation of Maxwell's equations. For a closed region whose surface is denoted by S and its volume is ΔV, the divergence operator on a differentiable vector function $\overline{A}$ is defined as

$$\nabla \cdot \overline{A} = \lim_{\Delta V \to 0} \frac{\oiint_s \overline{A} \cdot \overline{ds}}{\Delta V} \tag{6}$$

where $\overline{ds}$ is along the outward unit vector normal to the surface. Hence $\nabla \cdot \overline{A}$ represents the normalized total flux leaving a point. The curl of a differentiable vector $\overline{F}$ is defined in a similar manner by calculating the integral of $\overline{F}$ around a small closed contour. Consider a closed contour C representing the edge of a small surface whose area is denoted by ΔS and its unit normal vector is denoted by $\hat{n}$, then

$$\left(\nabla \times \overline{F}\right) \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint_c \vec{F} \cdot \overline{d\ell}}{\Delta S} \tag{7}$$

where $\overline{d\ell}$ is a differential length along the contour. The directions of $\overline{d\ell}$ and $\hat{n}$ must satisfy the right-hand rule. Hence $\nabla \times F$ represents the normalized circulation of vector $\overline{F}$ at a point.

With the help of Stokes' and divergence theorems the integral form of Maxwell's equations can easily be derived. For a differential operator $\overline{F}$ the Stokes' theorem relates a surface integral to a contour integral in the following manner

$$\iint_s \nabla \times \overline{F} \cdot \overline{ds} = \oint_c \overline{F} \cdot \overline{d\ell} \tag{8}$$

where C is the contour of an arbitrary surface S having C as its contour. The divergence theorem establishes a relationship between a volume and a surface integral for a differentiable vector A in the following manner

$$\iiint_v \nabla \cdot \overline{A}\, dv = \oiint_s \overline{A} \cdot \overline{ds} \tag{9}$$

where S is a closed surface enclosing V.

Applying Stokes' theorem to (1) and (2) results in, respectively, the Faraday's and modified Ampère's laws given by

$$\oint_c \overline{E} \cdot \overline{d\ell} = -\iint_s \frac{\partial \overline{B}}{\partial t} \cdot \overline{ds} \tag{10}$$

$$\oint_c \overline{H} \cdot \overline{d\ell} = \iint_s \left(\frac{\partial \overline{D}}{\partial t} + \overline{J}_e\right) \cdot \overline{ds} \tag{11}$$

Applying divergence theorem to (3) and (4) provides Gauss's law for the electric and magnetic flux densities

$$\oiint_s \overline{B} \cdot \overline{ds} = 0 \tag{12}$$

$$\oint\!\!\!\oint_s \overline{D} \cdot \overline{ds} = \iiint_v \rho \, dv \tag{13}$$

It should be noted here that the electric and magnetic field quantities are defined in terms of the physical electric and magnetic forces. In addition to the Maxwell's equations a fundamental force equation on a charged particle with electric charge q in an electromagnetic field is given by

$$\overline{F} = q\left(\overline{E} + \overline{u} \times \overline{B}\right)$$

where $\overline{u}$ is the velocity vector of the charged particle. This relationship is referred to as the Lorentz's force equation.

1 Constitutive Relations

The electric and magnetic field quantities are not independent parameter. Their relationship is defined by the constitutive relations enforced by the medium in which the field quantities exist. The most general form of constitutive relations is expressed by a 6×6 matrix which relate $\overline{D}$ and $\overline{B}$ vectors to $\overline{E}$ and $\overline{H}$ vectors, i.e.

$$\begin{bmatrix} \overline{D} \\ \overline{B} \end{bmatrix} = \overline{\overline{C}} \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} , \tag{14}$$

where $\overline{\overline{C}}$ may be expressed by four 3×3 block matrices of the following form

$$\overline{\overline{C}} = \begin{bmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{bmatrix} . \tag{15}$$

The most simple medium is an "isotropic medium" where $\overline{B}$ and $\overline{D}$ are uncoupled and $\overline{D}$ and $\overline{B}$ are linearly related to $\overline{E}$ and $\overline{H}$ respectively. That is,

$$\begin{aligned} \overline{D} &= \epsilon \overline{E} \qquad \epsilon = \text{permittivity} \\ \overline{B} &= \mu \overline{H} \qquad \mu = \text{permeability} \end{aligned} \tag{16}$$

Inside a material medium, the permittivity ϵ is proportional to the induced polarization vector $\overline{P}$ (electric dipole moments per unit volume) and the permeability μ is proportional to the induced magnetization vector $\overline{M}$ (magnetic dipole moments per unit volume). In fact

$$\begin{aligned} \overline{D} &= \epsilon_o \overline{E} + \overline{P} \qquad \epsilon_o = \text{free space permittivity} \\ \overline{B} &= \mu_o \overline{H} + \mu_o \overline{M} \qquad \mu_o = \text{free space permeability} \end{aligned} \tag{17}$$

In free space where $\overline{P} = M = 0$, the medium permittivity and permeability are respectively given by

$$\epsilon_o = 8.854 \times 10^{-12} \quad \text{Farad/m}$$
$$\mu_o = 4\pi \times 10^{-7} \quad \text{henry/m}$$

In isotropic materials where the electric and magnetic fields are relatively small $\overline{P}$ and $\overline{M}$ are parallel to and are linearly dependent on the electric and magnetic fields, i.e.

$$\begin{aligned} \overline{P} &= \epsilon_0 \chi_e \overline{E} \\ \overline{M} &= \chi_m \overline{H} \end{aligned} \tag{18}$$

where χ_e and χ_m are, respectively, the electric and magnetic susceptibility of the isotropic medium. The relative permittivity and permeability of the medium are hence related to the susceptibilities through

$$\begin{aligned} \epsilon_r &= \frac{\epsilon}{\epsilon_o} = 1 + \chi_e \\ \mu_r &= \frac{\mu}{\mu_o} = 1 + \chi_m \end{aligned} \tag{19}$$

The electric and magnetic dipole moments per unit volume ($\overline{P}$ and $\overline{M}$) for most materials produce electric and magnetic fields which are opposing the applied fields and hence the total field quantities inside materials are smaller than the applied fields. In this case ϵ_r and μ_r are positive quantities larger than unity. For some lossy materials at high frequencies (optical and higher) it is possible that $\overline{P}$ lags the applied electric field more than $90°$ which would result in negative values for ϵ_r.

Another material category is "anisotropic media" in which $\overline{B}$ and $\overline{D}$ are still decoupled. However, $\overline{E}$ and $\overline{D}$ and/or $\overline{H}$ and $\overline{B}$ are no longer parallel. The constitutive relations for anisotropic media are given by

$$\begin{aligned} \overline{D} &= \overline{\overline{\epsilon}} \cdot \overline{E} \\ \overline{B} &= \overline{\overline{\mu}} \cdot \overline{H} \end{aligned} \tag{20}$$

where $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are, in general, 3×3 matrices. A medium may be electrically, magnetically, or both electrically and magnetically anisotropic depending on whether the permittivity, permeability, or both are tensor quantities.

Periodic media such as crystals are described in general by symmetric permittivity tensors. Noting that for symmetric matrices the eigen vectors of the matrix forms a basis for $\mathcal{R}^3$ (three dimensional vector space) and the inverse of the eigen vector matrix is simply its transpose ($Q^{-1} = Q^T$) a coordinate transformation can be found to express $\overline{\overline{\epsilon}}$ in terms of a diagonal matrix, i.e.,

$$\overline{\overline{\epsilon}} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} . \tag{21}$$

The coordinate system in which $\overline{\overline{\epsilon}}$ is diagonal is called the "principal system" and the axes of this system are referred to as the principal axes of the medium. An anistropic medium with planar symmetry is called "uniaxial." In uniaxial media $\epsilon_x = \epsilon_y \neq \epsilon_z$ (choice of x and y here is arbitrary, i.e., in a uniaxial medium we may have $\epsilon_x = \epsilon_z \neq \epsilon_y$).

The media in which the electric and magnetic field quantities are coupled are referred to as "bianisotropic media". A bianisotropic medium becomes both polarized and magnetized when it is excited by an electric or magnetic field.

2 Time Harmonic Electromagnetic Waves

The Maxwell's equations given by (1) - (4) are a set of linear equations both in space and time. Since temporal functions can be expressed in terms of superposition of sinusoidal functions, solution of Maxwell's equations to sinusoidal functions of arbitrary frequency is sufficient to characterize the solution of arbitrary temporal excitations.

Assuming a time dependency of the form $e^{-i\omega t}$ for the field quantities, Maxwell's equations take the following form

$$\nabla \times \overline{E} = i\omega\mu\overline{H} \tag{22}$$

$$\nabla \times \overline{H} = -i\omega\epsilon\overline{E} + \overline{J}_e \tag{23}$$

$$\nabla \cdot \overline{B} = 0 \tag{24}$$

$$\nabla \cdot \overline{D} = \rho_e \tag{25}$$

Electric charges and currents are the sources of electromagnetic waves. Frequently it is found convenient to introduce the concepts of hypothetical magnetic charge (ρ_m) and current distributions ($\overline{J}_m$). By introducing ρ_m and $\overline{J}_m$ Maxwell's equations take a more symmetrical form. Basically (22) and (24) are, respectively, modified to

$$\nabla \times \overline{E} = i\omega\mu\overline{H} - \overline{J}_m \tag{26}$$

$$\nabla \cdot \overline{B} = \rho_m \tag{27}$$

This symmetry may be exploited in some cases to find the solution for Maxwell's equation. Consider a set of electromagnetic fields which satisfy Maxwell's equation given by (23), (25), (26), and (27). Now let us generate a new set of field quantities through the following transformation

$$\begin{array}{cccc} \overline{E}, \overline{D} \rightarrow \overline{H}', \overline{B}' & \overline{J} \rightarrow \overline{J}'_m & \rho \rightarrow \rho'_m & \epsilon \rightarrow \mu' \\ \overline{H}, \overline{B} \rightarrow -\overline{E}', -\overline{D}' & \overline{J}_m \rightarrow -\overline{J}' & \rho_m \rightarrow -\rho' & \mu \rightarrow \epsilon' \end{array} \tag{28}$$

It is obvious that the new set of field quantities (primed quantities) also satisfy the Maxwell's equations. The point of this exercise is that if the solution of Maxwell's equation to a set of source functions J, J_m, ρ, ρ_m is known the solution to a new problem with $\overline{J}' = -\overline{J}_m$, $\overline{J}'_m = \overline{J}$, $\rho' = -\rho_m$, and $\rho'_m = \rho$ when $\epsilon' = \mu$ and $\mu' = \epsilon$ is known $\left(\overline{E}' = -\overline{H}, \text{and } \overline{H}' = \overline{E}\right)$.

The duality relationship given by (28) is not unique and may not be the useful one. In situations where the medium parameters are not changed the following duality relations may be more useful:

$$\begin{array}{ccc} (\overline{E}, \overline{D}) \rightarrow \sqrt{\frac{\mu}{\epsilon}}(\overline{H}', \overline{B}') & \overline{J} \rightarrow \sqrt{\frac{\epsilon}{\mu}}\overline{J}'_m & \rho \rightarrow \sqrt{\frac{\epsilon}{\mu}}\rho'_m \\ (\overline{H}, \overline{B}) \rightarrow -\sqrt{\frac{\epsilon}{\mu}}(\overline{E}', \overline{D}') & \overline{J}_m \rightarrow -\sqrt{\frac{\mu}{\epsilon}}J' & \rho_m \rightarrow -\sqrt{\frac{\mu}{\epsilon}}\rho' \end{array} \tag{29}$$

Considering $\overline{J}_e$ and $\overline{J}_m$ as the sources in a homogeneous medium, the electric and magnetic field quantities can be obtained from the electric and magnetic Hertz vector potentials satisfying the following vector wave equations[2]

$$\nabla^2\overline{\Pi}_e + k^2\overline{\Pi}_e = -\frac{iZ}{k}\overline{J}_e \tag{30}$$

$$\nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m = -i\frac{Y}{k}\overline{J}_m \tag{31}$$

[2]Stratton, J.A., *Electromagnetic Theory*, pp. 28–32, New York: McGraw Hill, 1941.

where $k = \omega\sqrt{\mu\epsilon}$ is the propagation constant and $Z = \frac{1}{Y} = \sqrt{\frac{\mu}{\epsilon}}$ is the characteristic impedance of the medium. The advantage of Hertz vector potential is that, it combines the more traditional vector ($\overline{A}_e$) and scalar (ϕ_e) potentials and the Lorentz condition. The relations between the Hertz vector potential and $\overline{A}_e$ and ϕ_e are given by

$$\overline{A}_e = -i\omega\mu\epsilon\Pi_e \ , \ \ \phi_e = -\nabla \cdot \overline{\Pi}_e$$

Solution of (18) and (19) in an unbounded medium are given by

$$\overline{\Pi}_e(\overline{r}) = \frac{iZ}{4\pi k}\int_v \overline{J}_e(\overline{r}')\frac{e^{ik|\overline{r}-\overline{r}|}}{|\overline{r}-\overline{r}'|}\,dv' \tag{32}$$

$$\overline{\Pi}_m(\overline{r}) = \frac{iY}{4\pi k}\int_v \overline{J}_m(\overline{r}')\frac{e^{ik|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|}\,dv' \tag{33}$$

from which the electric and magnetic fields can be obtained from:

$$\overline{E}(\overline{r}) = \nabla\times\nabla\times\overline{\Pi}_e(\overline{r}) + ikZ\nabla\times\overline{\Pi}_m \tag{34}$$

$$\overline{H}(\overline{r}) = \nabla\times\nabla\times\overline{\Pi}_m(\overline{r}) - ikY\nabla\times\overline{\Pi}_e \tag{35}$$