Scattering from Periodic Surfaces[1]

Periodic surfaces are encountered in many active and passive microwave remote sensing application such as evaluation of scattering and emission from plowed field or rough water surfaces. In this section scattering of plane waves from one dimensional periodic surfaces is considered. First the scattering problem for two homogeneous dielectric media with an arbitrary periodic interface at oblique incidence is formulated using the extended boundary condition (EBC) approach. Then the problem of scattering from an inhomogeneous dielectric layer above a multi-layer dielectric half- space is analyzed using the volumetric polarization current. In both of these methods Floquet's theorem is invoked to derive the periodic Green's function that limits the extent of the integral equations just over one period of the distributed target.

Figure 1 shows the geometry of the scattering problem, where a plane wave is illuminating the periodic interface between two homogeneous dielectric layers. Without loss of generality a coordinate system is chosen so that the y-axis is parallel to the generating axis of the one-dimensional periodic surface.

In this coordinate system the surface height profile is only a function of x and satisfies:

$$f(x) = f(x + nL) \quad \forall\ n \in \ \mathcal{N}$$

where $\mathcal{N}$ is the set of natural numbers and L is the period of the surface.

Suppose a plane wave with an arbitrary propagation direction $\overline{k}_i$ is incident upon the interface and its electric field is given by

$$\overline{E}_i = \hat{e}_i E_o e^{i\overline{k}_i \cdot \overline{r}} \tag{1}$$

where

$$\overline{k}_i = k_{xi}\hat{x} + k_{yi}\hat{y} - k_{zi}\hat{z}$$

Since the geometry of the scatterer is independent of y, all field quantities must have a y- dependence similar to that of the incident wave, that is, $e^{ik_{yi}y}$. Similar to waveguide

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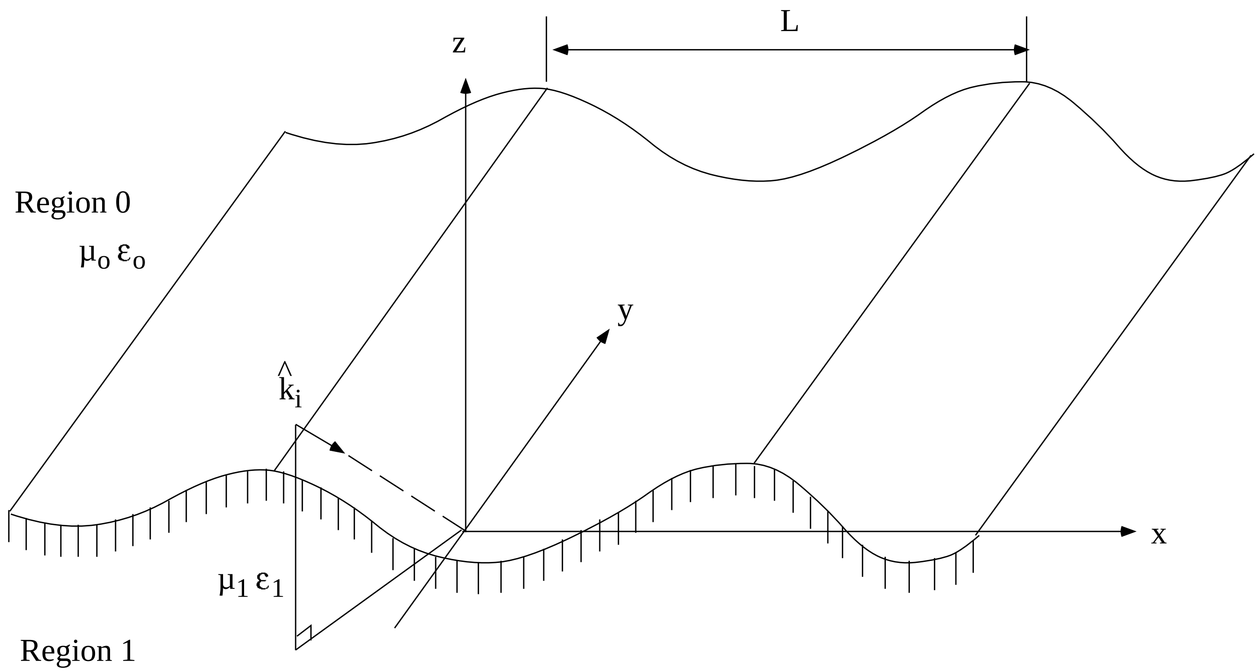

Figure 1: Geometry of a periodic surface separating two homogeneous media.

problems, in this case, the field components may be decomposed into TE and TM waves (with respect to the y-axis) and it can be shown that the electric and magnetic field quantities can be obtained once E_y and H_y are characterized.

Let us denote:

$$\overline{E}(\overline{r}) = \overline{E}_t(\overline{r}) + \hat{y}E_y(\overline{r}) \tag{2}$$
$$\overline{H}(\overline{r}) = \overline{H}_t(\overline{r}) + \hat{y}H_y(\overline{r}) \tag{3}$$

and also letting $\nabla = (\nabla_t + \hat{y}\frac{\partial}{\partial y}) = (\nabla_t + ik_{yi}\hat{y})$ where $\nabla_t = (\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial z}\hat{z})$. Maxwell's equations become

$$(\nabla_t + ik_{yi}\hat{y}) \times (\overline{E}_t(\overline{r}) + \hat{y}E_y(\overline{r})) = i\omega\mu(\overline{H}_t + \hat{y}H_y) \tag{4}$$
$$(\nabla_t + ik_{yi}\hat{y}) \times (\overline{H}_t(\overline{r}) + \hat{y}H_y(\overline{r})) = -i\omega\epsilon(\overline{E}_t + \hat{y}E_y) \tag{5}$$

Expanding (4) and (5) and collecting the transverse components, we get

$$\nabla_t \times \hat{y}E_y(\overline{r}) + ik_{yi}\hat{y} \times \overline{E}_t(\overline{r}) = i\omega\mu\overline{H}_t(\overline{r}) \tag{6}$$
$$\nabla_t \times \hat{y}H_y(\overline{r}) + ik_{yi}\hat{y} \times \overline{H}_t(\overline{r}) = -i\omega\epsilon\overline{E}_t(\overline{r}) \tag{7}$$

Now multiplying (6) by $\frac{ik_{yi}}{i\omega\mu}\hat{y}\times$, an expression for $ik_{yi}\hat{y}\times\overline{H}_t(\overline{r})$ is obtained which can be substituted in (7) to find $\overline{E}_t(\overline{r})$ in terms of $E_y(\overline{r})$ and $H_y(\overline{r})$. Noting that

$$\hat{y}\times(\nabla_t\times\hat{y}E_y(\overline{r})) = \hat{y}\times(\nabla_t E_y(\overline{r})\times\hat{y}) = \nabla_t E_y(\overline{r})$$

we have

$$\overline{E}_t(\overline{r}) = \frac{i}{k^2-k_{yi}^2}\left[k_{yi}\nabla_t E_y(\overline{r}) + \omega\mu\nabla_t H_y(\overline{r})\times\hat{y}\right] \tag{8}$$

Furthermore, upon the application of duality, it follows that

$$\overline{H}_t(\overline{r}) = \frac{i}{k^2-k_{yi}^2}\left[k_{yi}\nabla_t H_y(\overline{r}) + \omega\epsilon\,\hat{y}\times\nabla_t E_y(\overline{r})\right] \tag{9}$$

The longitudinal components E_y and H_y must satisfy the wave equation

$$\left(\nabla_t^2 + k_{j\rho}^2\right)\left\{\begin{array}{c} E_{jy} \\ H_{jy} \end{array}\right\} = 0 \tag{10}$$

where we have introduced a subscript j to denote each region ($j = 0$ or 1) and

$$k_{j\rho}^2 = k_j^2 - k_{yi}^2$$

The Green's function for the two-dimensional problem satisfies

$$\left(\nabla_t^2 + k_{j\rho}^2\right) g_j(\overline{\rho},\overline{\rho}') = -\delta(\overline{\rho}-\overline{\rho}') \tag{11}$$

and is given by

$$\begin{aligned} g_j(\overline{\rho},\overline{\rho}') &= \frac{i}{4}H_0^{(1)}\left(k_{j\rho}|\overline{\rho}-\overline{\rho}'|\right) \\ &= \frac{i}{4\pi}\int_{-\infty}^{+\infty}\frac{1}{k_{jz}}e^{i(k_x(x-x')+k_{jz}|z-z'|)}dk_x \end{aligned} \tag{12}$$

where

$$\begin{aligned} \overline{\rho} &= x\hat{x} + z\hat{z} \\ k_{jz} &= \sqrt{k_{j\rho}^2 - k_x^2} \end{aligned}$$

Applying Green's second identity to E_{jy} and $g_{jy}(\overline{\rho}, \overline{\rho}')$ over a volume that specifies Region 0 provides the following relation

$$\iint\limits_{\text{region }0} \left[g_o \nabla^2 E_{oy} - E_{oy} \nabla^2 g_o \right] ds' = -\int\limits_{C_\infty + S} \left(g_o \frac{\partial E_{oy}}{\partial n} - E_{oy} \frac{\partial g_o}{\partial n} \right) d\rho' = \begin{cases} \overline{E}_y(\overline{\rho}) & \overline{\rho} \text{ in region } 0 \\ 0 & \overline{\rho} \text{ in region } 1 \end{cases} \tag{13}$$

where C_∞ is a semi-circle of a very large radius and S is the interface contour. The negative sign on the right-hand-side integral is a result of defining the unit normal $\hat{n}$ inward to Region 0. Noting that E_{oy} in the zeroth region is the sum of the incident wave and scattered wave and that in the absence of the lower half-space

$$\overline{E}_{yi}(\overline{\rho}) = -\int\limits_{C_\infty} \left(g_o \frac{\partial E_{oy}}{\partial n} - E_{oy} \frac{\partial g_o}{\partial n} \right) d\rho' \tag{14}$$

which indicates that the source of incoming energy is somewhere in the upper half-space. Substitution of (14) into (13) results in

$$\overline{E}_{yi}(\overline{\rho}) - \int_{-\infty}^{+\infty} g_o(\overline{\rho}, \overline{\rho}') \hat{n} \cdot \nabla_t' E_{oy}(\overline{\rho}') - E_{oy}(\overline{\rho}') \hat{n} \cdot \nabla_t' g_o(\overline{\rho}, \overline{\rho}') \} \, d\ell' \tag{15}$$
$$= \begin{cases} E_{oy}(\overline{\rho}) & z > f(x) \\ 0 & z < f(x) \end{cases}$$

which is the statement of "extinction theorem." In (15) $d\ell$ is the differential arc length given by

$$d\ell = \sqrt{(dx)^2 + (dz)^2} = \sqrt{1 + \left(\frac{df}{dx} \right)^2} \, dx \ , \tag{16}$$

$\hat{n}$ is the upward unit normal to the interface given by

$$\hat{n} = \nabla_t (z - f(x)) = \frac{\left(\hat{z} - \frac{df}{dx} \hat{x} \right)}{\sqrt{1 + \left(\frac{df}{dx} \right)^2}} \tag{17}$$

In a similar manner applying the Green's second identity to the lower half-space we get

$$\int_{-\infty}^{+\infty} \{ g_1 \left(\overline{\rho} - \overline{\rho}' \right) \hat{n} \cdot \nabla_t' E_{1y}(\overline{\rho}') - E_{1y} \left(\overline{\rho}' \right) \hat{n} \cdot \nabla_t' g_1 \left(\overline{\rho}, \overline{\rho}' \right) \} \, d\ell' = \tag{18}$$
$$= \begin{cases} 0 & z > f(x) \\ E_{1y}(\overline{\rho}) & z < f(x) \end{cases} \ .$$

Equations (15) and (18) together with the boundary conditions are the necessary tool to calculate the scattered fields in both regions. However, since the extent of the integrals involved is infinite, these integral equations are not amenable to numerical solutions. So far we have not used the symmetry of the scatterer. If the y-z plane were the plane of incidence ($k_{xi} = 0$), then we expected that all field quantities be periodic with periodicity L. Hence it seems logical to postulate that apart from a progressive phase factor similar to that of the incident wave all the field quantities must be periodic. Mathematically

$$\begin{aligned} \overline{E}(x+nL, y, z) &= \overline{E}(x, y, z)e^{ik_{xi}(nL)} \\ \overline{H}(x+nL, y, z) &= \overline{H}(x, y, z)e^{ikx(nL)} \end{aligned} \tag{19}$$

For the problem at hand the dependency in y is explicit as well ($e^{ik_{yi}y}$) which will be suppressed throughout the rest of this analysis.

By dividing the integrals in (15) or (18) into a summation of integrals over one period we have

$$\begin{aligned} E_{jy}^{s}(\overline{\rho}) &= (-1)^{(j+1)} \sum_{n=-\infty}^{+\infty} \int_{x_o+nL}^{x_o+(n+1)L} \{g_j(\overline{\rho}, \overline{\rho}')\hat{n} \cdot \nabla_t' E_{jy}(\overline{\rho}') - E_{jy}(\overline{\rho}')\hat{n} \cdot \nabla_t' g_i(\overline{\rho}, \overline{\rho}')\} \\ &\cdot \sqrt{1 + (f'(x'))^2}\; dx' \end{aligned} \tag{20}$$

If variable x' is now changed to $x' + nL$ and the field property (19) is used, (20) can be written as

$$\begin{aligned} E_{jy}^{s}(\overline{\rho}) &= (-1)^{(j+1)} \int_{x_o}^{x_o+L} \{g_{jp}(\overline{\rho}, \overline{\rho}')\hat{n} \cdot \nabla_t' E_{jy}(\overline{\rho}') - E_{jy}(\overline{\rho}')\hat{n} \cdot \nabla_t' g_{jp}(\overline{\rho}, \overline{\rho}')\} \\ &\sqrt{1 + (f'(x'))^2}\; dx' \end{aligned} \tag{21}$$

where

$$g_{jp}(\overline{\rho}, \overline{\rho}') = \frac{i}{4} \sum_{n=-\infty}^{+\infty} H_o^{(1)} \left(k_{j\rho} \sqrt{(x - x' - nL)^2 + (z - z')^2} \right) e^{ik_{xi}nL} \tag{22}$$

is known as the periodic Green's function. The rate of convergence of the summation given by (22) is very poor and therefore it cannot be used directly in the numerical computation of the scattering problem. To rectify this deficiency, a method known as "Poisson summation formula"[2] will be used. In (22) the Fourier transform of the Hankel

[2]Davies, B., "*Integral Transforms and Their Applications,*" *Springer- Verlag, New York, 1985*

function given by (12) is used and the order of summation and integration is changed to obtain

$$g_{jp}(\overline{\rho},\overline{\rho}') = \frac{i}{4\pi}\int_{-\infty}^{+\infty}\frac{1}{k_{jz}}e^{ik_{jz}|z-z'|}\left(\sum_{n=-\infty}^{+\infty}e^{i(k_x(x-x')+(k_{xi}-k_x)nL}\right)dk_x \tag{23}$$

Noting that the summation in (23) represent the Fourier series representation of a periodic train of Dirac delta functions, i.e.

$$\sum_{n=-\infty}^{+\infty}e^{-in(k_x-k_{xi})L} = 2\pi\sum_{n=-\infty}^{+\infty}\delta\left[((k_x-k_{xi})L-2\pi n)\right],$$

equation (23) can be written as

$$g_{jp}(\overline{\rho},\overline{\rho}') = \frac{i}{2L}\sum_{n=-\infty}^{+\infty}\frac{1}{k_{jnz}}e^{ik_{nx}(x-x')}e^{ik_{jnz}|z-z'|} \tag{24}$$

where

$$\begin{aligned} k_{nx} &= k_{xi}+\frac{2\pi n}{L} \\ k_{jnz} &= \sqrt{k_j^2-k_{yi}^2-\left(k_{xi}+\frac{2\pi n}{L}\right)^2} \end{aligned} \tag{25}$$

Equation (24) can be interpreted as the field generated by a periodic array of line currents of equal amplitudes and progressive phase. Here it is shown that the array produces a discrete angular spectrum of plane waves propagating along.

$$\overline{K}_{jn} = k_{nx}\hat{x}+k_{yi}\hat{y}+k_{jnz}\hat{z} \tag{26}$$

A plane wave in this spectrum is called propagating if k_{jnz} is real (for purely real k_j) and is called non-propagating or evanescent if k_{jnz} is purely imaginary. The latter decay exponentially away from the surface and there are an infinite number of them. Since the decay is exponential the series converges very rapidly and therefore (24) is a much better representation for periodic Green's function than (22). The number of propagating waves is finite and depends on the period-over-wavelength ratio (L/λ_j) and the incidence angle. The nth mode is a propagating mode if n belongs to set $\mathcal{S}$ defined by

$$\mathcal{S} = \left\{n;\frac{-L}{2\pi}\left(\sqrt{k_j^2-k_{yi}^2}+k_{xi}\right) < n < \frac{L}{2\pi}\left(\sqrt{k_j^2-k_{yi}^2}-k_{xi}\right)\right\}$$

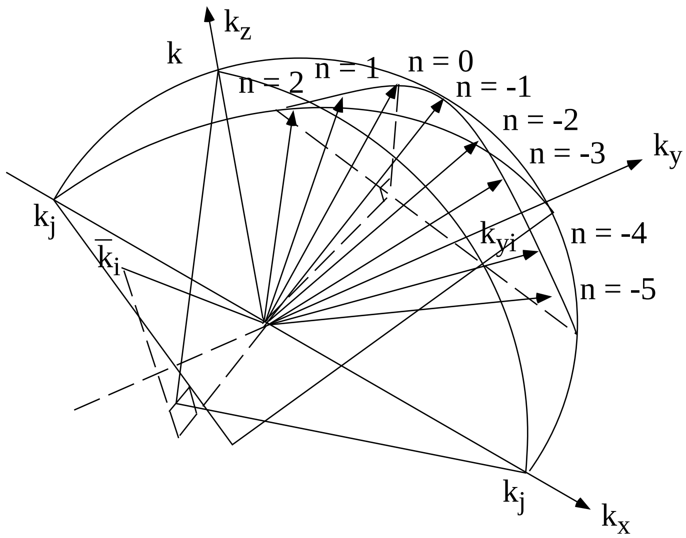

Figure 2: Discrete propagation vectors of propagating Floquet's modes in k-space for a periodic medium at oblique incidence.

Figure 2 shows the direction of propagating modes in k-space for the periodic surface for a given incidence angle and frequency.

Substituting the periodic Green's function into (15) and (18) the following equations are obtained where we use $(z - z')$ for $|z - z'|$ when $z > f(x)$ and $-(z - z')$ for $|z - z'|$ when $z < f(x)$ (for all x):

$$E_{0y}(\overline{\rho}) = E_{yi}(\overline{\rho}) + \sum_{n=-\infty}^{+\infty} b_n \; e^{i(k_{nx}x + k_{0nz}z)} \qquad z > f_{max} \tag{27}$$

$$0 = E_{yi}(\overline{\rho}) + \sum_{n=-\infty}^{+\infty} a_n \; e^{i(k_{nx}x - k_{0nz}z)} \qquad z < f_{min} \tag{28}$$

where the coefficients a_n and b_n are given by:

$$b_n = \frac{-i}{2Lk_{0nz}} \int_{x_o}^{x_o+L} \Big\{ \hat{n} \cdot \nabla_t' E_{oy}(\overline{\rho}') e^{-i(k_{nx}x' + k_{0nz}z')} - E_{oy}(\overline{\rho}') \; \hat{n} \cdot \nabla_t' e^{-i(k_{nx}x' + k_{onz}z')} \Big\} \sqrt{1 + (f'(x'))^2} \; dx' \tag{29}$$

$$a_n = \frac{-i}{2Lk_{0nz}} \int_{x_o}^{x_o+L} \Big\{ \hat{n} \cdot \nabla_t' E_{oy}(\overline{\rho}') e^{-i(k_{nx}x' - k_{0nz}z')} - E_{oy}(\overline{\rho}') \; \hat{n} \cdot \nabla_t' e^{-i(k_{nx}x' - k_{0nz}z')} \Big\} \sqrt{1 + (f'(x'))^2} \; dx' \tag{30}$$

It should be emphasized here that the integration is carried out over the surface, that is,

$$\overline{\rho}' = x'\hat{x} + f(x')\hat{z} \ ; \ \ z' = f(x')$$

The extended boundary condition can be applied to the y-component of the magnetic field as well. Using duality from (27) and (28) we have

$$\begin{aligned} H_{oy}(\overline{\rho}) &= H_{yi}(\overline{\rho}) + \sum_{n=-\infty}^{+\infty} d_n e^{i(k_{nx}x + k_{0nz}z)} \\ 0 &= H_{yi}(\overline{\rho}) + \sum_{n=-\infty}^{+\infty} c_n e^{i(k_{nx}x - k_{0nz}z)} \end{aligned} \tag{31}$$

where expressions for c_n and d_n are similar to those for a_n and b_n respectively with the exception that E_{oy} is replaced with H_{oy} in (29) and (30).

Now using the periodic Green's function in (18) we get:

$$0 = \sum_{n=-\infty}^{+\infty} B_n e^{i(k_{nx}x + k_{1nz}z)} \qquad z > f_{max} \tag{32a}$$

$$E_{1y}(\overline{\rho}) = \sum_{n=-\infty}^{+\infty} A_n e^{i(k_{nx}x - k_{1nz}z)} \quad z < f_{min} \tag{32b}$$

where

$$\begin{aligned} B_n &= \frac{+i}{2Lk_{1nz}} \int_{x_o}^{x_o+L} \Big\{ \hat{n} \cdot \nabla_t' E_{1y}(\overline{\rho}') e^{-i(k_{nx}x' + k_{1nz}z')} - E_{1y}(\overline{\rho}') \\ &\quad \hat{n} \cdot \nabla_t' e^{-i(k_{nx}x' + k_{1nz}z')} \Big\} \sqrt{1 + (f'(x'))^2}\, dx' \end{aligned} \tag{33}$$

and

$$\begin{aligned} A_n &= \frac{+i}{2Lk_{1nz}} \int_{x_o}^{x_o+L} \Big\{ \hat{n} \cdot \nabla_t' E_{1y}(\overline{\rho}') e^{-i(k_{nx}x' - k_{1nz}z')} - E_{1y}(\overline{\rho}') \\ &\quad \hat{n} \cdot \nabla_t' e^{-i(k_{nx}x' - k_{1nz}z')} \Big\} \sqrt{1 + (f'(x'))^2}\, dx' \end{aligned} \tag{34}$$

Applying the duality relationship to (32), the y-component of the magnetic field is given by

$$0 = \sum_{n=-\infty}^{+\infty} D_n e^{i(k_{nx}x + k_{1nz}z)} \qquad z > f_{max} \tag{35a}$$

$$H_{1y}(\overline{\rho}) = \sum_{n=-\infty}^{+\infty} C_n e^{i(k_{nx}x - k_{1nz}z)} \qquad z < f_{min} \tag{35b}$$

Where D_n and C_n can be obtained from (33) and (34) by replacing E_{1y} with H_{1y}. Noting that

$$E_{yi} = (\hat{y} \cdot \hat{e}_i) E_0 e^{i\overline{k}_i \cdot \overline{r}},$$

equation (28) can be solved for a_n's and it is evident that

$$\begin{aligned} a_n &= 0 \qquad \forall\, n \neq 0 \\ a_0 &= -E_0(\hat{y} \cdot \hat{e}_i) \end{aligned}$$

In a similar manner since $H_{yi} = \frac{(\overline{k}_i \times \hat{e}_i)\cdot\hat{y}}{k_0 Z_0} E_0 e^{i\overline{k}_i \cdot \overline{r}}$ equation (31) reveals that

$$\begin{aligned} c_n &= 0 \qquad \forall\, n \neq 0 \\ c_0 &= -\frac{E_0}{k_0 Z_0}(\overline{k}_i \times \hat{e}_i) \cdot \hat{y} \end{aligned}$$

Equations (32a) and (35a) also provide

$$B_n = D_n = 0 \qquad \forall\, n$$

Since a_n, c_n, B_n, and D_n are known, they can be used in (30) and (33) and their counterparts for the magnetic fields to provide four equations for the unknown surface fields $E_{oy} = E_{1y}, H_{oy} = H_{1y}$ and $\hat{n} \cdot \nabla_t E_{oy}, \hat{n} \cdot \nabla_t E_{1y}, \hat{n} \cdot \nabla_t H_{oy}$, and $\hat{n} \cdot \nabla_t H_{1y}$. It appears that there are six unknowns and four equations, however, the four latter unknown quantities ($\hat{n} \cdot \nabla_t \bullet$) are not independent and can be related to each other using the boundary conditions at the interface.

The boundary conditions mandate that $\hat{n} \times \overline{E}_t$ and $\hat{n} \times \overline{H}_t$ be continuos across the boundary. Using (8) and (9) it can be shown that

$$\begin{aligned} \frac{1}{k_{0\rho}^2}\left[k_{yi}\hat{n} \times \nabla_t E_{0y} + k_0 Z_0 \hat{n} \times (\nabla_t H_{0y} \times \hat{y})\right] = \frac{1}{k_{1\rho}^2}\left[k_{yi}\hat{n} \times \nabla_t E_{1y}\right. \\ \left. + k_1 Z_1 \hat{n} \times (\nabla_t H_{1y} \times \hat{y})\right] \end{aligned} \tag{36}$$

$$\frac{1}{k_{0\rho}^2}\left[k_{yi}\hat{n}\times\nabla_t H_{0y}+k_0 Y_0 \hat{n}\times(\hat{y}\times\nabla_t E_{0y})\right]=\frac{1}{k_{1\rho}^2}\left[k_{yi}\hat{n}\times\nabla_t H_{1y}\right.$$
$$\left.+\,k_1 Y_1 \hat{n}\times(\hat{y}\times\nabla_t E_{1y})\right] \tag{37}$$

Noting that $\hat{n}\cdot\hat{y}=0$, we have

$$\hat{n}\times(\nabla_t H_{jy}\times\hat{y}) = -\hat{n}\cdot\nabla_t H_{jy}\hat{y} \tag{38}$$
$$\hat{n}\times(\nabla_t E_{jy}\times\hat{y}) = -\hat{n}\cdot\nabla_t E_{jy}\hat{y} \tag{39}$$

substituting (38) into (36) and (39) into (37), it follows that

$$\hat{y}(\hat{n}\cdot\nabla_t H_{oy}) = \left[-\left(\frac{k_{0\rho}}{k_0}\right)^2+1\right]\frac{k_{yi}}{k_0 Z_0}\hat{n}\times\nabla_t E_{1y}+\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2\frac{k_1 Z_1}{k_0 Z_0}\left(\hat{n}\cdot\nabla_t H_{1y}\right)\hat{y} \tag{40}$$
$$\hat{y}\left(\hat{n}\cdot\nabla_t E_{oy}\right) = \left[\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2-1\right]\frac{k_{yi}}{k_0 Y_0}\hat{n}\times\nabla_t H_{1y}+\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2\frac{k_1 Y_1}{k_0 Y_0}\left(\hat{n}\cdot\nabla_t E_{1y}\right)\hat{y} \tag{41}$$

In derivation of (40) and (41) the following equalities were used

$$\hat{n}\times\nabla_t E_{0y} = \hat{n}\times\nabla_t E_{1y} \tag{42}$$
$$\hat{n}\times\nabla_t H_{0y} = \hat{n}\times\nabla_t H_{1y} \tag{43}$$

Scalar equations (40) and (41) together with the continuity condition of E_y and H_y are the necessary and sufficient boundary conditions for determining the unknown surface fields. All we have to do is to determine $\hat{n}\times\nabla_t E_{jy}$ and $\hat{n}\times\nabla_t H_{jy}$ in terms of surface fields E_{jy} and H_{jy}. Upon expansion of

$$\begin{aligned}\hat{n}\times\nabla_t E_{jy} &= \frac{(-f'(x)\hat{x}+\hat{z})}{\sqrt{1+(f')^2}}\times\left[\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial z}\hat{z}\right]E_{jy} \\ &= \frac{\left(\frac{\partial E_{jy}}{\partial x}+f'\frac{\partial E_{jy}}{\partial z}\right)}{\sqrt{1+(f')^2}}\;\hat{j}j=0,1\end{aligned} \tag{44}$$

Noting that on the surface where $z=f(x)$

$$\frac{dE_{0y}}{dx} = \frac{\partial E_{0y}}{\partial x} + \frac{\partial E_{0y}}{\partial z} f' \tag{45}$$

Similarly

$$\frac{dE_{1y}}{dx} = \frac{\partial E_{1y}}{\partial x} + \frac{\partial E_{1y}}{\partial z} f' \tag{46}$$

Noting that on the surface $E_{0y} = E_{1y}$ which implies $\frac{dE_{0y}}{dx} = \frac{dE_{1y}}{dx}$, in view of (44), (45), and (46)

$$\hat{n} \times \nabla_t E_{0y} = \hat{n} \times \nabla_t E_{1y} = \frac{dE_{0y}/dx}{\sqrt{1+(f')^2}} \hat{y} = \frac{dE_{1y}/dx}{\sqrt{1+(f')^2}} \hat{y} \tag{47}$$

The above equation is the proof for (42) and it also provides an expression for $n \times \nabla_t E_{1y}$ in terms of E_{0y} as needed in (40). In a similar manner it can be shown that

$$\hat{n} \times \nabla_t H_{0y} = \hat{n} \times \nabla_t H_{1y} = \frac{dH_{0y}/dx}{\sqrt{1+(f')^2}} \hat{y} = \frac{dH_{1y}/dx}{\sqrt{1+(f')^2}} \hat{y} \tag{48}$$

We are now in a position to solve for the unknown surface fields. As mentioned earlier, apart from the progressive phase factor $e^{ik_{xi}x}$ all field quantities including the surface fields are periodic functions of x with period L. Expanding the surface fields in terms of their Fournier series with unknown coefficients we have

$$E_{1y}(x, f(x)) = \sum_{m=-\infty}^{+\infty} \alpha_m e^{ik_{mx}x} \tag{49}$$

$$\sqrt{1+f'^2(x)}\, \hat{n} \cdot \nabla E_{1y}(x, f(x)) = \sum_{m=-\infty}^{+\infty} \beta_m e^{ik_{mx}x} \tag{50}$$

$$H_{1y}(x, f(x)) = \sum_{m=-\infty}^{+\infty} \gamma_m e^{ik_{mx}x} \tag{51}$$

$$\sqrt{1+f'^2(x)}\, \hat{n} \cdot \nabla H_{1y}(x, f(x)) = \sum_{m=-\infty}^{+\infty} \eta_m e^{ik_{mx}x} \tag{52}$$

Substituting (49) and (50) in (33) and noting that

$$\hat{n} \cdot \nabla'_t e^{-i(k_{nx}x' + k_{1nz}z')} = \frac{+i(k_{nx}f'(x') - k_{1nz})}{\sqrt{1+(f'(x))^2}} e^{-i(k_{nx}x' + k_{1nz}f(x))}$$

We have

$$B_n = 0 \quad = \quad \sum_{m=-\infty}^{+\infty} \beta_m \left\{ \frac{+i}{2Lk_{1nz}} \int_{x_o}^{x_o+L} e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} dx' \right\} \tag{53}$$

$$+ \quad \sum_{m=-\infty}^{+\infty} \alpha_m \left\{ \frac{1}{2Lk_{1nz}} \int_{x_o}^{x_o+L} (k_{nx}f'(x') - k_{1nz}) e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} dx' \right\}$$

Since $k_{mx} - k_{nx} = (m-n)\frac{2\pi}{L}$, the function $e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]}$ is a periodic function with period L. Therefore

$$\int_{x_o}^{x_o+L} k_{nx}f'(x') e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} dx' = \frac{ik_{nx}}{k_{1nz}} \int_{x_o}^{x_o+L} -ik_{1nz}f'(x') e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} dx'$$

$$= \frac{ik_{nx}}{k_{1nz}} \left\{ \int_{x_o}^{x_o+L} d\left(e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} \right) + i(k_{nx} - k_{mx}) \int_{x_o}^{x_o+L} e^{i[(k_{mx}-k_{nx})x'-k_{1nz}f(x')]} dx' \right\} \tag{54}$$

where the first integral in the right-hand side of (54) vanishes. Remembering that $k_{nx}^2 + k_{1nz}^2 = k_{1\rho}^2$, (53) can be written as

$$\sum_{m=-\infty}^{+\infty} Q_{nm}^{(1)} \beta_m + Q_{nm}^{(2)} \alpha_m = 0 \tag{55}$$

where

$$Q_{nm}^{(1)} = \frac{+i}{2Lk_{1nz}} I_1^- \quad ; \quad Q_{nm}^{(2)} = \frac{k_{nx}k_{mx} - k_{1\rho}^2}{2Lk_{1nz}^2} I_1^-$$

and

$$I_1^- = \int_{x_o}^{x_o+L} e^{i\left[2\pi(m-n)\frac{x'}{L} - k_{1nz}f(x')\right]} dx'$$

Also noting that $D_n = 0$, from (51) and (52) it can be shown that

$$\sum_{m=-\infty}^{+\infty} Q_{nm}^{(1)} \eta_m + Q_{nm}^{(2)} \gamma_m = 0 \tag{56}$$

Two more equations similar to (55) and (56) can be obtained using (30) and its counterpart for magnetic fields in which we use the boundary conditions on the tangential field components, namely, equations (40) and (41) together with $E_{0y} = E_{1y}$ and $H_{0y} = H_{1y}$.

According to (48)

$$(\hat{n} \times \nabla_t H_{1y}) \cdot \hat{y} = \sum_{m=-\infty}^{+\infty} \frac{i\gamma_m k_{mx}}{\sqrt{1+(f'(x))^2}} e^{ik_{mx}x}$$

and therefore from (41)

$$\begin{aligned} \sqrt{1+(f'(x))^2}\,\hat{n} \cdot \nabla_t E_{0y} &= \left[\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2 - 1\right] \frac{k_{yi}}{k_0 Y_0} \sum_{m=-\infty}^{+\infty} i\gamma_m k_{mx} e^{ik_{mx}x} + \\ &+ \left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2 \frac{k_1 Y_1}{k_0 Y_0} \sum_{m=-\infty}^{+\infty} \beta_m e^{ik_{mx}x} \end{aligned} \tag{57}$$

Substituting (57) and (49) into (30), results in

$$\begin{aligned} a_n &= \sum_{m=-\infty}^{+\infty} \gamma_m \left\{ \frac{k_{mx}}{2Lk_{0nz}} \left[\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2 - 1\right] \frac{k_{yi}}{k_0 Y_0} \int_{x_o}^{x_o+L} e^{i[(k_{mx}-k_{nx})x' + k_{0nz}f(x')]} dx' \right\} \\ &+ \sum_{m=-\infty}^{+\infty} \alpha_m \left\{ \frac{k_{nx}k_{mx} - k_{0\rho}^2}{2Lk_{0nz}^2} \int_{x_o}^{x_o+L} e^{i[(k_{mx}-k_{nx})x' + k_{0nz}f(x')]} dx' \right\} \\ &+ \sum_{m=-\infty}^{+\infty} \beta_m \left\{ \frac{-i}{2Lk_{0nz}} \left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2 \frac{k_1 Y_1}{k_0 Y_0} \int_{x_o}^{x_o+L} e^{i[(k_{mx}-k_{nx})x' + k_{0nz}f(x')]} dx' \right\} \end{aligned}$$

or equivalently

$$\sum_{m=-\infty}^{+\infty} \left(U_{nm}^{(1e)} \alpha_m + U_{nm}^{(2e)} \beta_m + U_{nm}^{(3e)} \gamma_m \right) = a_n \tag{58}$$

where

$$U_{nm}^{(1e)} = \frac{k_{nx}k_{mx} - k_{0\rho}^2}{2Lk_{0nz}^2} I_0^+ \tag{59}$$

$$I_0^+ = \int_{x_o}^{x_o+L} e^{i\left[2\pi(m-n)\frac{x'}{L} + k_{0nz}f(x')\right]} dx'$$

$$U_{nm}^{(2e)} = \frac{-i}{2Lk_{0nz}} \left[\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2\right] \frac{k_1 Y_1}{k_0 Y_0} I_0^+ \tag{60}$$

$$U_{nm}^{(3e)} = \frac{k_{mx}}{2Lk_{0nz}} \left[\left(\frac{k_{0\rho}}{k_{1\rho}}\right)^2 - 1\right] \frac{k_{yi}}{k_0 Y_0} I_0^+ \tag{61}$$

The fourth equation can easily be obtained using duality where $\alpha_m \to \gamma_m$, $\beta_m \to \eta_m$, $\gamma_m \to -\alpha_m$, $U_{nm}^{(je)} \to U_{nm}^{(jh)}$ and $a_n \to c_n$, hence

$$\sum_{m=-\infty}^{+\infty} \left(U_{nm}^{(1h)} \gamma_m + U_{nm}^{(2h)} \eta_m - U_{nm}^{(3h)} \alpha_m \right) = c_n \tag{62}$$

where the expressions for $U_{nm}^{(jh)}$ can be obtained from those of $U_{nm}^{(je)}$, given by (59)-(61), by replacing $Y_j \to Z_j (j = 0, 1)$. By truncating the infinite summations an approximate solution for the unknown surface field expansion coefficients, $\alpha_m, \beta_m, \gamma_m$ and η_m can be obtained. The accuracy of this approximate solution is a function of truncation number (M). It is obvious that higher accuracy can be achieved for higher values of M at the expense of computation time. In matrix notation, (55), (56), (58) and (62) can be presented by

$$\begin{bmatrix} \overline{\overline{U}}^{(1e)} & \overline{\overline{U}}^{(2e)} & \overline{\overline{U}}^{(3e)} & \overline{\overline{0}} \\ -\overline{\overline{U}}^{(3h)} & \overline{\overline{0}} & \overline{\overline{U}}^{(1h)} & \overline{\overline{U}}^{(2h)} \\ \overline{\overline{Q}}^{(2)} & \overline{\overline{Q}}^{(1)} & \overline{\overline{0}} & \overline{\overline{0}} \\ \overline{\overline{0}} & \overline{\overline{0}} & \overline{\overline{Q}}^{(2)} & \overline{\overline{Q}}^{(1)} \end{bmatrix} \begin{bmatrix} \overline{\alpha} \\ \overline{\beta} \\ \overline{\gamma} \\ \overline{\eta} \end{bmatrix} = \begin{bmatrix} \overline{a} \\ \overline{c} \\ \overline{0} \\ \overline{0} \end{bmatrix} \tag{63}$$

where $\overline{\overline{0}}$, and $\overline{0}$ are $(2M+1) \times (2M+1)$ and $(2M+1) \times 1$ zero matrix and vector respectively. Also, $\overline{\overline{U}}^{(je)}$, $\overline{\overline{U}}^{(jh)}$, $\overline{\overline{Q}}^{(1)}$, and $\overline{\overline{Q}}^{(2)}$ are $(2M+1) \times (2M+1)$ matrices whose elements were defined previously. The solution of the original complex scattering problem is reduced to solving the matrix equation given by (63). Once the unknown coefficients are obtained, they can be substituted in (29) and (34) to find the coefficients of upward traveling waves in region 0 and downward traveling waves in region 1.

From (29) we have

$$\begin{aligned} b_n &= \sum_{m=-\infty}^{+\infty} \alpha_m \left\{ \frac{k_{0\rho}^2 - k_{nx} k_{mx}}{2L k_{0nz}^2} I_0^- \right\} - \beta_m \left\{ \frac{i}{2L k_{0nz}} \left(\frac{k_{0\rho}}{k_{1\rho}} \right)^2 \frac{k_1 Y_1}{k_0 Y_0} I_0^- \right\} \\ &+ \gamma_m \left\{ \frac{k_{mx}}{2L k_{0nz}} \left(1 - \left(\frac{k_{0\rho}}{k_{1\rho}} \right)^2 \right) \frac{k_{yi}}{k_0 Y_0} I_0^- \right\} \end{aligned} \tag{64}$$

where

$$I_0^- = \int_{x_o}^{x_o+L} e^{i\left[2\pi(m-n)\frac{x'}{L} - k_{0nz} f(x')\right]} dx' \tag{65}$$

The coefficient of downward traveling wave (in region 1) is given by (34). After substituting the coefficients of tangential fields

$$A_n = \sum_{m=-\infty}^{+\infty} \beta_m \left\{ \frac{+i}{2Lk_{1nz}} I_1^+ \right\} - \alpha_m \left\{ \frac{k_{nx}k_{mx} - k_{1\rho}^2}{2Lk_{1nz}^2} I_1^+ \right\} \tag{66}$$

where

$$I_1^+ = \int_{x_o}^{x_o+L} e^{i\left[2\pi(m-n)\frac{x'}{L} + k_{1nz} f(x')\right]} dx' \tag{67}$$

The coefficient of upward and downward traveling magnetic fields can be obtained from (64) and (66) using duality.

$I_j^\pm$ given by (65) and (67) can be evaluated analytically for a sinusoidally varying height profile. Suppose the periodic height profile is

$$f(x) = +F \cos\left[\frac{2\pi}{L}(x - x_o)\right]$$

Then using the identity

$$\int_o^{2\pi} e^{im\phi} e^{\pm iv\cos\phi} d\phi = 2\pi(\pm i)^{|m|} J_{|m|}(v)$$

we find

$$I_j^\pm = L(\pm i)^{|m-n|} J_{|m-n|}(Fk_{jnz}) e^{i\frac{2\pi}{L}x_o}$$