

Statistics of Scattered Fields From Random Media

From electromagnetic scattering point of view a random medium is referred to an inhomogeneous medium where the permittivity and/or permeability of the medium are stochastic processes functions of the spatial and/or temporal variables. The scattered fields from such medium when illuminated by a monochromatic electromagnetic wave are random variables whose amplitude, phase, and polarization state fluctuates according to the statistics of the random medium and the attributes of the incident wave such as the frequency, polarization, and the incidence angle. For example the temporal response of a Doppler weather radar is a random process function of rain drop size, density, canting angle distribution of rain drops (orientation angle of the almost ellipsoidal particles) and the wind speed. Synthetic aperture radars record the spatial fluctuations of the backscatterer in the form of a radar image from which the radar backscatterer statistics can be estimated. Usually radar scenes are composed of many different types of statistically homogeneous distributed targets such as patches of bare surfaces, short vegetation covered surfaces, forested areas, urban areas, etc. For monochromatic radars the pixel-to-pixel backscatter fluctuations can be as high as 20-30 dB. This large fluctuation, which is known as radar fading, is a result of constructive and destructive interference of backscatter from many scattering points within each pixel.

The major goal in radar remote sensing is the extraction of the physical parameters of the targets from the measured radar backscatter. This process is known as the inverse problem. Because of the strong radar backscatter fluctuations, the inversion process cannot be attempted on a pixel-by-pixel basis. Instead the inversion process must be applied to the estimates of the statistical parameters of the radar backscatter response. To accomplish this task successfully the knowledge of the statistical properties of the radar backscatter is needed. Without going through detailed examinations of scattering processes within a random medium, in this chapter the statistical properties of scattered fields random media when illuminated by a monochromatic incident wave is discussed in a general sense.

1 Global Coordinate System and Scattering Matrix

For the theoretical analysis of scattering from random media and rough surfaces, it is often required to express the field quantities with respect to a coordinate system. Consider a Cartesian system shown in Fig. 1 where the x-y plane represent a horizontal

plane and the z-axis is along the vertical direction. In general, targets of interest in radar remote sensing can be categorized into two groups: 1) point targets and 2) distributed targets. Point targets usually refer to targets whose physical extent is much smaller than the footprint of the radar whereas a distributed target is a target whose physical extent is much larger than the antenna footprint. Let us consider a point target at the origin of the global coordinate system illuminated by a plane wave propagating along a unit vector \hat{k}_i whose electric field is given by

$$\overline{E}^i(\vec{r}) = \overline{E}_0 e^{ik_0 \hat{k}_i \cdot \vec{r}} \quad (1)$$

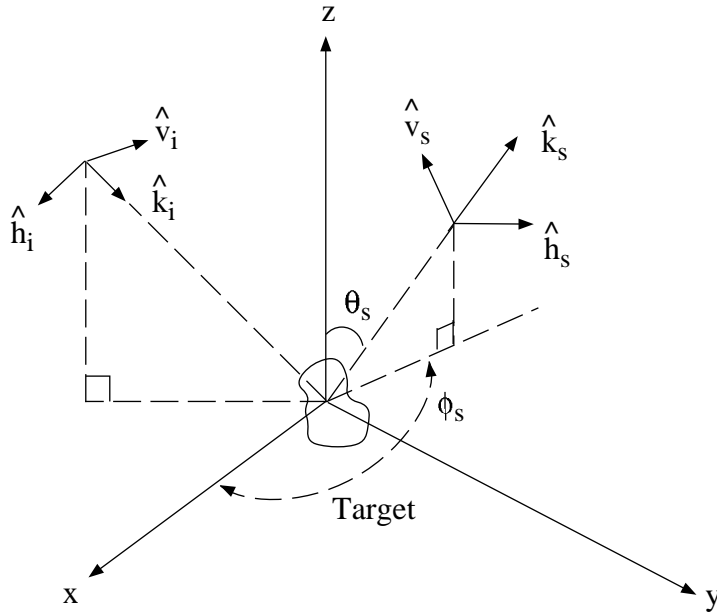


Figure 1: Incident and scattered fields principal polarizations in a global coordinate system.

Since $\overline{E}_0 \cdot \hat{k}_i = 0$, components of the incident field intensity may be expressed in terms of two perpendicular unit vectors in a plane perpendicular to the direction of propagation. We define a horizontal unit vector (parallel to the x-y plane) perpendicular to \hat{k}_i , as,

$$\hat{h}_i = \frac{\hat{k}_i \times \hat{z}}{|\hat{k}_i \times \hat{z}|} \quad (2)$$

hence the other unit vector (known as the vertical vector) which is perpendicular to both \hat{k}_i and \hat{h}_i can be defined by

$$\hat{v}_i = \hat{h}_i \times \hat{k}_i \quad (3)$$

Expressing \hat{k}_i in terms of the spherical coordinate angles θ_i, ϕ_i we have

$$\hat{k}_i = \sin \theta_i \cos \phi_i \hat{x} + \sin \theta_i \sin \phi_i \hat{y} + \cos \theta_i \hat{z} \quad (4)$$

Using (2) and (3)

$$\hat{h}_i = \sin \phi_i \hat{x} - \cos \phi_i \hat{y} \quad (5)$$

$$\hat{v}_i = -\cos \theta_i \cos \phi_i \hat{x} - \cos \theta_i \sin \phi_i \hat{y} + \sin \theta_i \hat{z} \quad (6)$$

The three unit vectors $\hat{v}_i, \hat{h}_i, \hat{k}_i$ form a coordinate system which will be referred to as incident coordinate system. In this coordinate system

$$\overline{E}_o = E_v^i \hat{v}_i + E_h^i \hat{h}_i \quad (7)$$

Suppose the observation point is in the far-field region of the scatterer at $\vec{r} = r \hat{k}_s$ where \hat{k}_s is unit vector defined by

$$\hat{k}_s = \sin \theta_s \cos \phi_s \hat{x} + \sin \theta_s \sin \phi_s \hat{y} + \cos \theta_s \hat{z} \quad .$$

The scattered field \overline{E}^s in the far-field region is a spherical wave and has no component along \hat{k}_s . A scattered coordinate system similar to the incident coordinate system can be defined where

$$\overline{E}^s = E_v^s \hat{v}_s + E_h^s \hat{h}_s \quad , \quad (8)$$

with

$$\hat{v}_s = -\cos \theta_s \cos \phi_s \hat{x} - \cos \theta_s \sin \phi_s \hat{y} + \sin \theta_s \hat{z} \quad , \quad (9)$$

and

$$\hat{h}_s = \sin \phi_s \hat{x} - \cos \phi_s \hat{y} \quad . \quad (10)$$

Due to the linearity of Maxwell's equations the components of the scattered field intensity (E_v^s and E_h^s) are linearly related to the components of the incident wave intensity (E_v^i and E_h^i). Apart from the spherical phase and amplitude factors ($\frac{e^{ikr}}{r}$), the linear relations can be expressed in terms of a matrix known as the scattering matrix. That is,

$$\overline{E}^s(\vec{r}) = \frac{e^{ik_o r}}{r} \overline{S} \overline{E}_o \quad (11)$$

where \bar{S} is a 2×2 complex matrix given by

$$\bar{S} = \begin{bmatrix} S_{vv} & S_{vh} \\ S_{hv} & S_{hh} \end{bmatrix}. \quad (12)$$

It should be noted here that, in this definition the reference phase point coincides with the scattering phase center of the scatterer. The 2×2 complex scattering matrix given by (12) was first defined by R. Clark Jones¹ and sometimes, in literature, is referred to as the Jones matrix.

The scattering matrix elements are complex quantities since the vector representation of the incident and scattered waves given, respectively, by (7) and (8) are complex. Apart from a normalization factor, the vector amplitude of a TEM wave uniquely specifies the polarization state of the wave defined by

$$\hat{p}_i = \frac{\bar{E}_0}{\sqrt{\bar{E}_0 \cdot \bar{E}_0^*}} \quad (13)$$

where \hat{p} is a complex unit vector. The normalizing factor in (13) is proportional to the power density carried by the incident wave.

Consider a TEM wave with a complex amplitude vector $\bar{E} = e_v \hat{v} + e_h \hat{h}$. The polarization of this wave is defined as the trace of the tip of the electric field at a given point in space in the plane perpendicular to the direction of propagation. It can easily be shown that this trace is in general an ellipse with certain tilt angle and axial ratio (AR). In phasor notation defining the ratio of

$$\frac{e_h}{e_v} = ae^{i\delta} \quad (14)$$

where both a and δ are real quantities, the time domain behavior of the electric field may be expressed by

$$\begin{aligned} \bar{E}(\bar{r}, t) &= e_v \cos(\omega t - \bar{k} \cdot \bar{r}) \hat{v} + e_v a \cos(\omega t - \bar{k} \cdot \bar{r} + \delta) \hat{h} \\ &= E_v(t, \bar{r}) \hat{v} + E_h(t, \bar{r}) \hat{h}. \end{aligned} \quad (15)$$

To specify the polarization, the temporal behavior of $\bar{E}(\bar{r}, t)$ at a fixed point $\bar{r} = \bar{r}_0$ in space must be determined. This can be done by eliminating t from the expressions for E_v and E_h . After some simple algebra it can be shown that ²

¹R.C. Jones, "A new calculus for the treatment of optical systems I: Description and discussion of the calculus," *J. Optical Society of America*, Vol. 31, pp. 488-493, 1941.

²J.D. Kraus, Electromagnetics, pp. 599, Fourth Edition, New York: McGraw-Hill, 1992.

$$\left(\frac{E_v}{e_v}\right)^2 - \frac{2E_v E_h}{ae_v^2} \cos \delta + \left(\frac{E_h}{ae_v}\right)^2 = \sin^2 \delta \quad (16)$$

Equation (3) specifies an ellipse in the $v-h$ plane as shown in Figure 2.

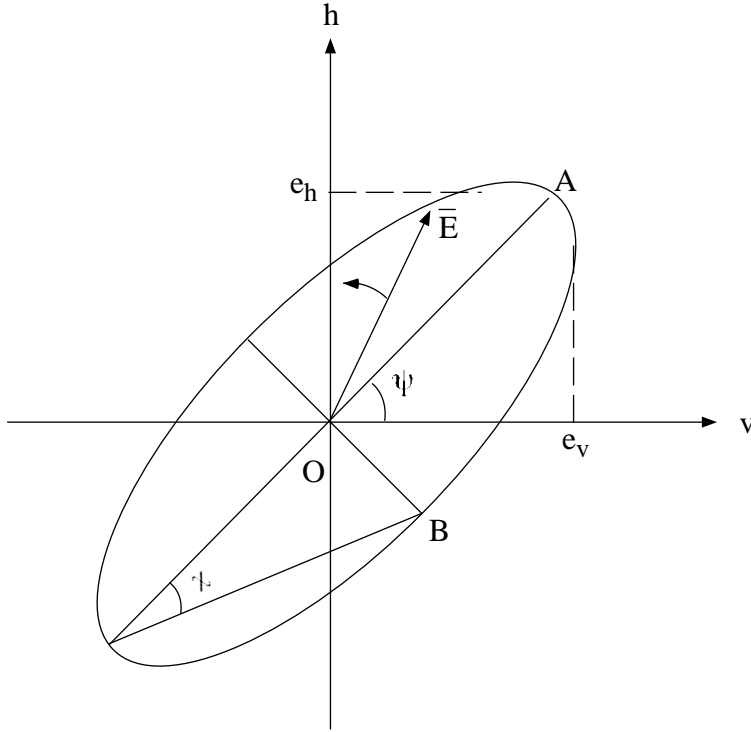


Figure 2: Polarization ellipse in $v-h$ plane specifying the trace of \overline{E} as a function of time at a given point in space.

The polarization ellipse (a part from a normalization factor) is uniquely specified by two angles χ and ψ where χ is known as the ellipticity angle given by

$$\chi = \tan^{-1} \frac{OB}{OA} = \tan^{-1} \frac{1}{AR}, \quad |\chi| \leq \frac{\pi}{4}$$

and ψ is known as the tilt angle ($-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$). The direction of rotation of the electric field as a function of time specifies the handedness of the polarization. For a right-hand elliptically polarized wave, where the fingers of the right hand follow the direction of rotation of \overline{E} , the thumb points towards the direction of propagation of the wave. The definition for a left-hand polarized wave is obvious. This definition for handedness is an IEEE standard and is opposite of what is used in the field of optics. To keep track of polarization handedness mathematically, the ellipticity angle χ is given a negative sign for right-handed and a positive sign for a left-handed polarization.

The tilt and orientation angles in terms of a and δ are given by

$$\begin{aligned}\tan(2\psi) &= \tan 2\gamma \cos \delta \\ \sin(2\chi) &= \sin 2\gamma \sin \delta\end{aligned}\tag{17}$$

where

$$\gamma = \tan^{-1} a$$

conversely knowing ψ and χ , one can determine a and δ from

$$\begin{aligned}\delta &= \tan^{-1} \left(\frac{\tan 2\chi}{\tan 2\psi} \right) \\ a &= \tan \left[\frac{1}{2} \cos^{-1}(\cos 2\chi \cos 2\psi) \right].\end{aligned}\tag{18}$$

Determination of δ from (18) is not unique and often leads to confusion. The following constraints remove the ambiguity in determining δ :

$$\begin{aligned}\text{if } \chi \leq 0 &\Rightarrow \delta \leq 0 \\ \text{if } \chi \geq 0 &\Rightarrow \delta \geq 0 \\ \text{if } \psi > 0 &\Rightarrow |\delta| < \frac{\pi}{2} \\ \text{if } \psi < 0 &\Rightarrow |\delta| > \frac{\pi}{2}\end{aligned}$$

As can be seen from Fig. 2, when the ellipticity angle $\chi = 0$, the polarization ellipse reduces to a line with orientation angle ψ . Equation (18) shows that when $\chi = +45^\circ$, δ is $\frac{\pi}{2}$ and the polarization of the wave is left-hand circular. Since the polarization of a wave is characterized by two angles, there is a one-to-one relationship between a point on the surface of a sphere and the polarization of the wave. Let us consider a sphere with the radius of $I_0 = |E|^2$ in a Q, U, V Cartesian coordinate system as shown in Fig. 3. This sphere is called Poincaré sphere. The Cartesian components of a point on the Poincaré sphere are components of a vector known as the Stokes vector defined by

$$F = \begin{bmatrix} I_0 \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I_0 \\ I_0 \cos 2\psi \cos 2\chi \\ I_0 \sin 2\psi \sin 2\chi \\ I_0 \sin 2\chi \end{bmatrix} = \begin{bmatrix} |e_v|^2 + |e_h|^2 \\ |e_v|^2 - |e_h|^2 \\ 2\Re[e_v e_h^*] \\ 2\Im[e_v e_h^*] \end{bmatrix}\tag{19}$$

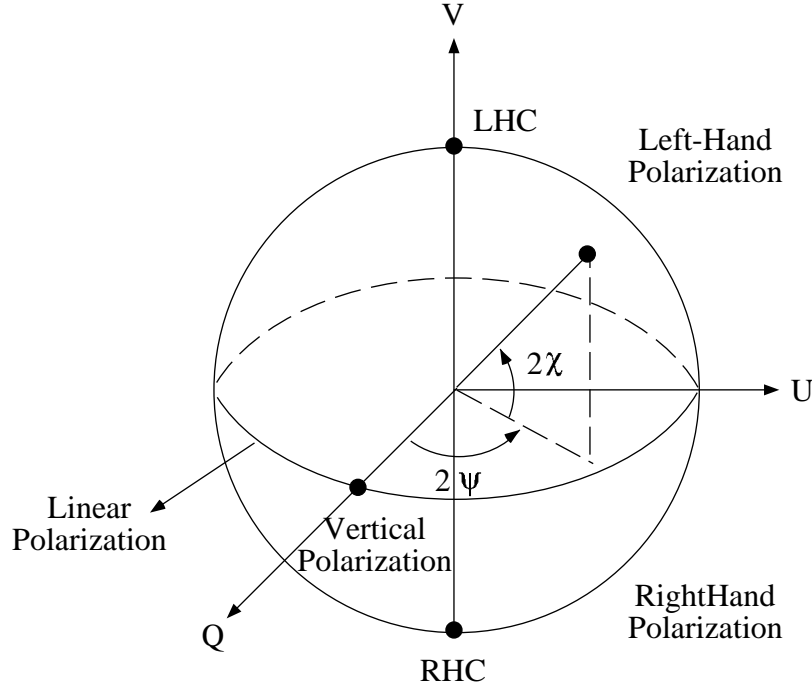


Figure 3: Poincaré sphere representing the polarization of a TEM wave with ellipticity angle χ and tilt angle ψ .

2 Radar Cross Section and Backscattering Coefficient

For a radar engineer the radar cross section (RCS) of the target which is proportional to the scattered power is of interest. The radar cross section of a target is defined as a fictitious area that can intercept the incident power density and reradiates the intercepted power isotropically into the surrounding space which produces the same power density at the receiver as the original target produces. Of course this fictitious area is a function of the target aspect angle with respect to the incoming wave and the direction of the observation point. Mathematically the radar cross section is defined as

$$\sigma_{pq}(\hat{k}_s, \hat{k}_i) = \lim_{r \rightarrow \infty} 4\pi r^2 \frac{|\hat{p} \cdot \overline{E}^s|^2}{|\hat{q} \cdot \overline{E}_o|^2} \quad (20)$$

where \hat{p} and \hat{q} are the complex polarization vectors of the receiver and transmitter. In terms of the elements of the scattering matrix and for the principal polarization configurations v and h the radar cross section of the target, σ_{mn} , is given by

$$\sigma_{mn} = 4\pi |S_{mn}|^2 \quad m, n = v, h \quad (21)$$

Interest in the notion of the scattering matrix stems from the fact that the RCS of the target for any arbitrary transmit and receive polarization configuration can be evaluated once the scattering matrix of the target is known.

This can be shown by inserting (12) in (20) using a matrix notation for the polarization vectors

$$\sigma_{pq}(\hat{k}_s, \hat{k}_i) = 4\pi |\hat{p}^t \bar{S} \hat{q}|^2 \quad (22)$$

where \hat{p}^t is a 1×2 matrix (superscript t indicates transpose operation) and \hat{q} is a 2×1 matrix given by

$$\begin{aligned} \hat{p}^t &= [p_v, p_h] \\ \hat{q} &= \begin{bmatrix} q_v \\ q_h \end{bmatrix} \end{aligned}$$

Representing the 2×2 scattering matrix in terms of a 4×1 matrix of the form

$$\bar{X} = \begin{pmatrix} S_{vv} \\ S_{vh} \\ S_{hv} \\ S_{hh} \end{pmatrix}$$

and defining radar polarization vector

$$\bar{U} = \begin{pmatrix} p_v q_v \\ p_v q_h \\ p_h q_v \\ p_h q_h \end{pmatrix}$$

(22) may be written as

$$\begin{aligned} \sigma_{pq}(\hat{k}_s, \hat{k}_i) &= 4\pi |\bar{U}^t \cdot \bar{X}|^2 \\ &= 4\pi (\bar{U}^t \cdot \bar{X})(\bar{U}^t \cdot \bar{X})^{*t} \\ &= 4\pi \bar{U}^t (\bar{X} \cdot \bar{X}^{*t}) \bar{U}^* \end{aligned} \quad (23)$$

where $(*)$ denotes complex conjugate operation. The 4×4 hermitian matrix $\bar{C} = \bar{X} \cdot \bar{X}^{*t}$ is known as the covariance matrix of the target which is given by

$$\bar{C} = \begin{pmatrix} S_{vv}S_{vv}^* & S_{vv}S_{vh}^* & S_{vv}S_{hv}^* & S_{vv}S_{hh}^* \\ S_{vh}S_{vv}^* & S_{vh}S_{vh}^* & S_{vh}S_{hv}^* & S_{vh}S_{hh}^* \\ S_{hv}S_{vv}^* & S_{hv}S_{vh}^* & S_{hv}S_{hv}^* & S_{hv}S_{hh}^* \\ S_{hh}S_{vv}^* & S_{hh}S_{vh}^* & S_{hh}S_{hv}^* & S_{hh}S_{hh}^* \end{pmatrix}. \quad (24)$$

Hence, either (22) or equivalent (23) can be used to find the RCS of a target to an arbitrary set of transmit and receive radar polarization once the target scattering matrix is known.

Distributed targets on the other hand are usually stochastic in nature. Basically, as the footprint of the radar is moved over the distributed target the scattered field fluctuates. As mentioned before, this fluctuation is caused by the random location, size, orientation, permittivity and permeability, and the number of scatterers within the illuminated area of the distributed scatterer. In this case the statistical parameters of the scattered field instead of RCS or the scattering matrix is sought. To make the statistical parameters of the scattered fields independent of the illumination area, normalized quantities such as average scattered power per unit area of the distributed target are evaluated or measured. A common quantity for characterizing scattering behavior of distributed targets is the scattering coefficient which is defined by

$$\sigma_{pq}^o(\hat{k}_s, \hat{k}_i) = \lim_{r \rightarrow \infty} \lim_{A \rightarrow \infty} \frac{4\pi r^2}{A} \frac{\langle |\hat{p} \cdot \bar{E}_s|^2 \rangle}{|\hat{q} \cdot \bar{E}_i|^2} \quad (25)$$

which is basically proportional to the second moment of the scattered field (average power). In (25) A is the illuminated area and hence the unit of the scattering coefficient is m^2/m^2 . Using the definition of the scattering matrix in (25), the backscattering coefficient for a pair of transmit and receive polarizations \hat{q} and \hat{p} can be obtained from

$$\sigma_{pq}^0 = \frac{4\pi}{A} \bar{U}^t \langle \bar{C} \rangle \bar{U}^* \quad (26)$$

In the computation of scattering coefficient for arbitrary polarizations \hat{p}, \hat{q} equation (26) is usually used, where a coherent polarimetric system is available for the measurement of the complex scattering matrix of the random media. Coherent polarimetric systems are capable of measuring the in-phase and phase quadrature components of the backscatter signal in all four basic transmit-receive radar polarization configurations. For an accurate scattering matrix measurement both the radar platform and the target must remain stationary (to within a small fraction of the wavelength) during all four measurements. At millimeter-wave frequencies and higher, it becomes practically very difficult to measure the scattering matrices of distributed targets directly. In this case the polarimetric

response of the random medium is obtained through the application of the modified Stokes vector and Mueller matrix.³

The modified Stokes vector is a real 4×1 vector that also describes the polarization state of a monochromatic TEM wave and is given by:

$$\bar{F}_m = \begin{bmatrix} |E_v|^2 \\ |E_h|^2 \\ 2R_e[E_v E_h^*] \\ 2\Im[E_v E_h^*] \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \cos 2\psi \cos 2\chi) \\ \frac{1}{2}(1 - \cos 2\psi \cos 2\chi) \\ \sin 2\psi \cos 2\chi \\ \sin 2\chi \end{bmatrix} I_o \quad (27)$$

where E_v and E_h are the principal components of the incident or scattered waves given by (7) or (8) and $I_o = |E_v|^2 + |E_h|^2$. As before ψ and χ are the tilt and ellipticity angles of the polarization ellipse.

The modified Stokes vector can be expressed in term of cross product of the field component vector \bar{G} using

$$\bar{F}_m = \bar{\nu} \bar{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \begin{pmatrix} E_v E_v^* \\ E_h E_h^* \\ E_v E_h^* \\ E_h E_v^* \end{pmatrix} \quad (28)$$

Using (11) it can easily be shown that

$$\bar{G}^s = \frac{1}{r^2} \bar{\Gamma} \bar{G}^i = \frac{1}{r^2} \begin{pmatrix} S_{vv} S_{vv}^* & S_{vh} S_{vh}^* & S_{vv} S_{vh}^* & S_{vh} S_{vv}^* \\ S_{hv} S_{hv}^* & S_{hh} S_{hh}^* & S_{hv} S_{hh}^* & S_{hh} S_{hv}^* \\ S_{vv} S_{hv}^* & S_{vh} S_{hh}^* & S_{vv} S_{hh}^* & S_{vh} S_{hv}^* \\ S_{hv} S_{vv}^* & S_{hh} S_{vh}^* & S_{hv} S_{vh}^* & S_{hh} S_{vv}^* \end{pmatrix} \bar{G}^i \quad (29)$$

Hence the scattered Stokes vector may be related to the incident Stokes vector via

$$\bar{F}_m^s = \frac{1}{r^2} \bar{M} \bar{F}_m^i$$

where

$$\bar{M} = \frac{1}{r^2} \bar{\nu} \bar{\Gamma} \bar{\nu}^{-1} \quad (30)$$

is the modified Mueller matrix. Explicitly, \bar{M} in terms of elements of \bar{S} is given by

³J. Jvan Zyl and F.T. Ulaby, "Scattering matrix representation for simple targets," in *Radar Polarimetry for Geoscience Applications*, Norwood: Artech House, Editors Ulaby and Elachi, 1990.

$$\bar{\bar{M}} = \begin{pmatrix} |S_{vv}|^2 & |S_{vh}|^2 & R_e[S_{vv}S_{vh}^*] & -\Im[S_{vv}S_{vh}^*] \\ |S_{hv}|^2 & |S_{hh}|^2 & R_e[S_{hv}S_{hh}^*] & -\Im[S_{hv}S_{hh}^*] \\ 2R_e[S_{vv}S_{hv}^*] & 2R_e[S_{vh}S_{hh}^*] & R_e[S_{vv}S_{hh}^* + S_{vh}S_{hv}^*] & -\Im[S_{vv}S_{hh}^* - S_{vh}S_{hv}^*] \\ 2\Im[S_{vv}S_{hv}^*] & 2\Im[S_{vh}S_{hh}^*] & \Im[S_{vv}S_{hh}^* + S_{vh}S_{hv}^*] & R_e[S_{vv}S_{hh}^* - S_{vh}S_{hv}^*] \end{pmatrix}. \quad (31)$$

Note that elements of $\bar{\bar{M}}$ are all real quantities.

To express RCS and backscattering coefficient of targets in terms of the Mueller matrix we note that⁴

$$\begin{aligned} \sigma_{pq} &= 4\pi |\hat{p}^t \bar{\bar{S}} \hat{q}|^2 \\ &= 4\pi \bar{g}_s^t \bar{\bar{\Gamma}} \bar{g}_i \end{aligned} \quad (32)$$

where $\bar{g}_s^t = (p_v p_v^*, p_h p_h^*, p_v p_h^*, p_h p_v^*)$ and

$$\bar{g}_i = \begin{pmatrix} q_v q_v^* \\ q_h q_h^* \\ q_v q_h^* \\ q_h q_v^* \end{pmatrix}.$$

But since $\bar{\bar{\Gamma}} = \bar{\bar{\nu}}^{-1} \bar{\bar{M}} \bar{\bar{\nu}}$

$$\sigma_{pq} = 4\pi \bar{\alpha}_s^t \bar{\bar{M}} \bar{\alpha}_i$$

where $\bar{\alpha}_s = (\bar{\bar{\nu}}^{-1})^t \bar{g}_s$ and $\bar{\alpha}_i = \bar{\bar{\nu}} \bar{g}_i$. Following a similar procedure the backscattering coefficient of a distributed target for an arbitrary set transmit and receive polarization \hat{q} and \hat{p} can be obtained from

$$\sigma_{pq}^\circ = \frac{4\pi}{A} \bar{\alpha}_s^t \langle \bar{\bar{M}} \rangle \bar{\alpha}_i.$$

The Mueller matrix for a distributed target can be obtained from measuring the response of a target for at least sixteen different transmit and receive polarization configurations. Basically at each polarization configuration many independent samples of the target are measured to evaluate the ensemble averages and then from the response of the target for different polarizations, $\langle \bar{\bar{M}} \rangle$ is evaluated.

⁴E.M. Kennaugh, "Effects of the type of polarization on echo characteristics," Report 389-9, Antenna Laboratory, Ohio State University, 1951.

3 Statistics of Polarimetric Response of Distributed Targets

In the past decade substantial effort within the microwave remote sensing community has been devoted to the development and improvement of polarimetry science. As mentioned in the previous section, polarimetric radars are capable of synthesizing the radar response of a target to any combination of the receive and transmit polarizations from either a coherent or incoherent measurements of the scattered field. Polarimetric radars have demonstrated their abilities in improving point-target detection and classification in a clutter background [Ioannidis and Hammers, 1979]. That is, for a point target with known scattering response, the transmit and receive polarizations can be chosen such that the target to clutter response is maximum. Also, different point targets in the radar scene can be classified according to their optimum polarization. To fully utilize the potentials of radar polarimeters in target detection and classification, the statistical properties of polarimetric response of distributed targets must be understood completely.

Experimental observations of phase difference statistics from a polarimetric SAR at L-band [Ulaby et al., 1987; Zebker et al., 1987] over agricultural terrain and bare soil surfaces indicate that the statistics of the co-polarized phase difference depends on the target type and its conditions. Recent measurements of bare soil surfaces by polarimetric scatterometers show that the variance of the co-polarized phase difference is a function of the roughness parameters and incidence angle but is less sensitive to moisture content [Sarabandi et al., 1991]. Apart from the magnitude of the scattering matrix element ($|S_{vv}|^2$, $|S_{vh}|^2$, ..., $|S_{hh}|^2$), polarimetric radar also provide the phase of the scattered fields. Considering the complexity involved in designing, manufacturing, and processing the data of an imaging polarimeter as opposed to a conventional imaging radar, it is necessary to examine the advantages that the imaging polarimeter provides about the targets of interest. For example, in retrieving the target biophysical parameters from the radar data one should ask whether there exists a relatively strong relationship between the target parameters and the measured phase of the scattering matrix components. If the answer is negative, obviously gathering polarimetric data for inversion of that parameter is a waste of effort. One way of confirming whether the phase difference of scattering matrix elements are sensitive to target parameters is by conducting controlled experiments and establishing the relationships empirically. This procedure is very difficult because of problems in repeatability of the experiment and difficulties in controlling the environmental conditions.

Another approach to examine the dependence of the radar response to the desired parameters of the targets is the application of theoretical models. For example, vector radiative transfer theory is a method commonly used for predicting the polarimetric response of random media [Tsang et al., 1985]. The solution of the radiative transfer equation relates the scattered-wave Stokes vector to the incident-wave Stokes vector *via* the Mueller matrix. Since the Mueller matrix is related to the scattering matrix,

through a nonlinear process, the statistical information about the phase difference of the scattering matrix elements are embedded in the Mueller matrix. In view of difficulties in measuring the scattering matrix at high frequencies and performing controlled experiments, it is necessary to establish a relationship between the Mueller matrix and the statistics of the phase differences of the scattering matrix elements. In the next section a procedure for the derivation of the probability density function of the co- and cross-polarized phase difference in terms of the Mueller matrix elements is provided. Here it is assumed that the scattering matrix elements are jointly Gaussian. Then the assumptions and final results are compared with the experimental data acquired by polarimetric scatterometers in Section 5.

4 Derivation of Phase Difference Statistics

The polarimetric response of a point or distributed target can be obtained by simultaneously measuring both the amplitude and phase of the scattered field using two orthogonal channels. When the radar illuminates a volume of a random medium or an area of a random surface, many point scatterers contribute to the total scattered energy received by the radar and therefore each element of the scattering matrix may be represented by

$$S_{pq} = |S_{pq}|e^{i\phi_{pq}} = \sum_{n=1}^N |s_{pq}^n|e^{i\phi_{pq}^n} \quad p, q = v, h \quad . \quad (33)$$

Here N is the total number of scatterers each having scattering amplitude $|s_{pq}^n|$ and phase ϕ_{pq}^n . It should be mentioned that the phase of each scatterer, as given in (33), includes a phase delay according to the location of the scatterer with respect to the center of the distributed target. Without loss of generality all multiple scattering over the surface or in the medium can be included in (33). Since the location of the scatterers within the illuminated area (volume) is random and if the differential path delays among scatterers are large compared with the wavelength, the process describing the phasor s_{pq} is a Wiener process (random walk) [Davenport, 1970]. If N is large enough, application of the central limit theorem shows that the real and imaginary parts of the scattering matrix element S_{pq} are independent, identically distributed, zero-mean Gaussian random variables. Equivalently it can also be shown that $|S_{pq}|$ and ϕ_{pq} are, respectively, Rayleigh and uniform independent random variables. The three elements of the scattering matrix, in general, can be viewed as a six-element random vector and it is again reasonable to assume that the six components are jointly Gaussian.

Observation of polarimetric data for a variety of distributed targets such as bare soil surfaces and different kinds of vegetation-covered terrain all indicate that the cross-polarized component of the scattering matrix (S_{hv}) is statistically independent of the co-polarized terms (S_{vv} and S_{hh}). Therefore the statistical behavior of S_{hv} can be obtained from a single parameter, namely the variance (σ_c^2) of the real or imaginary part of

$S_{hv} = X_5 + iX_6$, that is

$$f_{X_5, X_6}(x_5, x_6) = \frac{1}{2\pi\sigma_c^2} \exp\left[-\frac{x_5^2 + x_6^2}{2\sigma_c^2}\right]$$

or equivalently the joint density function $|S_{vh}|$ and ϕ_{vh} is

$$f_{|S_{vh}|, \phi_{vh}}(|s_{vh}|, \phi_{vh}) = \frac{1}{2\pi\sigma_c^2} |s_{vh}| \exp\left[-\frac{|s_{vh}|^2}{2\sigma_c^2}\right] \quad , \quad (34)$$

which indicates that ϕ_{vh} is uniformly distributed between $(-\pi, +\pi)$.

Since measurement of the absolute phase of the scattering matrix elements is very difficult, it is customary to factor out the phase of one of the co-polarized terms, for example S_{vv} , and therefore the phase *difference* statistics are of concern as opposed to the *absolute* phases. Since S_{hv} is assumed to be independent of S_{vv} (not a necessary assumption) and both ϕ_{hv} and ϕ_{vv} are uniformly distributed, it can be easily shown that the cross-polarized phase difference $\phi_c = \phi_{vh} - \phi_{vv}$ is also uniformly distributed between $(-\pi, +\pi)$.

The co-polarized elements of the scattering matrix, however, are dependent random variables which can be denoted by a four-component jointly Gaussian random vector \mathbf{X} . Let us define

$$S_{vv} = X_1 + iX_2 \quad , \quad S_{hh} = X_3 + iX_4$$

and since X_1, \dots, X_4 are Gaussian their joint probability density function can be fully determined by a 4×4 symmetric positive definite matrix known as covariance matrix (Λ) whose entries are given by [2]

$$\lambda_{ij} = \lambda_{ji} = \langle X_i X_j \rangle \quad , \quad i, j \in \{1, \dots, 4\} \quad .$$

The joint probability density function in terms of the covariance matrix takes the following form:

$$f_{\mathbf{X}}(x_1, \dots, x_4) = \frac{1}{4\pi^2 |\Lambda|^{\frac{1}{2}}} \exp\left[-\frac{1}{2} \tilde{\mathbf{X}} \Lambda^{-1} \mathbf{X}\right] \quad , \quad (35)$$

where $\tilde{\mathbf{X}}$ is transpose of the column vector \mathbf{X} . To characterize the covariance matrix the following observations are in order. First, it was shown that the real and imaginary parts of the scattering matrix elements are mutually independent and identically distributed zero-mean random variables, therefore

$$\lambda_{11} = \lambda_{22} = \langle X_1^2 \rangle = \langle X_2^2 \rangle \quad , \quad (36)$$

$$\lambda_{12} = \langle X_1 X_2 \rangle = 0 \quad , \quad (37)$$

$$\lambda_{33} = \lambda_{44} = \langle X_3^2 \rangle = \langle X_4^2 \rangle \quad , \quad (38)$$

$$\lambda_{34} = \langle X_3 X_4 \rangle = 0 \quad . \quad (39)$$

Second, it was shown that the absolute phase ϕ_{pp} is uniformly distributed and is independent of $|S_{pp}|$. Thus the random variable $\phi_{vv} + \phi_{hh}$ is also uniformly distributed and is independent of $|S_{vv}||S_{hh}|$ from which it can be concluded that

$$\begin{aligned} \langle |S_{vv}||S_{hh}| \cos(\phi_{vv} + \phi_{hh}) \rangle &= 0 \quad , \\ \langle |S_{vv}||S_{hh}| \sin(\phi_{vv} + \phi_{hh}) \rangle &= 0 \quad . \end{aligned} \quad (40)$$

In fact, the complex random variable $S_{vv}S_{hh}$ is obtained from a similar Wiener process which led to the random variables S_{vv} and S_{hh} . On the other hand

$$\begin{aligned} X_1X_3 - X_2X_4 &= |S_{vv}||S_{hh}| \cos(\phi_{vv} + \phi_{hh}) \quad , \\ X_1X_4 + X_2X_3 &= |S_{vv}||S_{hh}| \sin(\phi_{vv} + \phi_{hh}) \quad . \end{aligned} \quad (41)$$

In view of (40) and (41) it can easily be seen that

$$\lambda_{13} = \lambda_{24} \quad , \quad (42)$$

$$\lambda_{14} = -\lambda_{23} \quad . \quad (43)$$

The properties derived for the entries of the covariance matrix, as given by (36)-(39) and (42)-(43), indicates that there are only four unknowns left in the covariance matrix. The unknowns, namely λ_{11} , λ_{13} , λ_{14} , and λ_{33} , can be obtained directly from the Mueller matrix of the target as will be shown next.

In the case of a random medium we are dealing with a partially polarized scattered wave and the quantity of interest is the ensemble averaged Mueller matrix. Using the assumption that the co- and cross-polarized terms of the scattering matrix are independent (not a necessary assumption at this point) and employing the properties given by (36)-(39) and (42)-(43), the Mueller matrix in terms of the entries of the covariance matrix is given by

$$\mathcal{M} = \langle \mathbf{M} \rangle = \begin{bmatrix} 2\lambda_{11} & 2\sigma_c^2 & 0 & 0 \\ 2\sigma_c^2 & 2\lambda_{33} & 0 & 0 \\ 0 & 0 & 2\lambda_{13} + 2\sigma_c^2 & 2\lambda_{14} \\ 0 & 0 & -2\lambda_{14} & 2\lambda_{13} - 2\sigma_c^2 \end{bmatrix} . \quad (44)$$

Equation (44) provides enough equations to determine the unknown elements of the covariance matrix and variance of the cross-polarized component, i.e.

$$\begin{aligned} \sigma_c^2 &= \frac{\mathcal{M}_{12}}{2} \quad , \\ \lambda_{11} &= \frac{\mathcal{M}_{11}}{2} \quad , & \lambda_{33} &= \frac{\mathcal{M}_{22}}{2} \\ \lambda_{13} &= \frac{\mathcal{M}_{33} + \mathcal{M}_{44}}{4} \quad , & \lambda_{14} &= \frac{\mathcal{M}_{34} - \mathcal{M}_{43}}{4} \quad . \end{aligned}$$

With the covariance matrix, the joint density function of X_1, \dots, X_4 can be obtained as given by (35). Using a rectangular to polar transformation, i.e.

$$\begin{aligned} x_1 &= \rho_1 \cos \phi_{vv} \quad , & x_2 &= \rho_1 \sin \phi_{vv} \quad , \\ x_3 &= \rho_2 \cos \phi_{hh} \quad , & x_4 &= \rho_2 \sin \phi_{hh} \quad , \end{aligned}$$

the joint probability density function of the amplitudes and phases takes the following form

$$f_{\rho_1, \rho_2, \Phi_{vv}, \Phi_{hh}}(\rho_1, \rho_2, \phi_{vv}, \phi_{hh}) = \frac{1}{4\pi^2\sqrt{\Delta}}\rho_1\rho_2\exp\left\{-\frac{1}{2}[a_1\rho_1^2 + a_2\rho_2^2 - 2a_3\rho_1\rho_2]\right\} \quad , \quad (45)$$

where

$$\begin{aligned} \Delta &= |\Lambda| = (\lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2)^2 \quad , \\ a_1 &= \lambda_{33}/\sqrt{\Delta} \quad , \quad a_2 = \lambda_{11}/\sqrt{\Delta} \quad , \\ a_3 &= [\lambda_{13}\cos(\phi_{hh} - \phi_{vv}) + \lambda_{14}\sin(\phi_{hh} - \phi_{vv})]/\sqrt{\Delta} \quad . \end{aligned}$$

To obtain the co-polarized phase difference statistics the joint density function of ϕ_{vv} and ϕ_{hh} is needed which can be obtained from

$$f_{\Phi_{vv}, \Phi_{hh}}(\phi_{vv}, \phi_{hh}) = \int_0^\infty \int_0^\infty f_{\rho_1, \rho_2, \Phi_{vv}, \Phi_{hh}}(\rho_1, \rho_2, \phi_{vv}, \phi_{hh})d\rho_1d\rho_2 \quad . \quad (46)$$

Noting that a_1 is a positive real number, the integration with respect to ρ_1 can be carried out which results in

$$\begin{aligned} f_{\Phi_{vv}, \Phi_{hh}}(\phi_{vv}, \phi_{hh}) &= \frac{1}{4\pi^2\sqrt{\Delta}} \left\{ \frac{1}{a_1} \int_0^\infty \rho_2 e^{-\frac{a_2}{2}\rho_2^2} d\rho_2 \right. \\ &\quad \left. + \sqrt{\frac{\pi}{8a_1^3}} a_3 \int_0^\infty \rho_2^2 \left[1 \pm \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right) \right] e^{-\frac{1}{2a_1}(a_1a_2 - a_3^2)\rho_2^2} d\rho_2 \right\} \quad , \end{aligned} \quad (47)$$

where $\operatorname{erf}(\cdot)$ is the error function and the plus or minus sign is used according to the sign of a_3 . To evaluate the integrals in (47), we need to show that both a_2 and $a_1a_2 - a_3^2$ are positive numbers. By definition, a_2 is positive and to show $a_1a_2 - a_3^2$ is positive we note that Λ is a symmetric positive definite matrix, therefore its eigenvalues must be positive. It can be shown that Λ has two distinct eigenvalues γ_1 and γ_2 each with multiplicity 2 and their product is given by

$$\gamma_1\gamma_2 = \lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2 > 0.$$

Thus

$$a_1a_2 - a_3^2 = \gamma_1\gamma_2 + [\lambda_{13}\cos(\phi_{hh} - \phi_{vv}) - \lambda_{14}\sin(\phi_{hh} - \phi_{vv})]^2$$

is positive. After integrating the first integral and the first term of the second integral in (47) directly and using integration by parts on the second term of the second integral, (47) becomes

$$\begin{aligned} f_{\Phi_{vv}, \Phi_{hh}}(\phi_{vv}, \phi_{hh}) &= \frac{1}{4\pi^2\sqrt{\Delta}} \left\{ \frac{1}{a_1a_2} + \frac{a_3^2}{a_1a_2(a_1a_2 - a_3^2)} \right. \\ &\quad \left. + \frac{\sqrt{\pi}|a_3|}{\sqrt{8a_1}(a_1a_2 - a_3^2)} \int_0^\infty \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right) e^{-\frac{1}{2a_1}(a_1a_2 - a_3^2)\rho_2^2} d\rho_2 \right\} \quad . \end{aligned}$$

By expanding the error function in terms of its Taylor series, interchanging the order of summation and integration, then using the definition of the Gamma function it can be shown that

$$\int_0^\infty \operatorname{erf}\left(\frac{|a_3|}{\sqrt{8a_1}}\rho_2\right)e^{-\frac{1}{2a_1}(a_1a_2-a_3^2)\rho_2^2}d\rho_2 = \sqrt{\frac{2a_1}{\pi(a_1a_2-a_3^2)}} \tan^{-1}\left(\frac{|a_3|}{2\sqrt{a_1a_2-a_3^2}}\right) .$$

The joint density function of ϕ_{vv} and ϕ_{hh} is a periodic function of $\phi = \phi_{hh} - \phi_{vv}$ and therefore the random variable ϕ , after some algebraic manipulation, can be shown to have the following probability density function over the interval $(-\pi, +\pi)$

$$f_\Phi(\phi) = \frac{\lambda_{11}\lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2}{2\pi(\lambda_{11}\lambda_{33} - D^2)} \left\{ 1 + \frac{D}{\sqrt{\lambda_{11}\lambda_{33} - D^2}} \left[\frac{\pi}{2} + \tan^{-1} \frac{D}{\sqrt{\lambda_{11}\lambda_{33} - D^2}} \right] \right\} , \quad (48)$$

where we recall that

$$D = \lambda_{13} \cos \phi + \lambda_{14} \sin \phi$$

and the elements of the covariance matrix in terms of the Mueller matrix elements are given by

$$\begin{aligned} \lambda_{11} &= \frac{\mathcal{M}_{11}}{\mathcal{M}_{33}^2 + \mathcal{M}_{44}} , & \lambda_{33} &= \frac{\mathcal{M}_{22}}{\mathcal{M}_{34}^2 - \mathcal{M}_{43}} , \\ \lambda_{13} &= \frac{\mathcal{M}_{33} + \mathcal{M}_{44}}{4} , & \lambda_{14} &= \frac{\mathcal{M}_{34} - \mathcal{M}_{43}}{4} . \end{aligned}$$

Some limiting cases can be considered in order to check the validity of (48). For example, when S_{vv} and S_{hh} are uncorrelated, then both λ_{13} and λ_{14} are zero for which $f_\Phi(\phi) = 1/(2\pi)$, as expected. Also, for the case of completely polarized scattered wave where S_{vv} and S_{hh} are completely correlated, the determinant of Λ is zero and so $f_\Phi(\phi)$ is a delta function.

It is interesting to note that the p.d.f. of the phase difference is only a function of two parameters defined by

$$\alpha = \sqrt{\frac{\lambda_{13}^2 + \lambda_{14}^2}{\lambda_{11}\lambda_{33}}} , \quad \zeta = \tan^{-1} \frac{\lambda_{14}}{\lambda_{13}}$$

where α and ζ can vary from 0 to 1 and $-\pi$ to π respectively. In fact if the wave were completely polarized, ζ would have been the phase difference between the co-polarized terms. The parameter ζ will, henceforth, be referred to as the *polarized-phase-difference*. In terms of these parameters (48) can be written as

$$f_\Phi(\phi) = \frac{1 - \alpha^2}{2\pi [1 - \alpha^2 \cos^2(\phi - \zeta)]} \cdot \left\{ 1 + \frac{\alpha \cos(\phi - \zeta)}{\sqrt{1 - \alpha^2 \cos^2(\phi - \zeta)}} \left[\frac{\pi}{2} + \tan^{-1} \frac{\alpha \cos(\phi - \zeta)}{\sqrt{1 - \alpha^2 \cos^2(\phi - \zeta)}} \right] \right\} . \quad (49)$$

It can be shown that the maximum of the probability density function occurs at $\phi = \zeta$ independent of α . However, the width of the p.d.f. (e.g. the 3 dB angular width) is only a function of α which will be referred to as the *degree of correlation*. The probability distribution function given by (18) is the analog of Gaussian distribution for periodic random variables where ζ and α are, respectively, the counterparts of the mean and variance for Gaussian random variables. Figure 4 shows the p.d.f. for different values of ζ while keeping α constant, and Fig. 5 shows the p.d.f. for a fixed value of α while changing ζ as a parameter. The calculated mean and standard deviation of the phase difference as a function of both the polarized-phase-difference and the degree of correlation are depicted in Figs. 6 and 7 respectively.

Lastly, it is necessary to point out that the formulation of the co-polarized phase difference p.d.f., as given in (48), is not restricted to the backscattering case or to the co- and cross-polarized components being uncorrelated. In fact we can derive the cross-polarized phase difference statistics in a similar manner and the p.d.f. in this case for the backscattering case can be obtained from (48) upon the following substitution for the elements of the cross-polarized covariance matrix

$$\begin{aligned} \lambda_{11} &= \frac{\mathcal{M}_{11}}{2} \quad , & \lambda_{33} &= \frac{\mathcal{M}_{12}}{2} \quad , \\ \lambda_{13} &= \frac{\mathcal{M}_{13}}{2} \quad , & \lambda_{14} &= \frac{\mathcal{M}_{14}}{2} \quad . \end{aligned}$$

5 Probability Density Functions of Magnitudes and Stokes' Parameters

In this section analytical expressions for the joint probability density function of the magnitude of the scattering amplitudes are obtained, from which the first two components of the Stokes vector given by (19) can be derived. The starting point is equation (45) where by integrating the joint pdf $f_{\rho_1, \rho_2, \phi_{vv}, \phi_{hh}}(\rho_1, \rho_2, \phi_{vv}, \phi_{hh})$ over the entire domain of ϕ_{vv} and ϕ_{hh} , the joint pdf of ρ_1 and ρ_2 is obtained. That is,

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) = \int_0^{2\pi} \int_0^{2\pi} \frac{1}{4\pi^2 \sqrt{\Delta}} \rho_1 \rho_2 \exp \left\{ -\frac{1}{2} [a_1 \rho_1^2 + a_2 \rho_2^2 - 2a_3 \rho_1 \rho_2] \right\} d\phi_{vv} d\phi_{hh} \quad (50)$$

where as before

$$a_3 = [\lambda_{13} \cos(\phi_{hh} - \phi_{vv}) + \lambda_{14} \sin(\phi_{hh} - \phi_{vv})] / \sqrt{\Delta} . \quad (51)$$

Making a change of variable $\phi_{hh} - \phi_{vv} = \delta$ and $\phi_{vv} = \eta$, (51) may be written as

$$a_3 = \frac{\sqrt{\lambda_{13}^2 + \lambda_{14}^2}}{\sqrt{\Delta}} \cos(\delta - \xi)$$

where $\xi = \tan^{-1}(\lambda_{14}/\lambda_{13})$. Noting that the Jacobian is unity and cosine function is periodic with period 2π

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) = \frac{\rho_1 \rho_2}{2\pi \sqrt{\Delta}} \exp \left\{ -\frac{1}{2} [a_1 \rho_1^2 + a_2 \rho_2^2] \right\} \int_0^{2\pi} e^{\sqrt{\frac{\lambda_{13}^2 + \lambda_{14}^2}{\Delta}} \rho_1 \rho_2 \cos(\delta - \xi)} d\delta . \quad (52)$$

Recognizing the last integral as the modified Bessel function of zeroth order

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) = \frac{\rho_1 \rho_2}{\lambda_{11} \lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2} e^{\frac{-\lambda_{33} \rho_1^2 + \lambda_{11} \rho_2^2}{2(\lambda_{11} \lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2)}} I_0 \left[\frac{\sqrt{\lambda_{13}^2 + \lambda_{14}^2}}{\lambda_{11} \lambda_{33} - \lambda_{13}^2 - \lambda_{14}^2} \rho_1 \rho_2 \right] . \quad (53)$$

As mentioned earlier at frequencies where coherent measurement of fields is not possible, scattered Stokes vectors are measured. For Gaussian statistics the statistical behavior of Stokes vectors is studied by Steeger, et. al. [9] and later on by Barakat [10]. Let us first consider the first component of the Stokes vector. As before we may assume

$$E_v = X_1 + iX_2 \quad ; \quad E_h = X_3 + iX_4$$

where X_1, \dots, X_4 are jointly Gaussian as given by (35) with properties listed in (36)–(39). The first component of the Stokes vector is given by

$$F_1 = \rho_1^2 + \rho_2^2$$

where ρ_1 and ρ_2 have a joint pdf given by (53). Our objective here is to find pdf of F_1 formally given by

$$f_{F_1}(F_1) = \frac{d}{dF_1} [P(\rho_1^2 + \rho_2^2 < F_1)]$$

where $P(\rho_1^2 + \rho_2^2 < F_1)$ is the probability that $\rho_1^2 + \rho_2^2 < F_1$ as shown in Figure 8.

Using (53) and changing variables $\rho_1 = r \cos \theta$ and $\rho_2 = r \sin \theta$

$$\begin{aligned} P(\rho_1^2 + \rho_2^2 < F_1) &= \int_0^{\sqrt{F_1}} \int_0^{\pi/2} \frac{r^3 \sin \theta \cos \theta}{\sqrt{\Delta}} \exp \left[\frac{-r^2}{2\sqrt{\Delta}} (\lambda_{33} \cos^2 \theta + \lambda_{11} \sin^2 \theta) \right] \\ &\times I_0 \left[\frac{\sqrt{\lambda_{13}^2 + \lambda_{14}^2}}{\sqrt{\Delta}} r^2 \sin \theta \cos \theta \right] d\theta dr . \end{aligned}$$

Taking the derivative with respect to F_1 and making another change of variable $u = \cos 2\theta$

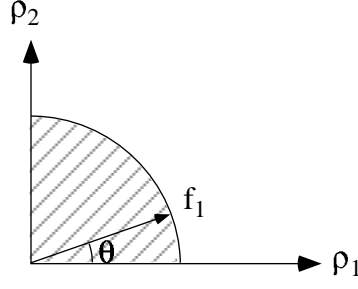


Figure 4: The integration region for the calculation of probability distribution function of F_1 .

$$\begin{aligned}
f_{F_1}(F_1) &= \frac{F_1}{8\sqrt{\Delta}} \exp\left[\frac{-F_1}{4\sqrt{\Delta}}(\lambda_{33} + \lambda_{11})\right] \int_{-1}^1 \exp\left[\frac{-F_1}{4\sqrt{\Delta}}(\lambda_{33} - \lambda_{11})u\right] \\
&\times I_0\left[\frac{\sqrt{\lambda_{13}^2 + \lambda_{14}^2}}{2\sqrt{\Delta}} f_1 \sqrt{1-u^2}\right] du
\end{aligned} \tag{54}$$

Starting from identity [11]

$$\int_0^{\pi/2} \sin x \cos(\beta' x) J_0(\alpha' \sin x) dx = \sqrt{\frac{\pi}{2\sqrt{\alpha'^2 + \beta'^2}}} J_{\frac{1}{2}}(\sqrt{\alpha'^2 + \beta'^2})$$

and changing $\beta' = i\beta$ and $\alpha' = i\alpha$, it can be shown that

$$\int_{-1}^1 e^{-\beta u} I_0(\alpha \sqrt{1-u^2}) du = \sqrt{\frac{\pi}{2i\sqrt{\alpha^2 + \beta^2}}} J_{1/2}(i\sqrt{\alpha^2 + \beta^2}).$$

Knowing that [12]

$$J_{1/2}(iz) = i \left(\frac{2}{i\pi z}\right)^{1/2} \sinh(z)$$

(54) may be written as

$$f_{F_1}(F_1) = \exp\left[\frac{-F_1}{4\sqrt{\Delta}}(\lambda_{33} + \lambda_{11})\right] \frac{\sinh\left[\frac{\sqrt{(\lambda_{11} + \lambda_{33})^2 - 4\sqrt{\Delta} \frac{F_1}{4\sqrt{\Delta}}}}{\sqrt{(\lambda_{11} + \lambda_{33})^2 - 4\sqrt{\Delta}}}\right]}{\sqrt{(\lambda_{11} + \lambda_{33})^2 - 4\sqrt{\Delta}}}. \tag{55}$$

Under a condition that the two components of the scattered fields are uncorrelated, that is, $\lambda_{13} = \lambda_{14} = 0$ then $\sqrt{\Delta} = \lambda_{11}\lambda_{33}$ and

$$f_{F_1}(F_1) = \frac{1}{2(\lambda_{11} - \lambda_{33})} \left[e^{\frac{-F_1}{2\lambda_{11}}} - e^{\frac{-F_1}{2\lambda_{33}}} \right] .$$

In a situation where $\lambda_{33} \rightarrow \lambda_{11}$, using L'Hopital rule, we can show

$$f_{F_1}(F_1) = \frac{F_1}{4\lambda_{11}^2} e^{\frac{-F_1}{2\lambda_{11}}} .$$

For conditions where the two components are highly correlated $\lambda_{11}\lambda_{33} \simeq \lambda_{13}^2 + \lambda_{14}^2$ or $\Delta \rightarrow 0$. In this case

$$\sqrt{(\lambda_{11} + \lambda_{33})^2 - 4\sqrt{\Delta}} \simeq (\lambda_{11} + \lambda_{33}) - \frac{2\sqrt{\Delta}}{(\lambda_{11} + \lambda_{33})}$$

and (55) reduces to

$$f_1(F_1) = \frac{1}{2(\lambda_{11} + \lambda_{33})} \exp \left[\frac{-F_1}{2(\lambda_{11} + \lambda_{33})} \right] .$$

6 Comparison with Measurements

Using the polarimetric data gathered by scatterometers from a variety of natural targets, the assumptions leading to the probability density function of phase differences as derived in the previous section are examined. Also by generating the histograms, means, and standard deviations of the phase differences from the data and comparing them with the results based on the p.d.f. derived from the measured Mueller matrices validity of the model is also examined. The polarimetric radar measurements of bare soil surfaces were performed at L-, C-, and X-band frequencies for a total of eight different soil surface conditions (four roughness and two moisture conditions). For this experiment we tried to preserve the absolute phase of the measured scattering matrix by calibrating the surface data with a metallic sphere located at the same distance from the radar as the center of the surface target. For each frequency, surface condition, and incidence angle a minimum of 700 independent samples were collected. The detailed procedure of the data collection and calibration is given in reference [4].

By generating the histograms of the real and imaginary parts of the elements of the scattering matrix for all surfaces, it was found that they have a zero-mean Gaussian distribution as we assumed. Figure 8 represents a typical case where the histogram of the real and imaginary parts of S_{vv} and S_{hh} of a dry surface with rms height 0.32 cm

and correlation length 9.9 cm at C-band have a bell-shaped distribution. The properties of the covariance matrix as given by (5)-(8) and (42)-(43) are verified by calculating the covariance matrices of the data for all cases. Table 1 represents a typical situation where the covariance matrix for the same rough surface at C-band possesses the mentioned properties approximately, that is $\lambda_{11} \approx \lambda_{22}$, $\lambda_{12} \approx \lambda_{34} \approx 0$, $\lambda_{33} \approx \lambda_{44}$, $\lambda_{13} \approx \lambda_{24}$, and $\lambda_{14} \approx -\lambda_{23}$. The small discrepancies are due to the fact that the measurement of the scattering matrix with absolute phase has an uncertainty of ± 30 degrees.

Table 2 gives the Mueller matrix of the typical surface (Table 1) at C-band from which the co- and cross-polarized phase difference probability density functions are calculated using (48) and are compared with the measured phase histograms in Figs. 9 and 10 respectively. Similar comparisons were also made for the rest of surfaces, frequencies, and incidence angles and it was found that the expression (48) predicts the density functions very accurately. Some example of these comparisons are shown in Figs.11 and 12. Figures 11 and 12 compare the mean and standard deviation of the co-polarized phase difference *versus* incidence angle at L- and X-band for a surface with rms height 0.4 cm and correlation length 8.4 cm in dry condition using the results based on the direct calculation and the results derived from (48).

7 Conclusions

Prompted by the experimental observations which show strong dependence of phase differences of the scattering matrix elements on the physical parameters of random media, the statistical behavior of the phase differences for distributed targets is studied. The probability density functions of the phase differences are derived from the Mueller matrix of the target. In derivation of the density functions it is assumed that the real and imaginary parts of the co- and cross-polarized terms of the scattering matrix are jointly Gaussian and their covariance matrices are found in terms of the Mueller matrix elements. The functional form of the co- and cross-polarized density functions are similar and are obtained independently. It is shown that the density function of the phase difference is completely determined in terms of only two parameters. The assumptions and final expressions are verified by using a set of polarimetric data acquired by scatterometers from rough surfaces.

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1.00	0.03	0.75	-0.12
0.03	0.90	0.08	0.68
0.75	0.08	0.77	0.05
-0.12	0.68	0.05	0.69

Table 1: Normalized covariance matrix of co-polarized terms of scattering matrix for a surface with rms height 0.3 cm and correlation length 9 cm at C-band and at 30 degrees incidence angle.

1.000	0.030	0.000	0.000
0.028	0.767	0.000	0.000
0.000	0.000	0.770	-0.11
0.000	0.000	0.110	0.711

Table 2: Normalized Mueller matrix for a surface with rms height 0.3 cm and correlation length 9 cm at C-band and at 30 degrees incidence angle.

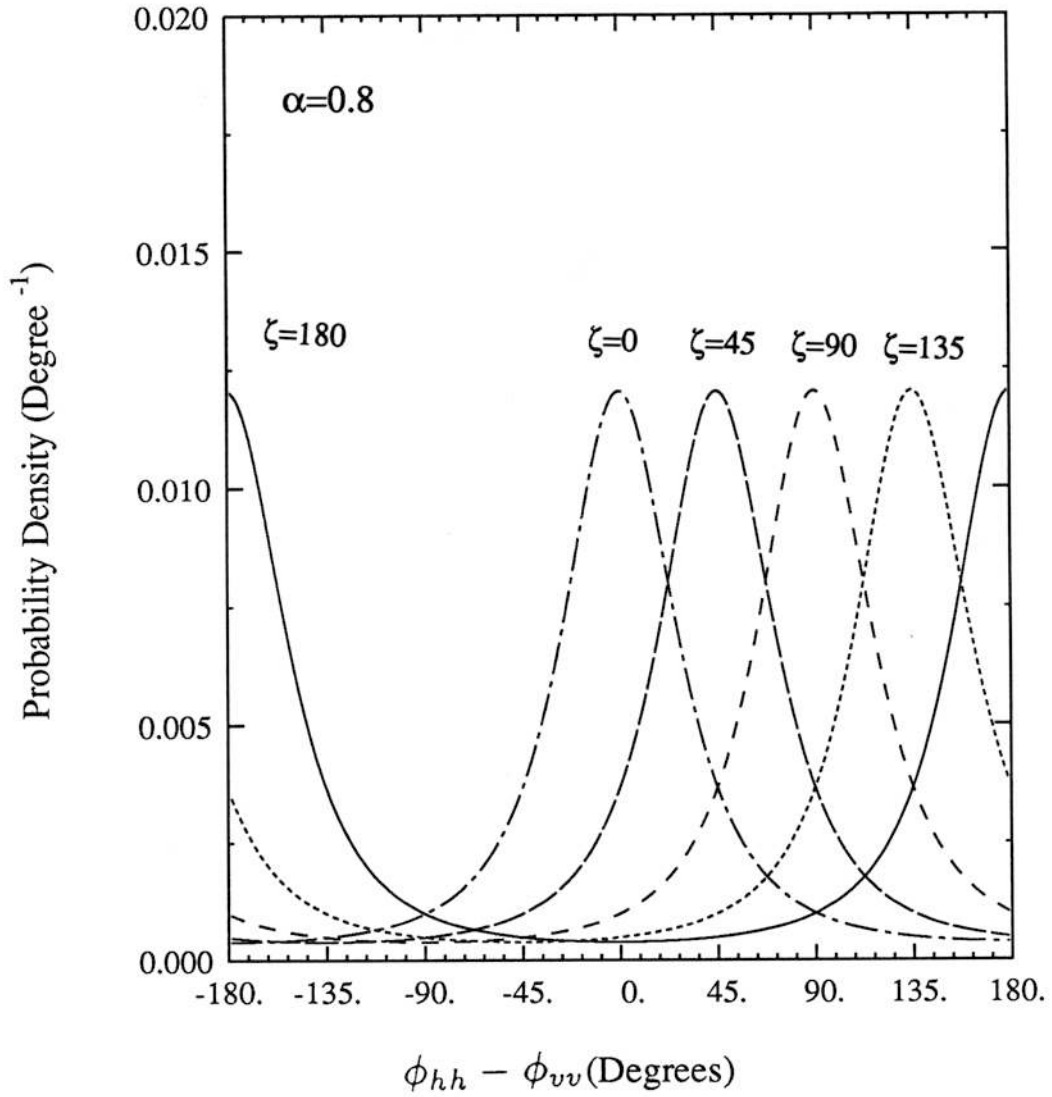


Figure 5: The probability density function of the co-polarized phase difference for a fixed value of α (degree of correlation) and five values of ζ (coherent-phase-difference).

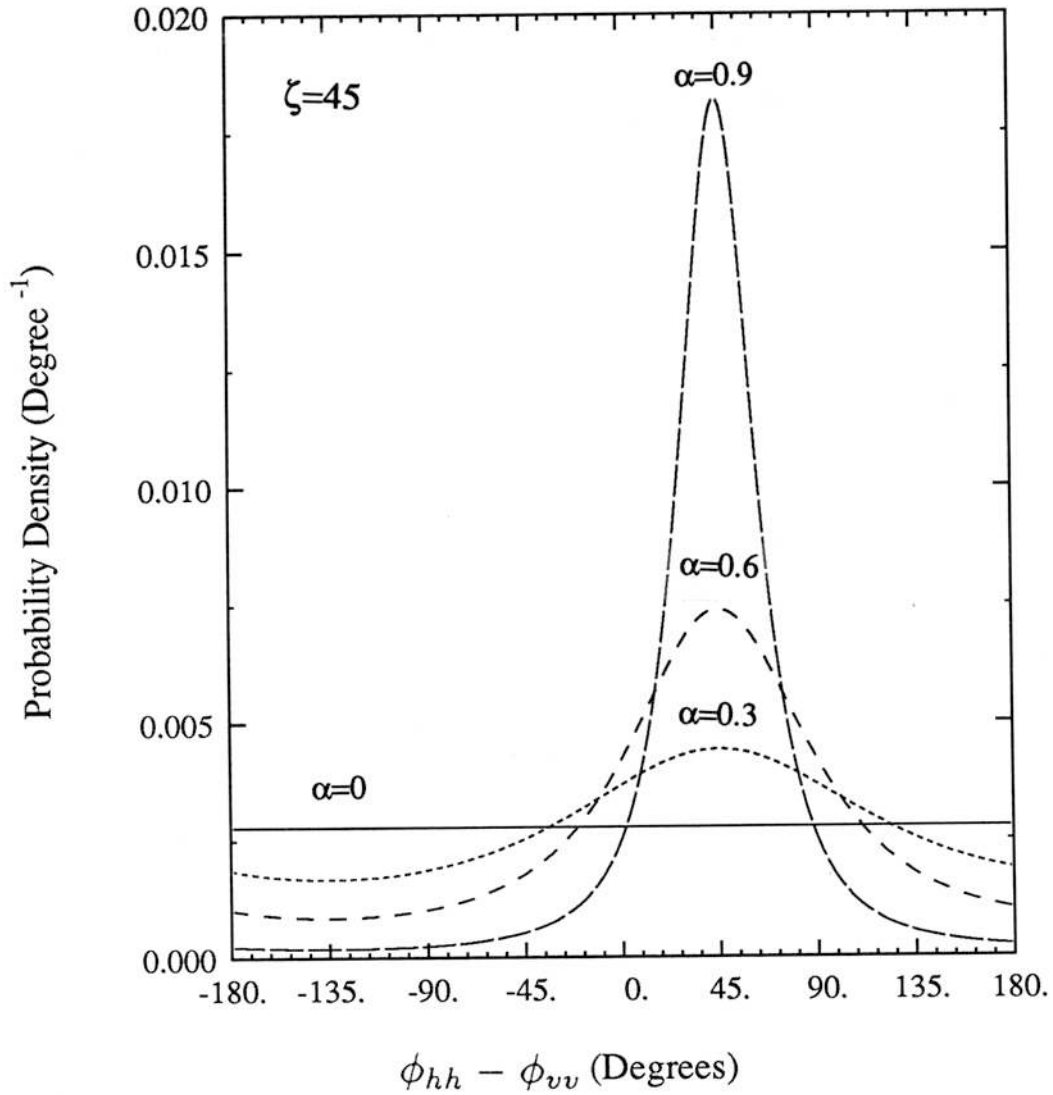


Figure 6: The probability density function of the co-polarized phase difference for a fixed value of ζ (coherent-phase-difference) and four values of α (degree of correlation).

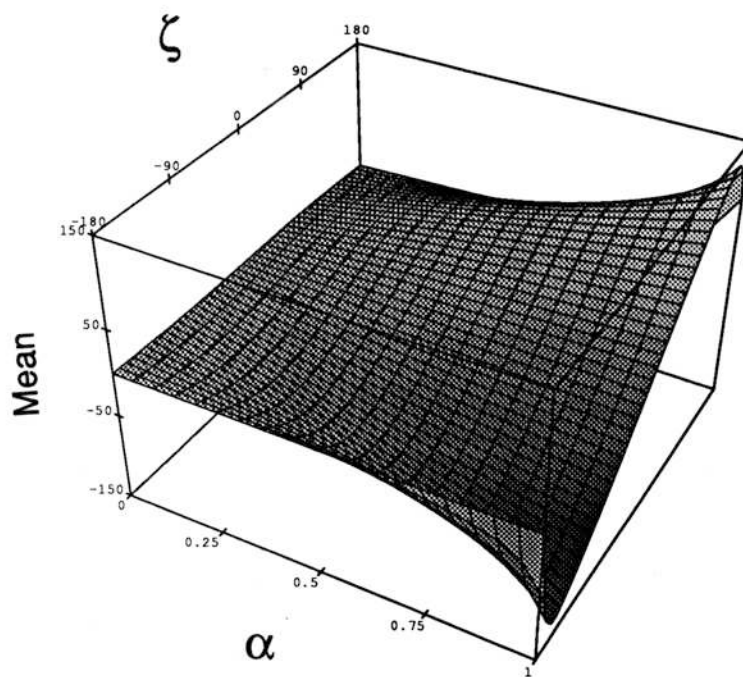


Figure 7: The mean value of the co-polarized phase difference as a function of α (degree of correlation) and ζ (coherent-phase-difference).

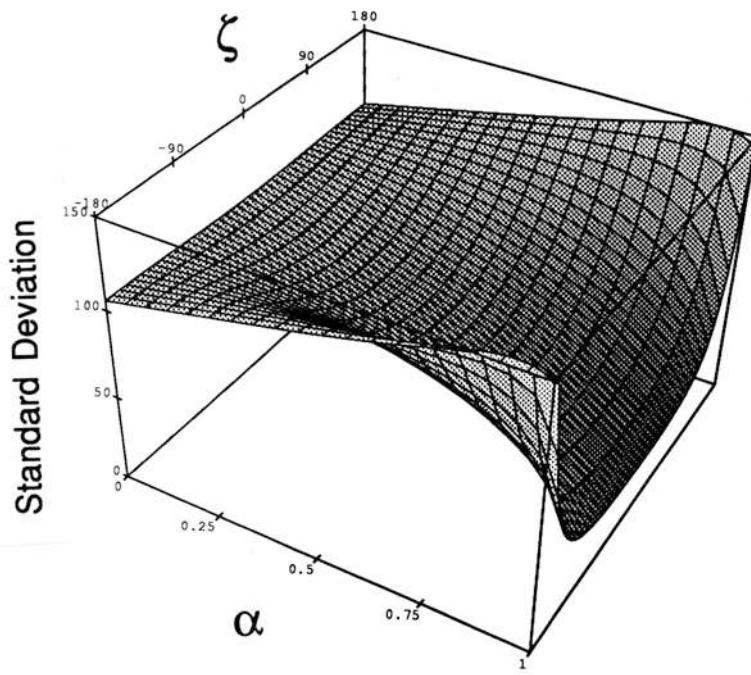


Figure 8: The standard deviation of the co-polarized phase difference as a function of α (degree of correlation) and ζ (coherent-phase-difference).

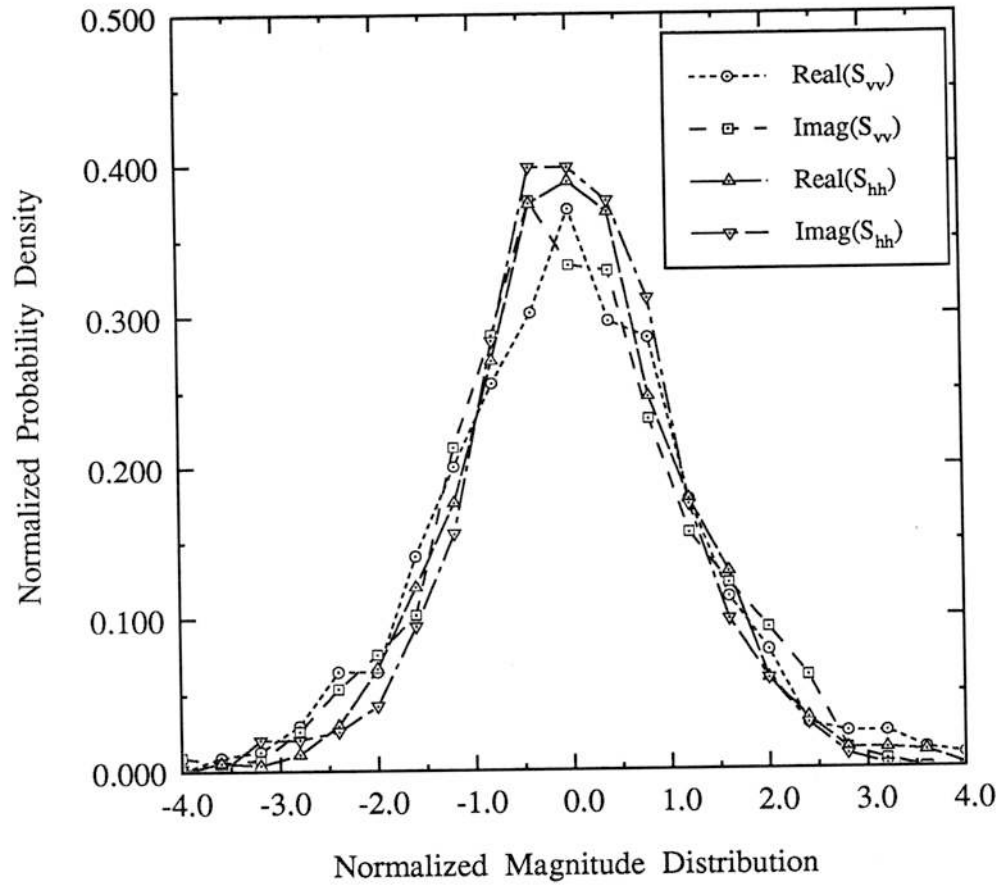


Figure 9: The histogram of the real and imaginary parts of S_{vv} and S_{hh} for a rough surface with rms height 0.32 cm and correlation length 9.9 cm at C-band and 30° incidence angle.

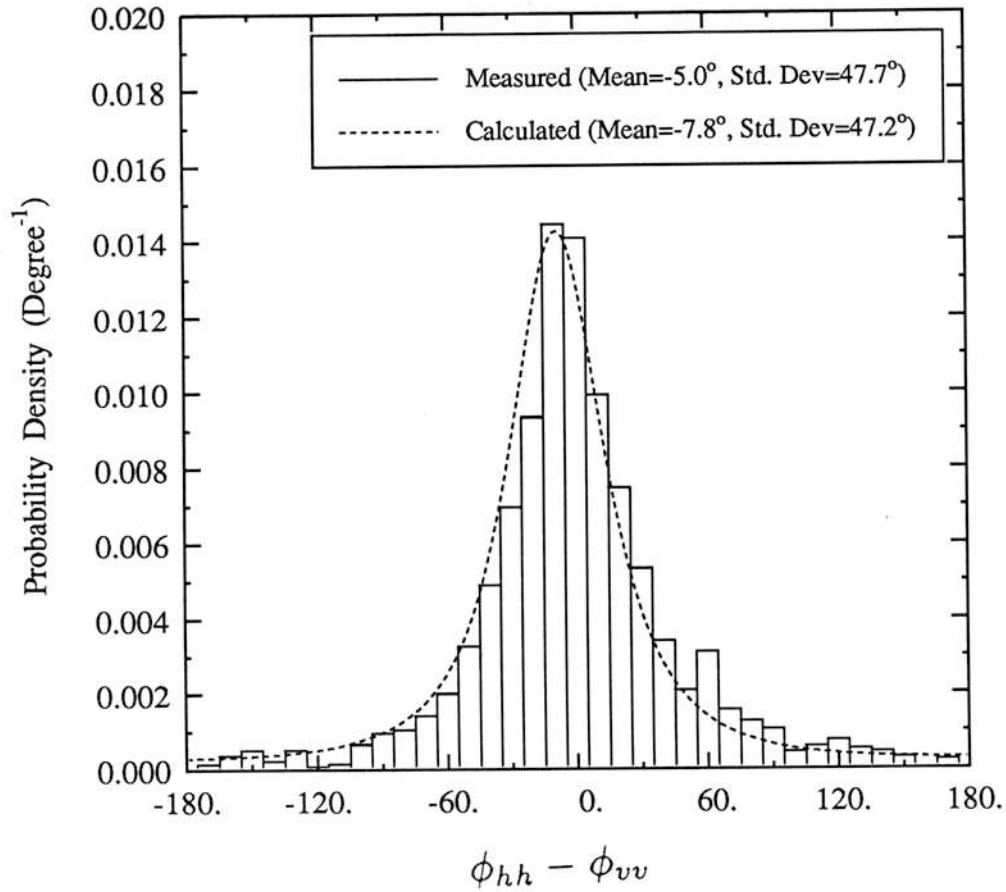


Figure 10: The histogram and p.d.f. of the co-polarized phase difference for a rough surface with rms height 0.32 cm and correlation length 9.9 cm at C-band and 30° incidence angle.

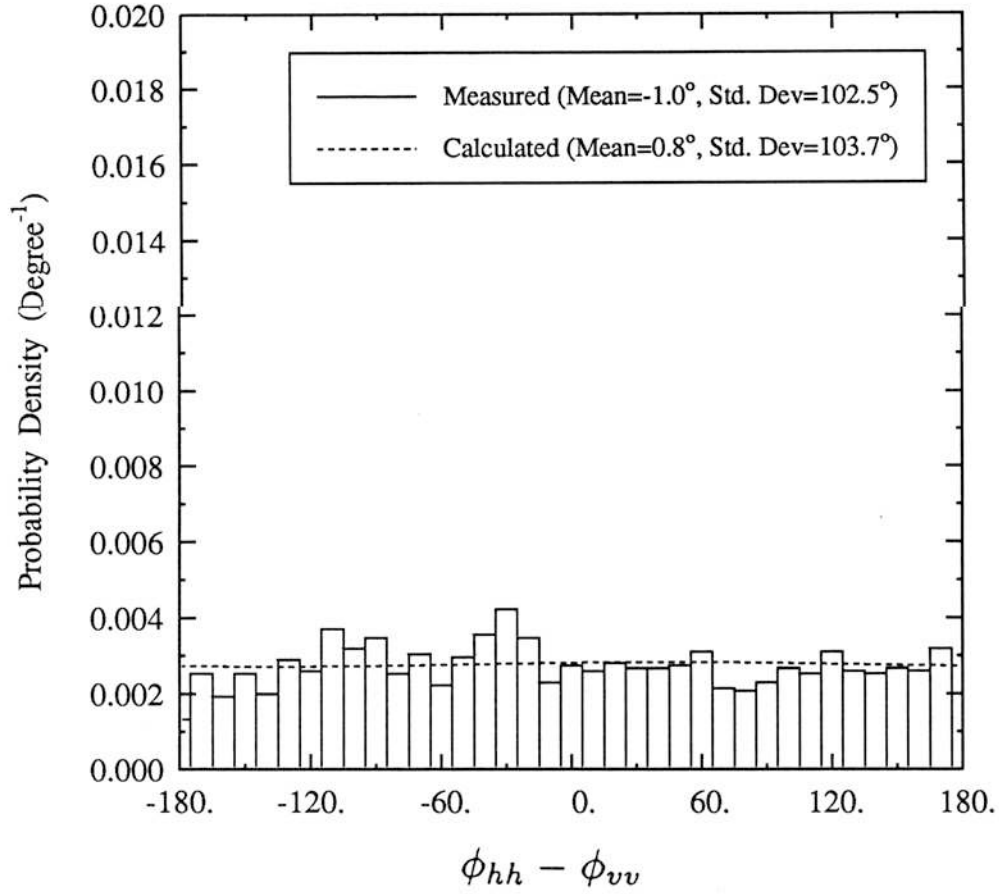


Figure 11: The histogram and p.d.f. of the cross-polarized phase difference for a rough surface with rms height 0.32 cm and correlation length 9.9 cm at C-band and 30° incidence angle.

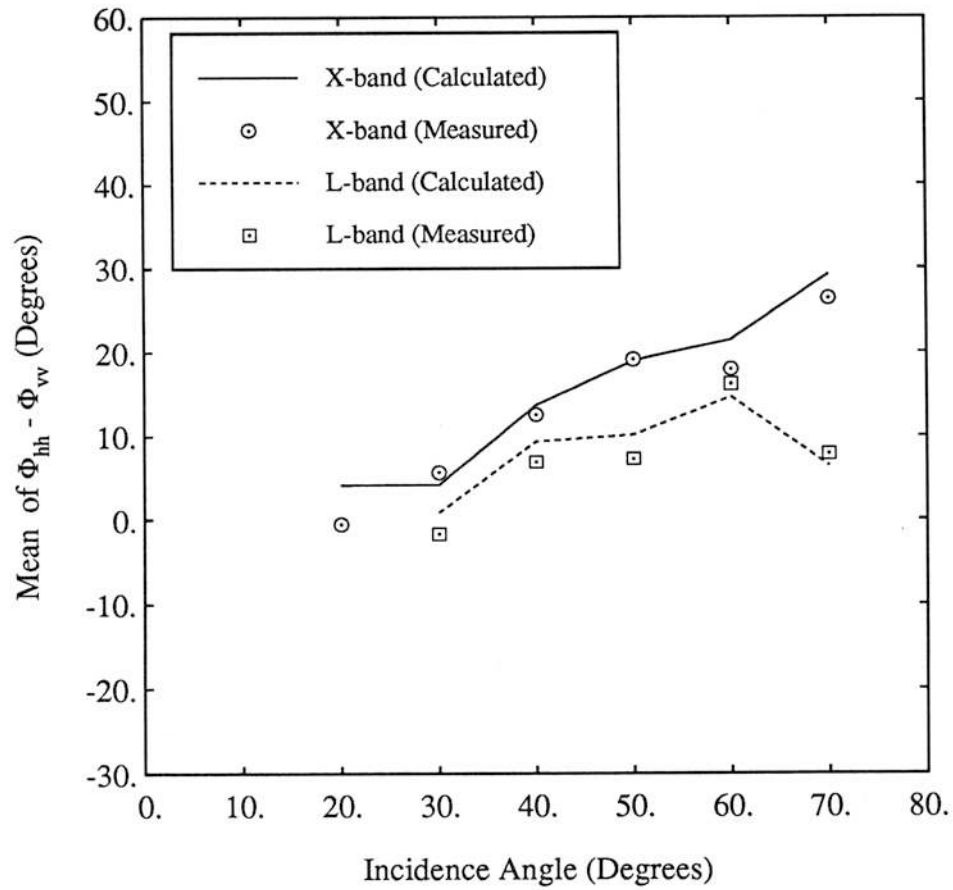


Figure 12: Angular dependency of the mean of the co-polarized phase difference for a dry rough surface with rms height 0.4 cm and correlation length 8.4 cm at L- and X-band.

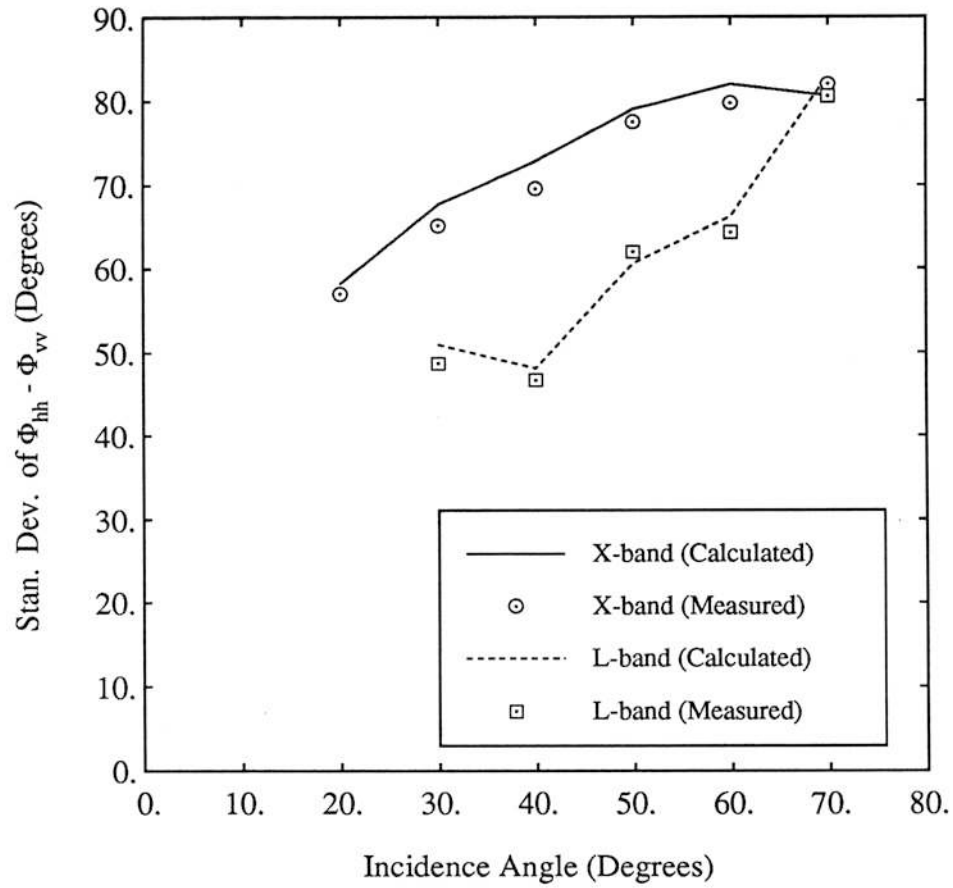


Figure 13: Angular dependency of the standard deviation of the co-polarized phase difference for a dry rough surface with rms height 0.4 cm and correlation length 8.4 cm at L- and X-band.