# Dyadic Analysis[1]

In this section a brief review of dyadic analysis is presented. Dyadic operations and theorems provide an effective tool for manipulation of field quantities. [2] Dyadic notation was first introduced by Gibbs in 1884 which later appeared in literature.[3] Consider a vector function $\overline{F}$ having three scalar component $f_i$ with (i = 1,2,3) in a Cartesian system, that is

$$\overline{F} = \sum_{i=1}^{3} f_i \hat{x}_i$$

Now consider three different vector functions $\overline{F}_j$, given by

$$\overline{F}_j = \sum_{i=1}^{3} f_{ij} \hat{x}_i \quad j = 1, 2, 3 \tag{1}$$

which constitute a dyad $\overline{\overline{F}}$ given by

$$\overline{\overline{F}} = \sum_{j=1}^{3} \overline{F}_j \hat{x}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} f_{ij} \hat{x}_i \hat{x}_j \tag{2}$$

The doublets $\hat{x}_i \hat{x}_j$ (i, j = 1, 2, 3) form the nine-unit dyad basis. In matrix notation

$$\overline{\overline{F}} = \left(\overline{F}_1, \overline{F}_2, \overline{F}_3\right) = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

It should be emphasized that

$$\hat{x}_i \hat{x}_j \neq \hat{x}_j \hat{x}_i \quad \text{for i} \neq j$$

[1] Copyright K. Sarabandi, 1997.

[2] Tai, C.T., "Dyadic Green's Functions in Electromagnetic Theory," New York: IEEE Press, 2nd ed., 1993.

[3] Gibbs, J.W., "The scientific papers of J. Willard Gibbs" Vol. 2, pp. 84-90, New York: Dover, 1961.

In general a dyad can be formed from the product of two arbitrary vectors $\bar{a}$ and $\bar{b}$ to form $\bar{\bar{F}}= \bar{a}\bar{b}$. Components of $\bar{\bar{F}}$ can be obtained from a matrix product of $\bar{a}$ denoted by a $3 \times 1$ matrix with $1 \times 3$ matrix $\bar{b}$. Note that the converse may not be necessarily true. That is, a general dyad may not be expressable in terms of product of two vector like $\overline{a}\overline{b}$.

Transpose of $\bar{\bar{F}}$, denoted by $[\bar{\bar{F}}]^T$, is defined by

$$[\bar{\bar{F}}]^T = \sum_j x_j \overline{F}_j = \sum_i \sum_j f_{ij} \hat{x}_j \hat{x}_i.$$

In matrix form $[\bar{\bar{F}}]^T$ is simply the transpose of matrix $\bar{\bar{F}}$. A symmetric dyad is a dyad for which

$$[\bar{\bar{F}}_s]^T = \bar{\bar{F}}_s \ .$$

A particular case of a symmetric dyad is the "idemfactor" $(\bar{\bar{I}})$ described by

$$f_{ij} = \delta_{ij}$$

where $\delta_{ij}$ is the Kronecker delta function. The idemfactor is explicitly expressed by

$$\bar{\bar{I}}= \sum_{i=1}^{3} \hat{x}_i \hat{x}_i.$$

An antisymmetric dyad is defined by

$$[\bar{\bar{F}}_a]^T = - \bar{\bar{F}}_a$$

It is easy to show that any dyad can be decomposed into a symmetric and antisymmetic dyad components

$$\bar{\bar{F}}= \frac{1}{2}[\bar{\bar{F}} + \bar{\bar{F}}^T] + \frac{1}{2}[\bar{\bar{F}} - \bar{\bar{F}}^T]$$

# 1 The Scalar Products

The "anterior scalar product" of a vector and a dyad is defined by

$$\overline{a} \cdot \bar{\bar{F}}= \sum_{j=1}^{3} \left(\overline{a} \cdot \overline{F}_j\right) \hat{x}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i f_{ij} \hat{x}_j \tag{3}$$

which is a vector. The "posterior scalar product" is defined by

$$\overline{\overline{F}} \cdot \overline{a} = \sum_{j=1}^{3} \overline{F}_j \left(\hat{x}_j \cdot \overline{a}\right) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_j f_{ij} \hat{x}_i \tag{4}$$

which is also a vector. It is obvious that for a symmetric dyad

$$\overline{a} \cdot \overline{\overline{F}}_s = \overline{\overline{F}}_s \cdot \overline{a}$$

and

$$\overline{a} \cdot \overline{\overline{I}} = \overline{\overline{I}} \cdot \overline{a} = \overline{a}$$

# 2 The Vector Products

Like scalar products, there are two different vector products between a vector a dyad. The "anterior vector product" is defined by

$$\overline{a} \times \overline{\overline{F}} = \sum_{j=1}^{3} (\overline{a} \times \overline{F}_j) \hat{x}_j \tag{5}$$

and the "posterior vector product" is defined by

$$\overline{\overline{F}} \times \overline{a} = \sum_{j=1}^{3} \overline{F}_j (\overline{x}_j \times \overline{a}). \tag{6}$$

# 3 Vector Dyadic Products

Consider two vectors $\overline{a}$ and $\overline{b}$ and a dyad $\overline{\overline{C}}$ with vector components $\overline{C}_j$. Since

$$\overline{a} \cdot (\overline{b} \times \overline{C}_j) = -\overline{b} \cdot (\overline{a} \times \overline{C}_j) = (\overline{a} \times \overline{b}) \cdot \overline{C}_j \quad j = 1, 2, 3 \tag{7}$$

it can easily be shown that

$$\overline{a} \cdot (\overline{b} \times \overline{\overline{C}}) = -\overline{b} \cdot (\overline{a} \times \overline{\overline{C}}) = (\overline{a} \times \overline{b}) \cdot \overline{\overline{C}} \tag{8}$$

Equation (8) is obtained from the three equations given by (7) after juxtaposing $\hat{x}_j$ at the posterior position of each equation and adding them up.

# 4 Differential Operations on Dyadic Functions

Differential operators such as divergence, gradient, and curl are often encountered in analysis of electromagnetic field quantities. Hence it is useful to define these operators when applied to dyads. The divergence of a dyadic function is defined by:

$$\nabla\cdot \overline{\overline{F}}=\sum_{j=1}^{3}\nabla\cdot\overline{F}_j\hat{x}_j \tag{9}$$

The resulting quantity is a vector. In a similar manner to the anterior vector product, the curl of a dyadic is defined by

$$\nabla\times \overline{\overline{F}}=\sum_{j=1}^{3}(\nabla\times\overline{F}_j)\hat{x}_j \tag{10}$$

which forms a dyad. Gradient of a vector function form a dyadic function which is given by:

$$\nabla\overline{F}=\sum_{j=1}^{3}(\nabla f_j)\hat{x}_j=\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial f_j}{\partial x_i}\right)\hat{x}_i\hat{x}_j \tag{11}$$

# 5 Vector-Dyadic Green's Second Identity

The Green's second identity for vector functions can be used to develop the vector-dyadic version of the theorem. For any two vector functions $\overline{P}$ and $\overline{Q}_j$ which together with their first and second derivatives are continuous it can be shown that[4]

$$\begin{aligned}\iiint_v[\overline{P}\cdot\nabla\times\nabla\times\overline{Q}_j-(\nabla\times\nabla\times P)\cdot Q_j]dv &= \oiint[\overline{Q}_j\times\nabla\times\overline{P}-\overline{P}\times\nabla\times Q_j]\cdot\hat{n}ds \\ &= \oiint_s[(\nabla\times\overline{P}\times\hat{n})\cdot\overline{Q}_j+\overline{P}\cdot(\hat{n}\times\nabla\times\overline{Q}_j)]ds\end{aligned} \tag{12}$$

Suppose $\overline{Q}_j$ is one of the constituent vectors of dyad $\overline{\overline{Q}}$. By juxtaposing $\hat{x}_j$ at the posterior position of (12) and adding the resulting equations, it can easily be shown that

$$\iiint_v[\overline{P}\cdot\nabla\times\nabla\times\overline{\overline{Q}}-(\nabla\times\nabla\times\overline{P})\cdot\overline{\overline{Q}}]dv=\oiint[\overline{P}\cdot(\hat{n}\times\nabla\times\overline{\overline{Q}})-(\hat{n}\times\nabla\times\overline{P})\cdot\overline{\overline{Q}}]ds \tag{13}$$

[4]Stratton, J.A., "Electromagnetic Theory," pp. 250, New York: Mc Graw-Hill, 1941.

which is the statement of vector-dyadic Green's second identity.

# 6 Dyadic-Dyadic Green's Second Identity

Another useful form of the Green's second identity is its dyadic-dyadic form. This form can be obtained directly from (13). Let us consider three vectors $\overline{P}_j$ for $j = 1, 2, 3$. Noting that

$$\overline{P}_j \cdot (\hat{n} \times \nabla \times \overline{\overline{Q}}) = -(\hat{n} \times \overline{P}_j) \cdot \nabla \times \overline{\overline{Q}} \ ,$$

using $\overline{P}_j$ in (13), and then transposing this resulting equation, we get

$$\begin{aligned} &\iiint_v \left\{ [\nabla \times \nabla \times \overline{\overline{Q}}]^T \cdot \overline{P}_j - [\overline{\overline{Q}}]^T \cdot \nabla \times \nabla \times \overline{P}_j \right\} dv = \\ &- \oiint_s \left\{ [\nabla \times \overline{\overline{Q}}]^T \cdot (\hat{n} \times \overline{P} j) + [\overline{\overline{Q}}]^T \cdot (\hat{n} \times \nabla \times \overline{P}_j) \right\} ds \end{aligned} \tag{14}$$

Juxtaposing $\hat{x}_j$ at the posterior position of (14) and summing the results, the dyadic-dyadic Green's second identity can be obtained and is given by

$$\begin{aligned} &\iiint_v \left\{ [\nabla \times \nabla \times \overline{\overline{Q}}]^T \cdot \overline{\overline{P}} - [\overline{\overline{Q}}]^T \cdot \nabla \times \nabla \times \overline{\overline{P}} \right\} dv = \\ &- \oiint_s \left\{ [\nabla \times \overline{\overline{Q}}]^T \cdot (\hat{n} \times \overline{\overline{P}}) + [\overline{\overline{Q}}]^T \cdot (\hat{n} \times \nabla \times \overline{\overline{P}}) \right\} ds \end{aligned} \tag{15}$$