Green's Function for Tenuous Random Media

In the previous section we demonstrated the application of wave theory to a tenuous medium excited by a plane wave for evaluating the bistatic scattered fields. In certain conditions propagation of electromagnetic waves in turbulent medium is of interest. For these situations quantities such as the effective propagation constant and variance of the field fluctuations for an arbitrary source function are sought for. In this section a theoretical analysis is provided for evaluating the dyadic Green's function in a tenuous random media using a perturbation method similar to what was presented in the previous section. Feynman diagrams will be introduced to compactly represent complex equations involved [Frisch, 1968].

1 Dyson's Equation for the Mean Field

The Dyadic Green's Function for a turbulent medium with a small permittivity fluctuation is obtained in this section. Suppose an infinitesimal source point is embedded in a tenuous medium with permittivity $\epsilon(\overline{r}) = \langle\epsilon\rangle + \tilde{\epsilon}(\overline{r})$. The Dyadic Green's Function for this problem must satisfy the wave equation given by

$$\nabla\times\nabla\times \overline{\overline{G}}(\overline{r}_1,\overline{r}_0) - k_m^2\, \overline{\overline{G}}(\overline{r}_1,\overline{r}_0) = \Delta f(\overline{r}_1)\, \overline{\overline{G}}(\overline{r}_1,\overline{r}_0) + \overline{\overline{I}}\,\delta(\overline{r}_1-\overline{r}_0) \qquad (1)$$

where as before

$$\begin{aligned} k_m^2 &= \omega^2\mu\langle\epsilon(\overline{r})\rangle \\ \Delta &= \omega^2\mu\sqrt{\langle\tilde{\epsilon}(\overline{r})^2\rangle} \\ f(\overline{r}) &= \frac{\omega^2\mu\tilde{\epsilon}(\overline{r})}{\Delta} \end{aligned} \qquad (2)$$

and Δ is assumed to be a small quantity. If the medium were homogeneous with permittivity $\epsilon_m = \langle\epsilon(\overline{r})\rangle$, then the Green's function for the unbounded homogeneous problem ($\overline{\overline{G}}^{(0)}(\overline{r},\overline{r}')$) satisfies

$$\nabla \times \nabla \times \overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r}) - k_m^2 \overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r}) = \overline{\overline{I}} \delta(\overline{r}_1 - \overline{r}). \quad (3)$$

Applying the Dyadic Green's second identity to $\overline{\overline{G}}(\overline{r}_1, \overline{r}_0)$ and $\overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r})$, and noting that both $\overline{\overline{G}}(\overline{r}_1, \overline{r}_0)$ and $\overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r})$ satisfy the radiation condition, it can readily be shown that

$$\overline{\overline{G}}(\overline{r}, \overline{r}_0) = \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_0) + \Delta \int \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_1) \cdot \overline{\overline{G}}(\overline{r}_1, \overline{r}_0) f(\overline{r}_1) d^3 r_1 \quad (4)$$

Integral equation (4) for $\overline{\overline{G}}(\overline{r}, \overline{r}_0)$ is exact. However, exact solutions can't be obtained easily. Under the tenuous medium assumption an iterative solution for $\overline{\overline{G}}(\overline{r}, \overline{r}_0)$ can be obtained. Expanding the Green's fFunction in terms of a perturbation series

$$\overline{\overline{G}}(\overline{r}, \overline{r}_0) = \sum_{n=0}^{\infty} G^{(n)}(\overline{r}, \overline{r}_0) \Delta^n \quad (5)$$

and substituting this expression in (4), then collecting terms of equal power in Δ we find

$$\overline{\overline{G}}^{(n)}(\overline{r}, \overline{r}_0) = \int \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_1) \cdot \overline{\overline{G}}^{(n-1)}(\overline{r}_1, \overline{r}_0) f(\overline{r}_1) d^3 r_1 \quad n = 1, 2, \cdots \quad (6)$$

Substituting (6) in (5), the Dyadic Green's Function in explicit form, can be written as

$$\begin{aligned}
\overline{\overline{G}}(\overline{r}, \overline{r}_0) &= \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_0) + \Delta \int \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_1) \cdot f(\overline{r}_1) \overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r}_0) d^3 r_1 \quad (7) \\
&+ \Delta^2 \iint \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_1) \cdot f(\overline{r}_1) \overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r}_2) \cdot f(\overline{r}_2) \overline{\overline{G}}^{(0)}(\overline{r}_2, \overline{r}_0) d^3 r_1 d^3 r_2 \\
&+ \Delta^3 \iiint \overline{\overline{G}}^{(0)}(\overline{r}, \overline{r}_1) \cdot f(\overline{r}_1) \overline{\overline{G}}^{(0)}(\overline{r}_1, \overline{r}_2) \cdot f(\overline{r}_2) \overline{\overline{G}}^{(0)}(\overline{r}_2, \overline{r}_3) \cdot f(\overline{r}_3) \\
&\quad \overline{\overline{G}}^{(0)}(\overline{r}_3, \overline{r}_0) d^3 r_1 d^3 r_2 d^3 r_3 \\
&+ \cdots
\end{aligned}$$

Equation (7) is exact so long as the perturbation expansion (5) converges. The physical interpretation of (7) is rather simple. The first term is the solution to a homogeneous medium where there is no scattering. The next term, the first-order solution, represents single scattering from dielectric inhomogeneities throughout the medium radiating in homogeneous medium as depicted in Figure 1.

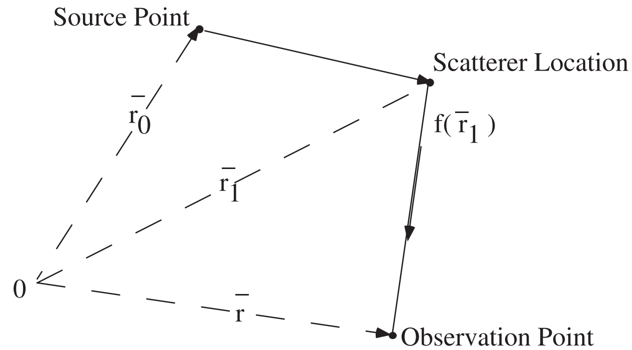


Figure 1: Physical interpretation of the first-order perturbation solution in a random medium where the signal from the source point $\overline{r}_0$ to the observation point $\vec{r}$ experiences a single scattering at $\overline{r}_1$.

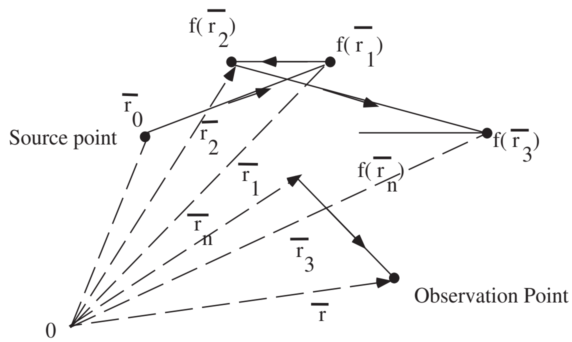


Figure 2: Physical interpretation of the n-th order perturbation solution in a random media.

The second-order and higher order terms represent multiple scattering where the signal path between the source point and observation point, depending on the order, goes through many scattering locations as shown in Figure 2.

Since $f(\overline{r})$ is a random process, the dyadic Green's function given by (7) is also a random process. In this case the statistical behavior of the dyadic Green's function are of interest. The statistical average of the Dyadic Green's Function represent an effective homogeneous medium that describes the propagation of the "mean-field" in the medium. Associated with this mean-field are the effective phase velocity and attenuation rate. Assuming that the permittivity fluctuations is a Gaussian process simplifies the analysis drastically. It can be shown that for a zero-mean Gaussian random vector $(f(\overline{r}_1), \cdots f(\overline{r}_n))$

$$\langle f(\overline{r}_1) f(\overline{r}_2) \cdots f(\overline{r}_n) \rangle = 0 \qquad n = \text{odd number}$$

and for n even

$$\langle f(\overline{r}_1) f(\overline{r}_2) \cdots f(\overline{r}_n) \rangle = \sum_{\text{distinct pairs}} \langle f(\overline{r}_i) f(\overline{r}_j) \rangle \cdots \langle f(\overline{r}_m) f(\overline{r}_n) \rangle$$

for example for $n = 6$

$$\begin{aligned} \langle f(r_1) \cdots f(\overline{r}_6) \rangle \;\; = \;\; & \langle f(\overline{r}_1) f(\overline{r}_2) \rangle \langle f(\overline{r}_3) f(\overline{r}_4) \rangle \langle f(\overline{r}_5) (f \overline{r}_6) \rangle \\ & + \langle f(\overline{r}_1) f(\overline{r}_2) \rangle \langle f(\overline{r}_3) f(\overline{r}_5) \rangle \langle f(\overline{r}_4) f(\overline{r}_6) \rangle \\ & + \langle f(\overline{r}_1 f(\overline{r}_2) \rangle \langle f(\overline{r}_3) f(\overline{r}_6) \rangle \langle f(\overline{r}_4) f(\overline{r}_5) \rangle \\ & + \cdots \end{aligned}$$

Taking the ensemble average of (7) we have

$$\begin{aligned} \langle \overline{\overline{G}} (\overline{r}, \overline{r}_0) \rangle \;\; = \;\; & \overline{\overline{G}}^{(0)} (\overline{r}, \overline{r}_0) + \Delta^2 \iint \overline{\overline{G}}^{(0)} (\overline{r}, \overline{r}_1) \cdot \overline{\overline{G}}^{(0)} (\overline{r}_1, \overline{r}_2) \cdot \overline{\overline{G}}^{(0)} (\overline{r}_2, \overline{r}_0) \\ & \langle f(\overline{r}_1) f(\overline{r}_2) \rangle d^3 r_1 d^3 r_2 \\ + \;\; & \Delta^4 \iiiint \overline{\overline{G}}^{(0)} (\overline{r}, \overline{r}_1) \cdot \overline{\overline{G}}^{(0)} (r_1, \overline{r}_2) \cdot \overline{\overline{G}}^{(0)} (\overline{r}_2, \overline{r}_3) \cdot \overline{\overline{G}}^{(0)} (\overline{r}_3, \overline{r}_4) \cdot \overline{\overline{G}}^{(0)} (r_4, \overline{r}_0) \\ & [\langle f(\overline{r}_1) f(\overline{r}_2) \rangle \langle f(\overline{r}_3) f(\overline{r}_4) \rangle + \langle f(\overline{r}_1) f(\overline{r}_3) \rangle \langle f(\overline{r}_2) f(\overline{r}_4) \rangle \\ + \;\; & \langle f(\overline{r}_1) f(\overline{r}_4) \rangle \langle f(\overline{r}_2) f(\overline{r}_3) \rangle \,] \, d^3 r_1 d^3 r_2 d^3 r_3 d^3 r_4 \\ + \;\; & \cdots \end{aligned} \tag{8}$$

A close form for (8) can be found because of its iterative nature. However, it should be pointed out that the closed form expression for (8) does not necessarily render a simple solution. To derive a closed form expression, we resort to Feynman diagrams which enables us to keep track of long mathematical expressions in term of a symbolic diagram. Let us introduce the following notions:

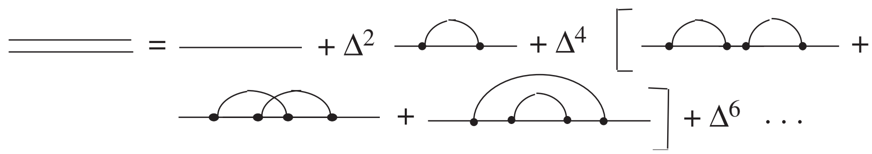
• $\triangleq$ A node defining a scattering point within the random medium ($f(\overline{r})$).

 $\triangleq$ Correlation operator ($\langle f(\overline{r}_1) f(\overline{r}_2) \rangle$) between two scattering points.

——— $\triangleq$ Zeroth-order propagation operator ($\overline{\overline{G}}^{(0)} (\overline{r}_1, \overline{r}_2)$).

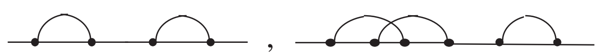
═══ $\triangleq$ Ensemble average operator for the dyadic Green's function $\langle \overline{\overline{G}} (\overline{r}, \overline{r}_0) \rangle$.

Using the above-defined operators (8) can be pictorially represented by

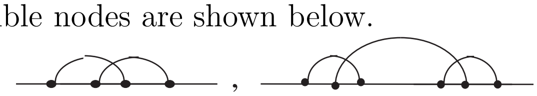


(9)

To reduce (9), each term is categorized as either being irreducible (strongly connected) term or reducible term. The reducible terms are those that contain at least one pair of adjacent nodes so that when they are disconnected from the original chain of nodes, two distinct acceptable chains of nodes are generated. For example, the following nodes are reducible



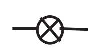
and examples of irreducible nodes are shown below.



The sum of all strongly connected nodes in (9) without the end connectors are lumped into an operator known as <u>mass operator</u> which is graphically shown by $\otimes$ i.e.,

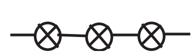
(10)

With this definition all the strongly connected nodes of (9) is given by

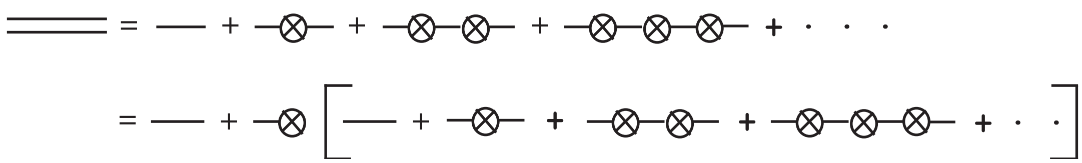


The sum of all reducible nodes in (9) which is composed of two strongly connected elements can be represented by

In a similar manner



represents the sum of all reducible elements composed of three strongly connected elements. Hence (9) can be represented by



The operator in the bracket is recognized as the ensemble average operator itself, thus

(11)

In mathematical form

$$\begin{aligned} \langle \overline{\overline{G}} \, (\overline{r}, \overline{r}_o) \rangle &= \overline{\overline{G}}^{(0)} \, (r, r_o) + \iint d^3 r_1 d^3 r_2 \, \overline{\overline{G}}^{(0)} \, (\overline{r}, \overline{r}_1) \cdot \, \overline{\overline{M}} \, (\overline{r}_1, \overline{r}_2) \\ & \cdot \langle \overline{\overline{G}} \, (\overline{r}_2, \overline{r}_o) \rangle \end{aligned} \tag{12}$$

which is the statement of Dyson's equation. In (12) $\overline{\overline{M}} \, (\overline{r}_1, \overline{r}_2)$ is the dyadic mass operator. The dyadic mass operator given by (10) is an infinite series for which an exact solution is not found, however, by terminating the infinite series for sufficiently small values of Δ an approximate solution can be obtained.

2 Approximate Solutions

In this section two approximate solutions for the dyadic mass operator are periodic. Both of these two approximations are based on truncation of the infinite series given by (10).

2.1 Bilocal Approximation

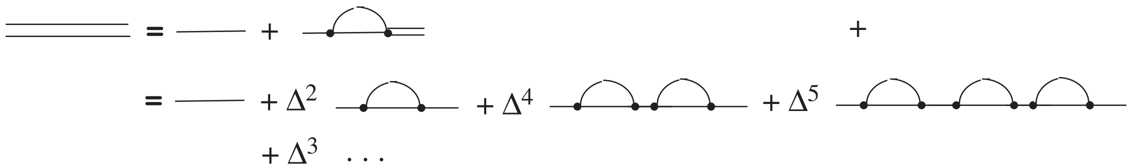
In the bilocal approximation only the first term of (10) is retained. Basically

(13a)

or mathematically

$$\overline{\overline{M}} \, (\overline{r}_1, \overline{r}_2) \simeq \Delta^2 C \, (\overline{r}_1 - \overline{r}_2) \, \overline{\overline{G}}^{(0)} \, (\overline{r}_1, \overline{r}_2) \tag{13b}$$

Using approximation (13) Dyson's equation given by (8) and (9) simplifies to



(14)

and

$$\langle \overline{\overline{G}}(\overline{r},\overline{r}_0)\rangle = \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_0) + \Delta^2 \iint \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_1)\cdot G^{(0)}(\overline{r}_1,\overline{r}_2)\cdot \langle \overline{\overline{G}}(\overline{r}_2,\overline{r}_0)\rangle \, C(\overline{r}_1-\overline{r}_2)d^3r_1d^3r_2 \quad (15)$$

It is obvious from (14) that (15) contains the first-order solution and partial components of higher-order solutions. The next step is to demonstrate how (15) can be used to evaluate the effective propagation constant in the random medium. Let us first consider a scalar case. Dyson's equation under bilocal approximation for scalar wave propagation can be directly deduced from (15) and is given by [Tatarskii and Gertsenshtein, 1963].

$$\langle g(\overline{r},\overline{r}_0)\rangle = g^{(0)}(\overline{r},\overline{r}_0) + \Delta^2 \iint g^{(0)}(\overline{r},\overline{r}_1)g^{(0)}(\overline{r}_1,\overline{r}_2)\langle g(\overline{r}_2,\overline{r}_0)\rangle C(\overline{r}_1-\overline{r}_2)d^3r_1d^3r_2 \quad (16)$$

where

$$(\nabla^2 + k_m^2)g^{(0)}(\overline{r},\overline{r}_0) = -\delta(\overline{r}-\overline{r}_0) \quad (17)$$

Applying the operator $(\nabla^2 + k_m^2)$ on (16) and using (17) we get

$$\begin{aligned}(\nabla^2 + k_m^2)\langle g(\overline{r},\overline{r}_0)\rangle &= -\delta(\overline{r},\overline{r}_0) - \Delta^2 \iint \delta(\overline{r}-\overline{r}_1)g^{(0)}(\overline{r}_1,\overline{r}_2)\langle g(\overline{r}_2,\overline{r}_0)\rangle \quad (18)\\ &\quad \times C(\overline{r}_1-\overline{r}_2)d^3r_1d^3r_2 \\ &= -\delta(\overline{r}-\overline{r}_0) - \Delta^2 \int g^{(0)}(\overline{r},\overline{r}_2)\langle g(\overline{r}_2,\overline{r}_0)\rangle C(\overline{r}-\overline{r}_2)d^3r_2\end{aligned}$$

Assuming that $C(\overline{r}-\overline{r}_2)$ drops rapidly in a neighborhood around $\overline{r}$ then (18) may be approximated by

$$(\nabla^2 + k_m^2)\langle g(\overline{r},\overline{r}_0)\rangle = -\delta(\overline{r}-\overline{r}_0) - \Delta^2\langle g(\overline{r},\overline{r}_0)\rangle \int g^{(0)}(\overline{r},\overline{r}_2)C(\overline{r}-\overline{r}_2)d^3\overline{r}_2 \quad (19)$$

Here we are assuming that the rate of change of C is much faster than the spatial variation of $\langle g^{(0)}(\overline{r}_2, \overline{r})\rangle$. Noting that $g^{(0)}(\overline{r}, \overline{r}_2) = \frac{e^{ik_m|\overline{r}-\overline{r}_2|}}{4\pi|\overline{r}-\overline{r}_2|}$, and denoting $|\overline{r} - \overline{r}_2| = R$, (19) takes the following form

$$[\nabla^2 + k_m^2 + \Delta^2 \int g^{(0)}(R)C(R)d^3R]\langle g(\overline{r}, \overline{r}_0)\rangle = -\delta(\overline{r} - \overline{r}_0) \tag{20}$$

If the effective propagation constant of the medium is represented by K_{eff}, close inspection of (20) reveals that

$$K_{eff}^2 = k_m^2 + \Delta^2 \int g^{(0)}(R)C(R)d^3R \tag{21}$$

which is the quantity of interest. Assuming an exponential correlation function

$$C(R) = e^{-R/a}$$

it can easily be shown that

$$K_{eff}^2 = k_m^2 + \frac{\Delta^2 a^2}{(1 - ik_m a)^2} \tag{22}$$

and if $k_m a << 1$

$$K_{eff}^2 = k_m^2 + \Delta^2 a^2 (1 + 2ik_m a), \tag{23}$$

Recalling that (see (2))

$$\frac{\Delta}{k_m^2} = \frac{\sqrt{\langle\tilde{\epsilon}(r)^2\rangle}}{\epsilon_m} << 1$$

From (22) the effective propagation constant can be calculated approximately from

$$K_{eff} = k_m \left[1 + \frac{\Delta}{2k_m^2}\Delta a^2(1 + 2ik_m a)\right] \tag{24}$$

Equation (24) indicates that the real part of the propagation constant is slightly higher that obtained from the average dielectric constant and an imaginary component is introduced which can be attributed, physically, to the scattering loss in the medium.

Equation (19) was solved assuming a sharply varying correlation function to establish some physical understanding. However (19) can be solved exactly noting that the right-hand-side integral is a convolution type. Denoting the Fourier transform of $\langle g(\overline{r}, \overline{r}_0)\rangle$ by $\tilde{g}(\overline{K}, \overline{r}_0)$ and the Fourier transform of $C(\overline{r})$ by $W(\overline{K})$ (18) can be written as

$$(k_m^2 - |\overline{K}|^2)\tilde{g}(\overline{K}, \overline{r}_0) = -1 - \Delta^2 \tilde{g}(\overline{K}, \overline{r}_0) \left[W(\overline{K}) * \frac{1}{|\overline{K}|^2 - k_m^2} \right] \frac{1}{(2\pi)^3}$$

from which $\tilde{g}(\overline{K}, \overline{r}_0)$ can be obtained

$$\tilde{g}(\overline{K}, \overline{r}_0) = \frac{-1}{(k_m^2 - |\overline{K}|^2) + \frac{\Delta^2}{(2\pi)^3} \left[W(\overline{K}) * \frac{1}{|\overline{K}|^2 - k_m^2} \right]} \tag{25}$$

where we have used $\frac{1}{|\overline{K}|^2 - k_m^2}$ as the Fourier Transform of $g^{(0)}(r, r_0)$. The approximate form of the propagation constant can be obtained from (25) easily by assuming that $C(\overline{r}) = e^{-|\overline{r}|/a}$ for which we have

$$\frac{1}{(2\pi)^3} W(\overline{K}) * \frac{1}{|\overline{K}|^2 - k_m^2} = \frac{1}{|\overline{K}|^2 - (k_m + \frac{i}{a})^2} \tag{26}$$

If a is a very small quantity then (26) can be approximated by

$$\frac{1}{|\overline{K}|^2 - (k_m + \frac{i}{a})^2} \simeq \frac{a^2}{(1 - ik_m a)^2} \simeq a^2(1 + 2ik_m a)$$

and (25) becomes

$$\tilde{g}(K, \overline{r}_0) \simeq \frac{1}{|\overline{K}|^2 - k_m^2 - \Delta^2 a^2 (1 + 2ik_m a)}$$

Hence the effective propagation constant is recognized as

$$K_{eff}^2 = k_m^2 + \Delta^2 a^2 (1 + 2ik_m a)$$

which is the same as (23) and thus (24) follows. The bilocal approximation can now be applied to the vector wave equation for estimating the propagation constant of the mean-field. A similar procedure that led to derivation of (21) can be pursued to get[1]

$$\overline{\overline{K}}_{eff}^{\,2} = k_m^2 \,\overline{\overline{I}} + \Delta^2 \int d^3R \,\overline{\overline{G}}^{(0)}(R) C(R) \tag{27}$$

The Dyadic Green's Function is singular at $R = 0$, and it can be shown that over a small volume around $R = 0$ the integral of $\overline{\overline{G}}^{(0)}$ is nonvanishing even as the volume shrinks to zero. In this sense the Dyadic Green's Function can be represented in terms

[1] Tatarskii, V.I., Propagation of electromagnetic waves in a medium with strong dielectric constant fluctuations, *Sov. Phys. JE TP*, Vol. 19, pp. 946-953, 1964.

of a delta function and a "principal value" expression with the understanding that the volume integral of the principal value component excludes a small volume around $R = 0$. The geometrical shape of the exclusion volume determines the coefficient of the delta function. It can be shown that for a spherical exclusion volume

$$\overline{\overline{G}}^{(0)}(R) = \overline{\overline{G}}_{ps}^{(0)}(R) - \frac{\overline{\overline{I}}}{3k_m^2}\delta(\overline{R}) \tag{28}$$

where subscript ps denote the principal value. The second term of (28) may be considered as the self-cell contribution in the context of numerical analysis.

The explicit form of the Dyadic Green's Function was found to be

$$\overline{\overline{G}}^{(0)}(R) = \left[(1 + \frac{i}{k_m R} - \frac{1}{k_m^2 R^2})\,\overline{\overline{I}} + (-1 - \frac{3i}{k_m R} + \frac{3}{k_m^2 R^2})\hat{R}\hat{R}\right]\frac{e^{ik_m R}}{4\pi R} \tag{29}$$

In the vicinity of $R = 0$ the contribution of the integrand of (27) mostly comes from the delta function, and for large values of R the components of $\overline{\overline{G}}_{ps}^{(0)}(R)$ proportional to $\frac{1}{R}$ are dominant. Noting that

$$\int_0^{\pi}\int_0^{2\pi}\hat{R}\hat{R}\sin\theta d\theta d\phi = \frac{4\pi}{3}\,\overline{\overline{I}} \tag{30}$$

(27) can be written as

$$\overline{\overline{K}}_{eff}^{2} = [k_m^2 - \frac{\Delta^2}{3k_m^2} + \frac{2}{3}\Delta^2\int_0^{\infty} R\,C(R)e^{ik_m R}dR]\;\overline{\overline{I}} \tag{31}$$

Assuming exponential correlation function ($C(R) = e^{-R/a}$) we get

$$K_{eff}^2 = k_m^2 - \frac{\Delta^2}{3k_m^2} + \frac{2}{3}\Delta^2 a^2(1 + 2ik_m a) \tag{32}$$

where as before we have assumed that $k_m a << 1$. There are some differences between (32) and (23) which indicates the difference between scalar and vector wave propagation in tenuous media. Accuracy of (32) depends on Δ where it is required that

$$\Delta << k_m^2 \tag{33}$$

and

$$\Delta a^2 << 1 \tag{34}$$

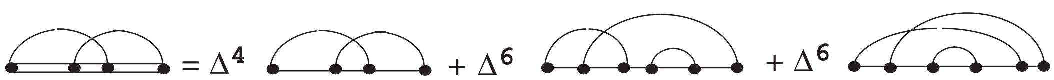
Under situations where (33) is the restrictive criterion, theory of strong permittivity fluctuation can be employed to remove this limitation. However, it should be noted that the solution given by (32) is derived from a perturbation solution which has a severe restriction on Δ to begin with.

2.2 Nonlinear Approximation

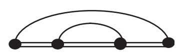
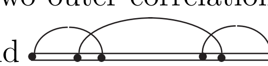
In this section an alternative expression for the mass operator is derived and then using only the first term of the infinite series an approximate solution is obtained. Let us first consider an operator defined by , which using (9) is given by

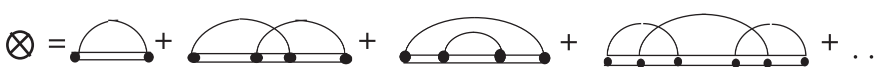
(35)

This operator includes all the strongly connected links of the mass operator which has a single outer correlation. Now let us consider operator  which has the following expansion



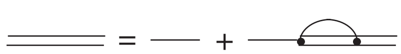
(36)

(36) represents the sum of all strongly connected terms with two outer correlations intersecting at one point. Similarly operators like  and  and higher order would include all strongly connected nodes of the mass operator, therefore



(37)

In the nonlinear approximation only the first term of (37) is retained in (11) for the mass operator and thus we have



(38)

In mathematical form (38) can be written as

$$\begin{aligned}\langle \overline{\overline{G}}(\overline{r},\overline{r}_0)\rangle &= G^{(0)}(\overline{r},\overline{r}_0) + \Delta^2 \iint d^3r_1 d^3r_2\, \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_1)\langle \overline{\overline{G}}(\overline{r}_1,\overline{r}_2)\rangle \\ &\quad \times \langle \overline{\overline{G}}(\overline{r}_2,\overline{r}_0)\rangle C(\overline{r}_1-\overline{r}_2)\end{aligned} \tag{39}$$

This equation can be solved using the Fourier transform method as shown before. To demonstrate that the results based on the nonlinear approximation may differ from the bilocal approximation, consider the scalar Green's function for the scalar wave propagation case considered before. The counterpart of equation (19) under the nonlinear approximation takes the following form:

$$\left(\nabla^2 + k_m^2\right)\langle g(\overline{r},\overline{r}_0)\rangle \;=\; -\delta(\overline{r}-\overline{r}_0) - \Delta^2 \int \langle g(\overline{r},\overline{r}_2)\rangle\langle g(\overline{r}_2,\overline{r}_0)\rangle C(\overline{r}-\overline{r}_2) d^3r_2$$

Again assuming that the correlation function is sharply varying at $\overline{r}_2$ near $\overline{r}$, the counterpart of (21) is found to be

$$K_{eff}^2 = k_m^2 + \Delta^2 \int \langle g(R)\rangle C(R) d^3R \tag{40}$$

For an exponential correlation function

$$K_{eff}^2 = k_m^2 + \Delta^2 a^2 (1 + 2iK_{eff}a) \tag{41}$$

where it is assumed that $K_{eff}a << 1$. The effective propagation constant can be obtained from:

$$K_{eff} = k_m \left[1 + \frac{\Delta^2 a^2}{2k_m^2}(1 + 2iK_{eff}a)\right] \tag{42}$$

To the zeroth order in $\Delta, K_{eff} = k_m$ and to the first order in Δ^2 is

$$K_{eff} = k_m \left[1 + \frac{\triangle^2 a^2}{2k_m^2}(1 + 2ik_m a)\right] \tag{43}$$

which is the result based on the bilocal approximation. Therefore the difference between the nonlinear and bilocal approximation is of order of $\triangle^4$ and higher.

3 Bethe-Salpeter Equation

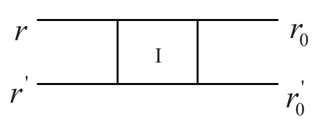
The perturbation series was used to derive an expression for the mean field in a tenuous random medium. Basically the dyadic Green's function of the medium with permittivity fluctuations was derived in terms of an infinite series of multiple scattering terms in the unperturbed medium. Suppose instead of the mean-field, the field covariance is of interest. In the most general configuration the covariance of fields at point $\overline{r}$ and $\overline{r}'$ due to source points $\overline{r_0}$ and $\overline{r_0}'$ is sought. This can be obtained once the ensemble average of the dyadic Green's function is calculated. Starting from equation (7) we have

$$
\begin{aligned}
\langle \overline{\overline{G}}(\overline{r},\overline{r_0})\, \overline{\overline{G}}^{*}(\overline{r}',\overline{r}_0')\rangle = \langle \overline{\overline{G}}^{(0)}(\overline{r},\overline{r_0})\, \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0') + \Delta\{\overline{\overline{G}}^{(0)}(\overline{r},\overline{r_0}) \int \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0')\, \cdot \\
f(\overline{r}_1')\, \overline{\overline{G}}^{(0)*}(\overline{r}_1',\overline{r}_0') d^3\overline{r}_1' + \left(\int \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_1)\cdot f(\overline{r}_1)\, \overline{\overline{G}}^{(0)}(\overline{r}_1,\overline{r}_0) d^3\overline{r}_1 \right) \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0')\} \\
+\Delta^2\{\overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_0) \iint \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_1')\cdot f(\overline{r}_1')\cdot \overline{\overline{G}}^{(0)*}(\overline{r}_1',\overline{r}_2')\cdot f(\overline{r}_2')\, \overline{\overline{G}}^{(0)*}(\overline{r}_2',\overline{r}_0') d^3\overline{r}_1' d^3 r_2' \\
+ \left(\iint G^{(0)}(\overline{r},\overline{r}_1) f(\overline{r}_1) G^{(0)}(\overline{r}_1,\overline{r}_2) f(\overline{r}_2) G(\overline{r}_2,\overline{r}_0) d^3\overline{r}_1 d^3\overline{r}_2 \right) \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0') \\
+ \iint \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_1)\cdot f(\overline{r}_1)\cdot \overline{\overline{G}}^{(0)}(\overline{r}_1,\overline{r}_0)\cdot \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_1')\cdot f(\overline{r}_1')\cdot \overline{\overline{G}}^{(0)*}(\overline{r}_1',\overline{r}_0') d^3\overline{r}_1, d^3\overline{r}_1'\} \\
+\Delta^3\{\cdots\} + \Delta^4\{\cdots\} + \cdot\rangle \quad (44)
\end{aligned}
$$

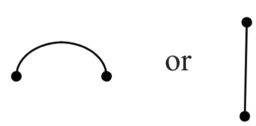
where we have assumed $f(r)$ is a real randome process. Further if $f(r)$ is a zero-mean Gaussian process the ensemble average of all odd power of Δ disappears. The terms in even power of Δ can be decomposed into three categories: 1) integrals involving cross correlation in unprimed coordinates only, 2) integrals involving cross-correlation in primed coordinateds only, and 3) integrals involving cross correlation between the primed and unprimed coordinates. Hence (43) can be written as

$$
\begin{aligned}
\langle \overline{\overline{G}}(\overline{r},\overline{r}_0)\, \overline{\overline{G}}^{*}(\overline{r}',\overline{r}_0')\rangle = \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_0)\langle \overline{\overline{G}}^{*}(\overline{r}',\overline{r}_0')\rangle + \langle \overline{\overline{G}}(\overline{r},\overline{r}_0)\rangle\, \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0') \\
- \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_0)\, \overline{\overline{G}}^{(0)*}(\overline{r}',\overline{r}_0') + \Delta^2\{\iint \overline{\overline{G}}^{(0)}(\overline{r},\overline{r}_1)\, \overline{\overline{G}}^{(0)}(\overline{r}_1,\overline{r}_0)\, \overline{\overline{G}}^{*}(\overline{r}',\overline{r}_1') \\
\overline{\overline{G}}^{(0)*}(\overline{r}_1',\overline{r}_0')\langle f(\overline{r}_1) f(\overline{r}_1')\rangle d^3\overline{r}_1 d^3\overline{r}_1'\} + \Delta^4\{\quad\} + \cdots \qquad (45)
\end{aligned}
$$

Equation (45) can be represented by a diagram which provides a convenient tool for further analysis. Let us introduce a two-level diagram where one level is the unprimed and the other level is the primed coordinate. We define the following notations:

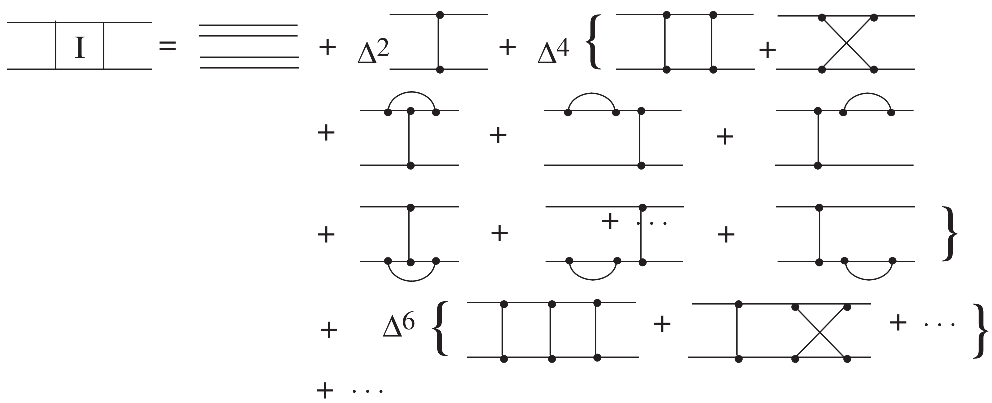


$\triangleq \quad \langle \overline{\overline{G}}(r, r_0)\rangle$



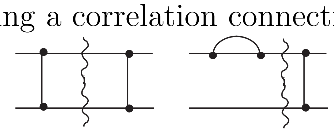
$\triangleq \quad \langle f(r_1) f(r_2)\rangle$

and the symbol is defined as $\langle \overline{\overline{G}}(\overline{r},\overline{r}_0)\rangle\langle \overline{\overline{G}}(\overline{r}',\overline{r}_0')\rangle$ as they don't contain any correlations between the two layers. Therefore,

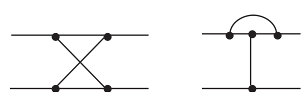


(46)

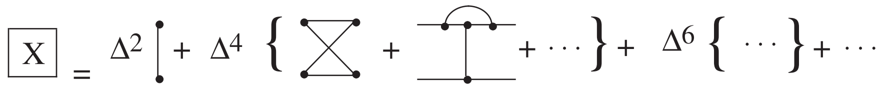
As before the diagrams are categorized into weakly connected or reducible diagrams and strongly connected or irreducible diagrams. The weakly connected diagrams are those that can be generated by breaking a correlation connection for example:



All other diagrams are strongly connected. The following diagrams are examples of strongly connected diagrams.

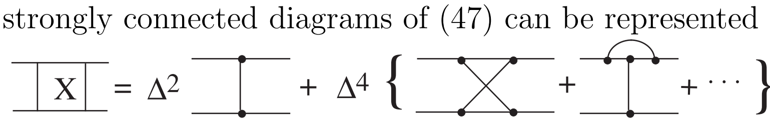


The intensity operator, similar to the mass operator, is defined as the sum of all strongly connected diagrams minus the end connectors and is represented by:

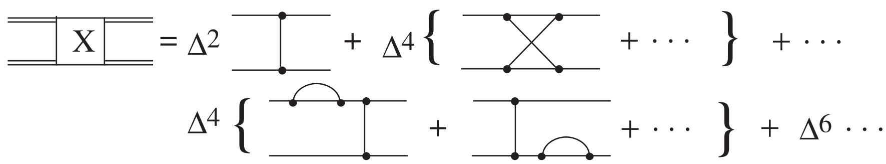


(47)

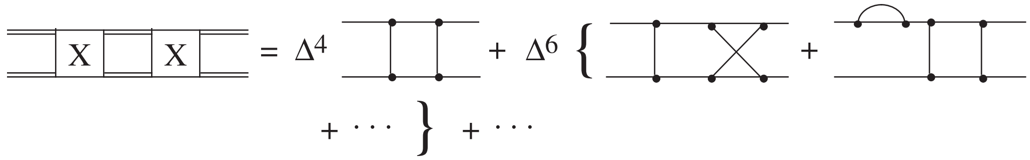
The sum of all strongly connected diagrams of (47) can be represented by



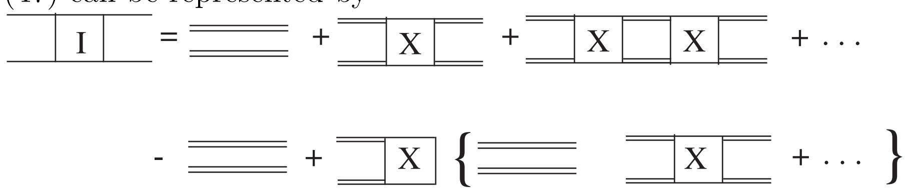
The sum of all strongly connected and weakly connected diagrams of (46) composed of only one of the elements of the intensity operators is represented by



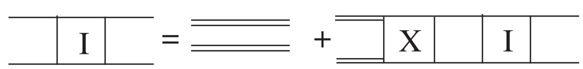
and the sum of all weakly connected diagrams of (46) composed of two of the elements of the intensity operators is given by



In general (47) can be represented by



or equivalently



(48)

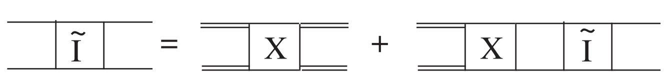
Equation (48) is the exact expression for (44) which can be written mathematically as

$$\begin{aligned}\langle \overline{\overline{G}}\,(\bar r,\bar r_0)\ \overline{\overline{G}}^{\,*}\,(\bar r',\bar r'_0)\rangle &= \langle \overline{\overline{G}}\,(\bar r,\bar r_0)\rangle\langle \overline{\overline{G}}^{\,*}\,(\bar r',\bar r'_0)\rangle \\ + \iiiint d^3\bar r_1 d^3\bar r_2 d^3\bar r'_1 d^3\bar r'_2 \langle \overline{\overline{G}}\,(\bar r,\bar r_1)\rangle\langle \overline{\overline{G}}^{\,*}\,(\bar r',\bar r'_1)\rangle\ &I(\bar r_1,\bar r_2,\bar r'_1.\bar r'_2) \\ &\langle \overline{\overline{G}}\,(\bar r_2,\bar r_0)\ \overline{\overline{G}}^{\,*}\,(\bar r'_2,\bar r'_0)\rangle \end{aligned} \tag{49}$$

where $I(\bar r_1,\bar r_2,\bar r'_1.\bar r'_2)$ is the intensity operator given by (47). Equation (49) is referred to as the Bethe-Salpeter equation. The field covariance defined by $\langle \overline{\overline{G}}\,(\bar r,\bar r_0)\ \overline{\overline{G}}^{\,*}\,(\bar r',\bar r'_0)\rangle - \langle \overline{\overline{G}}\,(\bar r,\bar r_0)\rangle\langle \overline{\overline{G}}^{\,*}\,(\bar r',\bar r'_0)\rangle$, is represented by

(50)

which can also be represented by



(51)

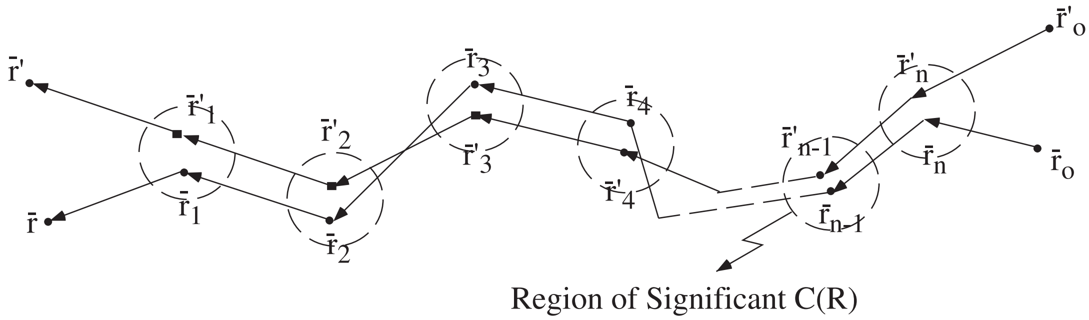
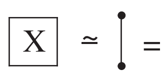


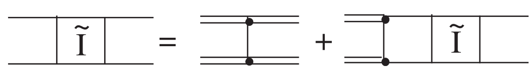
Figure 3: The physical interpretation of the n-th order term in the ladder approximation.

A closed form solution for the intensity operator [X] cannot be obtained, hence an approximate form must be used. The simplest approximation to the intensity operator can be obtained by retaining the first term of the expansion, that is,



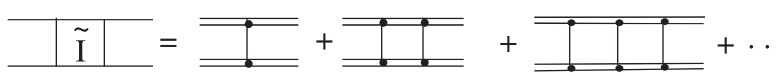
$$= \delta(\bar{r}_1 - \bar{r}_2)\delta(\bar{r}'_1 - \bar{r}'_2)\, C(\bar{r}_1 - \bar{r}'_1) \tag{52}$$

Under this approximation the field covariance is given by



(53)

Expanding (53), we get



(54)

Because of the geometry of the diagram, this solution is referred to as the ladder approximation. The ladder approximation can be used in conjunction with the bilocal or nonlinear approximations. Mathematically the ladder approximate of (54) is given by

$$
\begin{aligned}
&\langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_0)\, \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_0)\rangle = \langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_0)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_0)\rangle + \\
&\Delta^2 \int d^3\bar{r}_1 \int d^3\bar{r}'_1 \langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_1)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_1)\rangle C(\bar{r}_1, \bar{r}'_1)\langle \bar{\bar{G}}\,(\bar{r}_1, \bar{r}_0)\, \bar{\bar{G}}^*\,(\bar{r}'_1, \bar{r}'_0)\rangle \\
= \;&\langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_0)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_0)\rangle + \Delta^2 \int d^3\bar{r}_1 \int d^3\bar{r}'_1 \langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_1)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_1)\rangle \\
&C(\bar{r} - \bar{r}'_1)\langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_0)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_0)\rangle + \triangle^4 \int d^3\bar{r}_1 \int d^3\bar{r}_2 \int d^3\bar{r}'_1 \int d^3\bar{r}'_2 \\
&\langle \bar{\bar{G}}\,(\bar{r}, \bar{r}_1)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}', \bar{r}'_1)\rangle C(\bar{r}_1 - \bar{r}'_1)\langle \bar{\bar{G}}\,(\bar{r}_1, \bar{r}_2)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}'_1, \bar{r}'_2)\rangle C(\bar{r}_2 - \bar{r}'_2) \\
&\langle \bar{\bar{G}}\,(\bar{r}_2 - \bar{r}_0)\rangle\langle \bar{\bar{G}}^*\,(\bar{r}'_2 - \bar{r}'_0)\rangle + \Delta^6 \cdots
\end{aligned} \tag{55}
$$

Consider the n-th order term of the above series. For a given set of $\overline{r}_1, \overline{r}_2, \cdots, \overline{r}_n$ the contribution from the integrals $\overline{r}_1, \overline{r}_2, \cdots, \overline{r}_n$ comes from the vicinities of $\overline{r}_1, \overline{r}_2, \cdots, \overline{r}_n$ noting that $C(R)$ decays rapidly away from $R = 0$. This phenonmenon is depicted in Figure 3.

In order to demonstrate the application of the ladder approximation we consider a simple case of scalar wave propagation in a random medium with a sharply varying correlation coefficient $C(\overline{r}_1 - \overline{r}_2)$. This correlation coefficient represents a medium with point scatterers. Suppose there are two point sources at points $\overline{r}_0$ and $\overline{r}'_0$ and we are interested in finding the field correlation at a point $\overline{r}$ in the medium. The Dyson's equation under the bilocal approximation for scalar wave propagation was obtained and is given by

$$\langle g(\overline{r}, \overline{r}_1) \rangle \simeq \frac{e^{iK_{eff}|\overline{r} - \overline{r}_1|}}{4\pi|\overline{r} - \overline{r}_1|}$$

where $k_{eff} = k_m + i\Delta^2 a^3$ assuming an exponential correlation function $C(R) = e^{-R/a}$. Substituting this correlation function in the ladder approximation (counterpart of (50) for scalra Green's function) and noting the $C(\overline{r}_1 - \overline{r}'_2)$ is rapidly varying function we get

$$\begin{aligned} \langle g(\overline{r}, \overline{r}_0) g^*(\overline{r}, \overline{r}'_0) \rangle \;\; = \;\; & \frac{e^{iK_{eff}|\overline{r} - \overline{r}_0|}}{4\pi|\overline{r} - \overline{r}_0|} \cdot \frac{e^{-K_{eff}|\overline{r} - \overline{r}'_0|}}{4\pi|\overline{r} - \overline{r}'_0|} \\ & + \Delta^2 \int d^3\overline{r}_1 \langle g(\overline{r}_1, \overline{r}_0) g^*(\overline{r}_1, \overline{r}'_0) \rangle \int d^3\overline{r}'_1 \frac{e^{iK_{eff}|\overline{r} - \overline{r}_0|}}{4\pi|\overline{r} - \overline{r}_0|} \cdot \\ & \frac{e^{-iK_{eff}|\overline{r} - \overline{r}_1|}}{4\pi|\overline{r} - \overline{r}'_1|} \, e^{\frac{-|\overline{r}_1 - \overline{r}'_1|}{a}} \end{aligned}$$

The embedded integral is a function of $\overline{r}$ and $\overline{r}_1$ and constitutes the kernel of the integral equation. However, the solution of the integral equation may not be straightforward.