Exact Asymptotics in Channel Coding

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(Joint work with Yücel Altuğ)

Cornell University
School of Electrical and Computer Engineering

I.I.D. Sums

- Let $\{X_i\}$ be i.i.d. with zero mean and unit var.

$$\text{Asymptotic behavior of } \sum_{i=1}^{N} X_i ?$$

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- Central Limit Theorem:

$$\lim_{N \to 0} \Pr\left(\sum_{i=1}^{N} X_i > \epsilon \sqrt{N}\right) = Q(\epsilon)$$

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- Moderate Deviations: if β is in (1/2, 1):

$$\lim_{N\to 0} \frac{1}{N^{2\beta-1}} \log \Pr\left(\sum_{i=1}^{N} X_i > \epsilon N^{\beta}\right) = \Lambda^*_{\mathcal{N}}(\epsilon)$$

Channel Coding

Discrete memoryless channel: *W(y|x)*

$x \in \mathcal{X}$

$y \in \mathcal{Y}$

$$\boxed{W}$$

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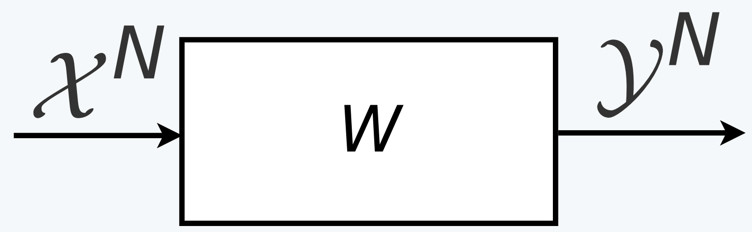

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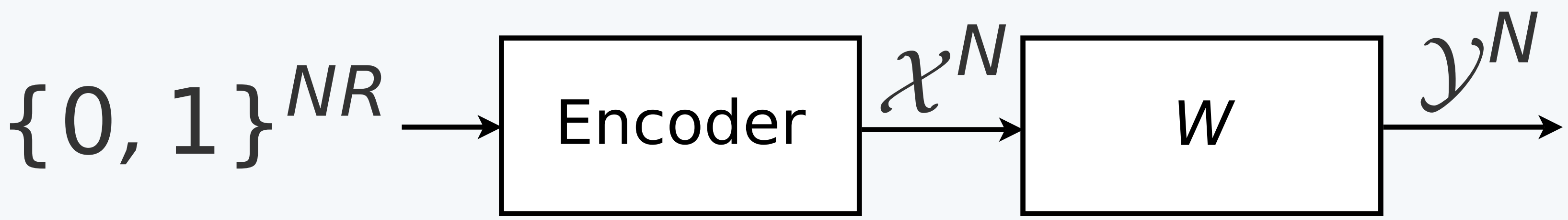

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$$\{0,1\}^{NR} \longrightarrow \boxed{\text{Encoder}} \xrightarrow{\mathcal{X}^N} \boxed{W} \xrightarrow{\mathcal{Y}^N} \boxed{\text{Decoder}} \longrightarrow \{0,1\}^{NR}$$

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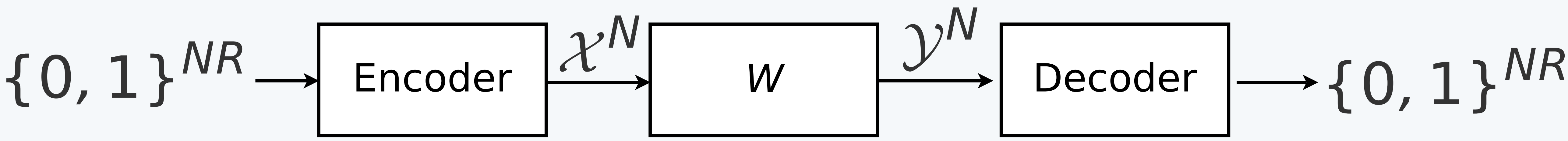

Parameters:

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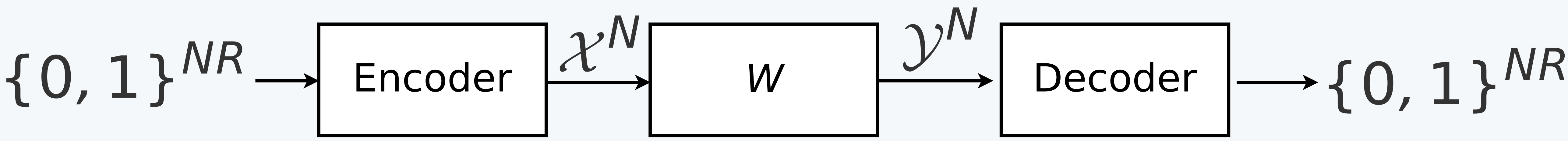

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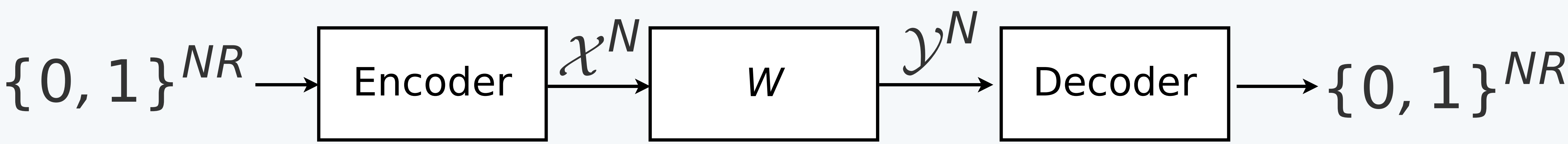

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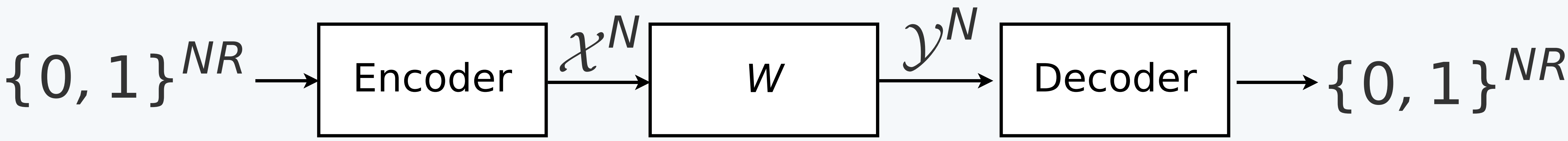

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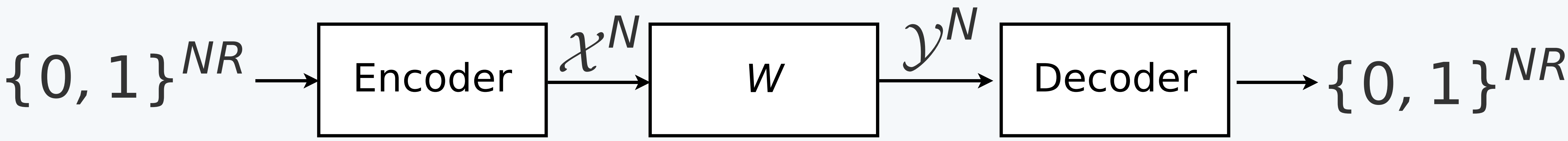

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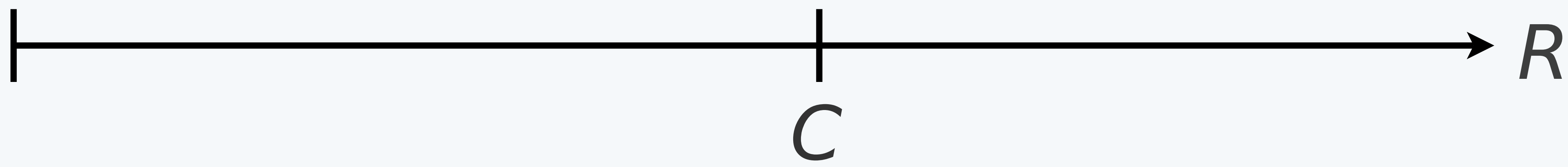
$$P_e(N,R) := \min\{P_e : \text{ code is } (N,R)\}$$

Channel Coding Theorem [Shannon '48]

Channel capacity: $C := \max_{P_X} \mathrm{I}(P_X; W)$



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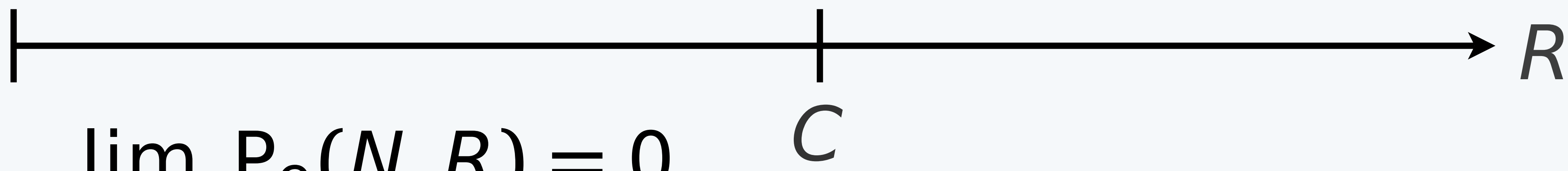

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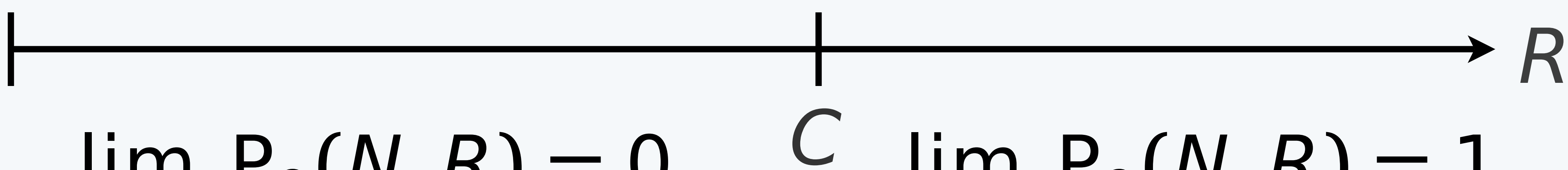

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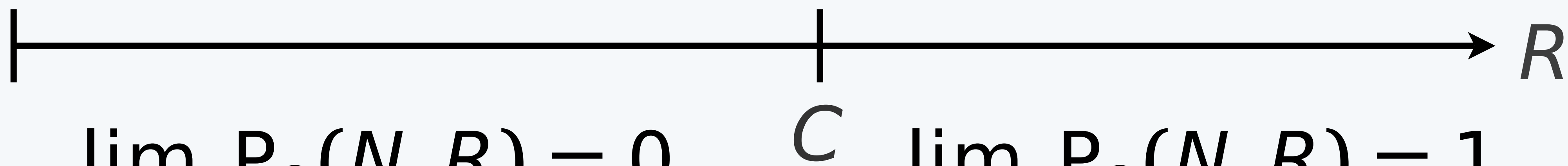

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Speed of convergence?

Error Exponents

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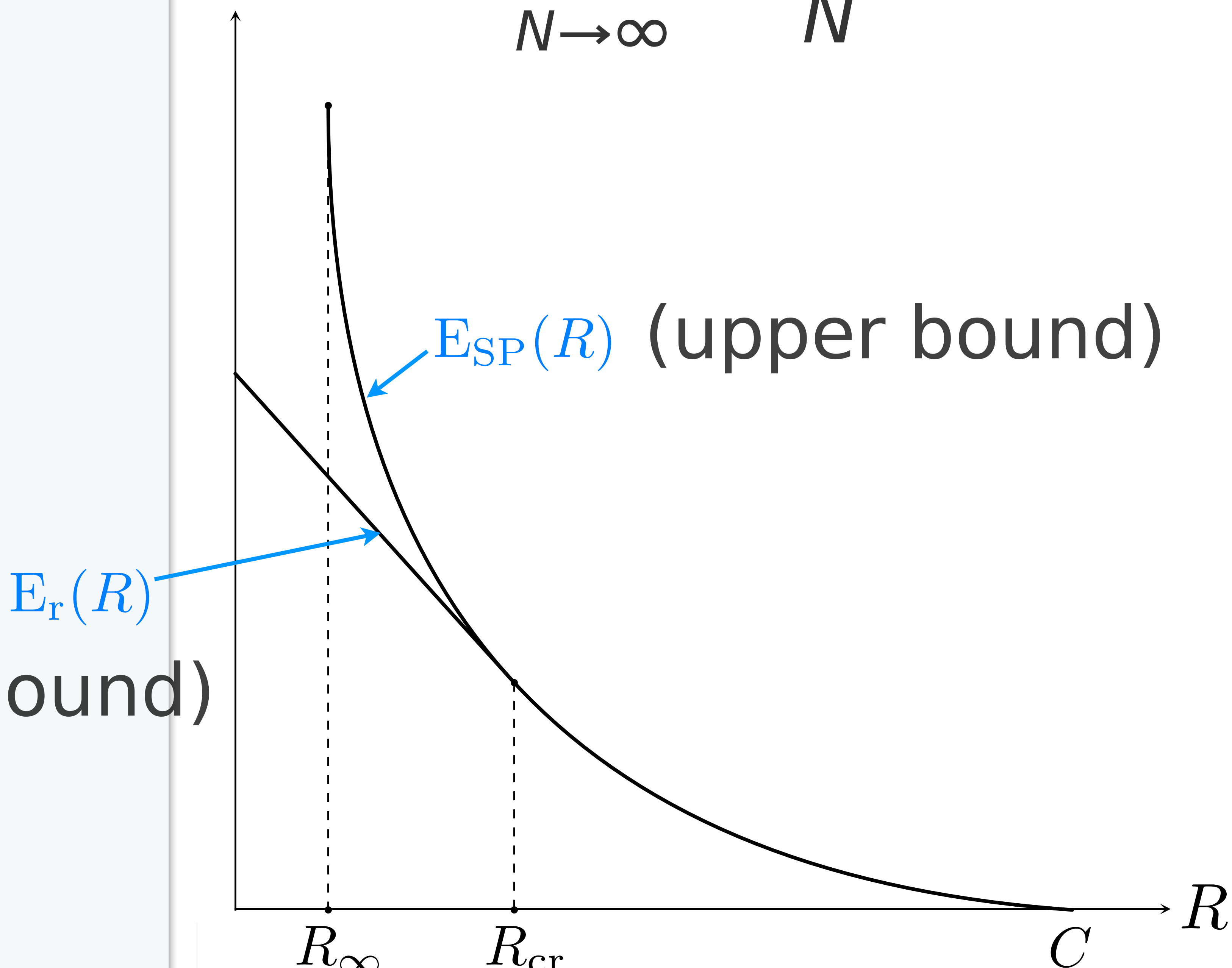

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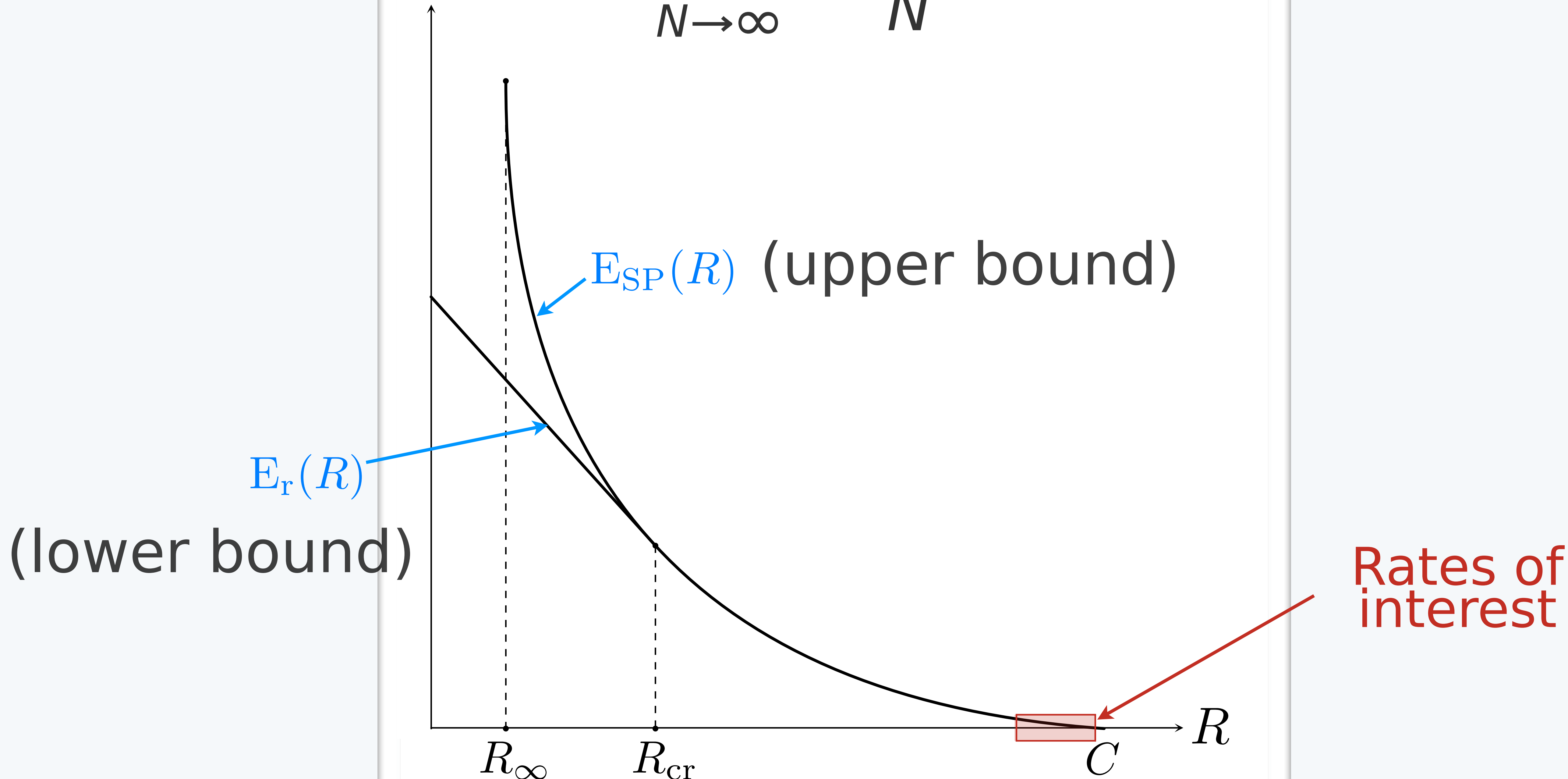

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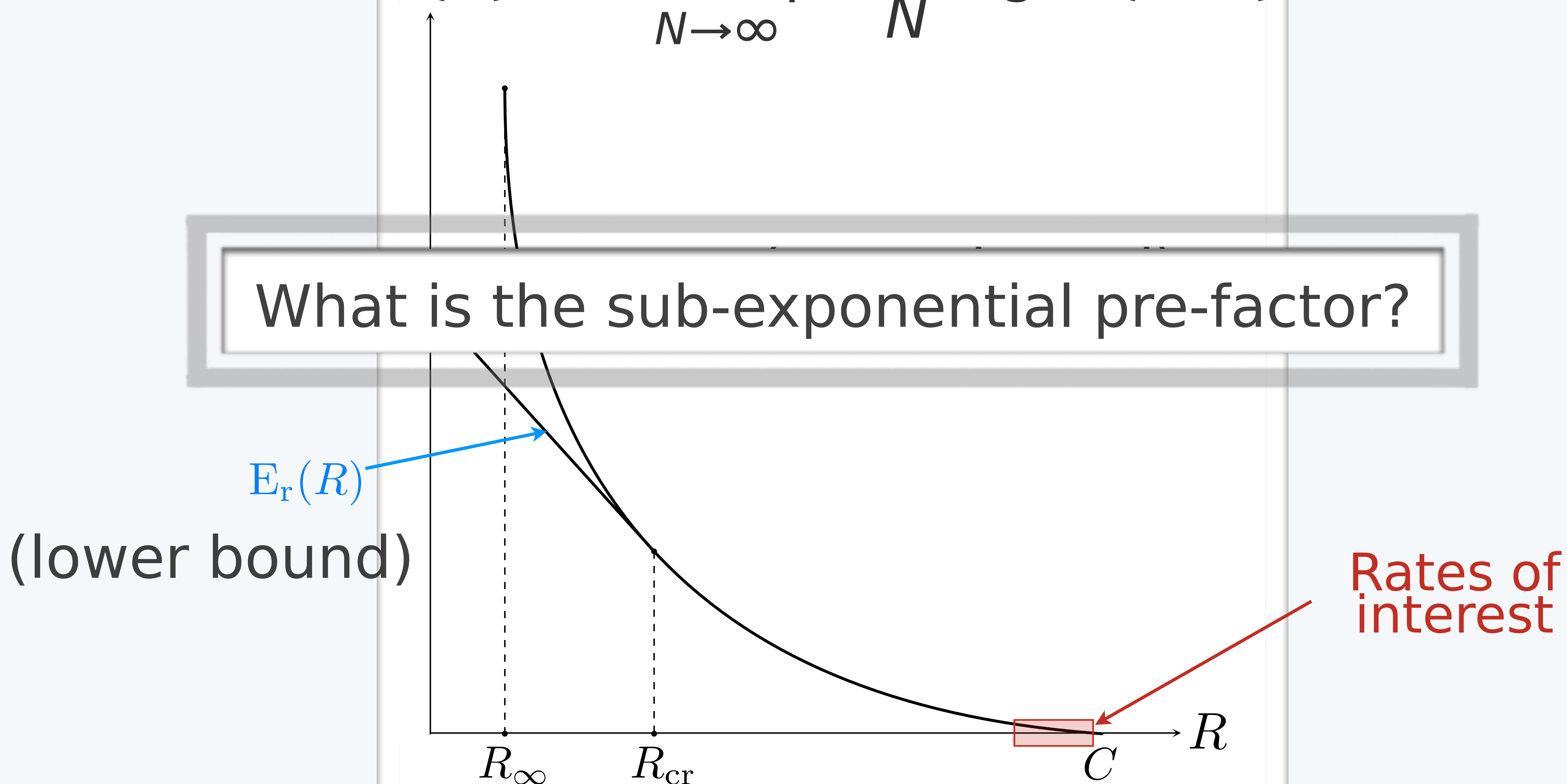

What is the sub-exponential pre-factor?

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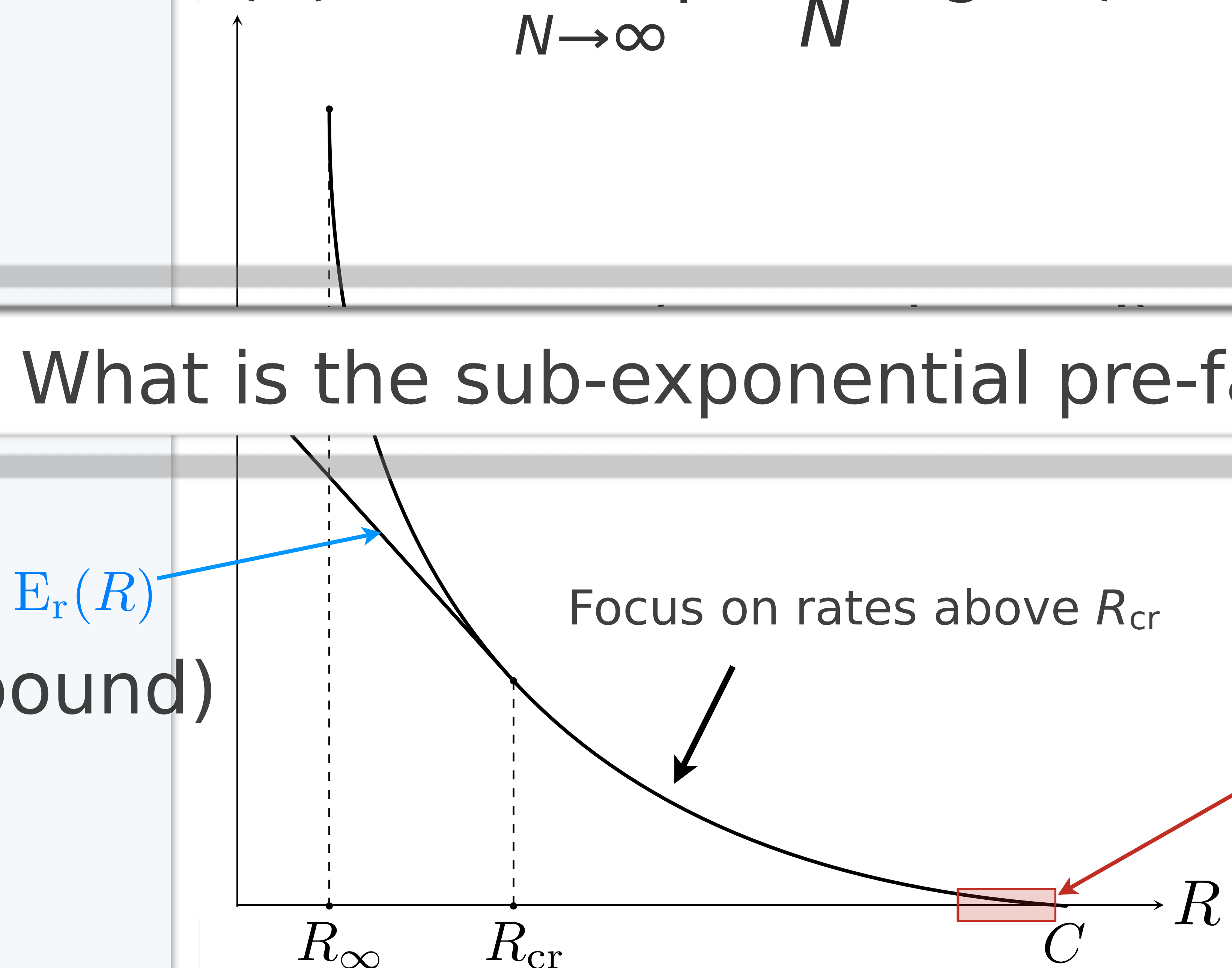

(lower bound)

Rates of interest

Pre-factors From the Literature

Upper bounds

Lower bounds

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Fano '61: $2 \cdot 2^{-NE(R)}$

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Initial Guess

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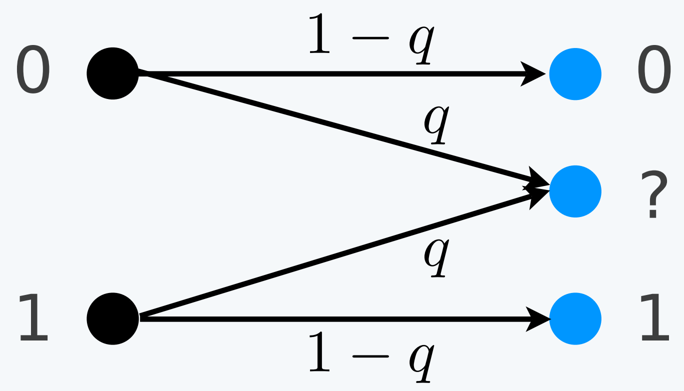
Elias '55

- Binary Erasure Channel (BEC):



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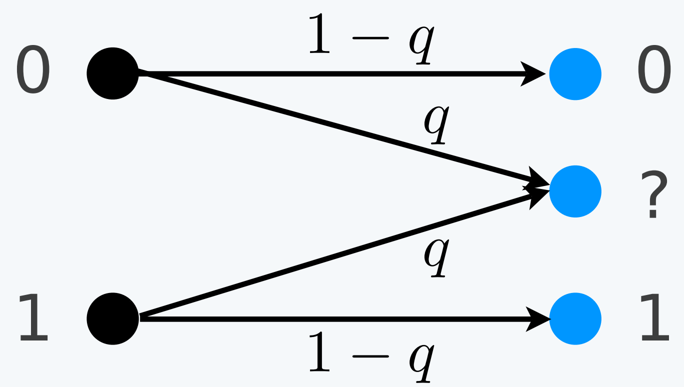

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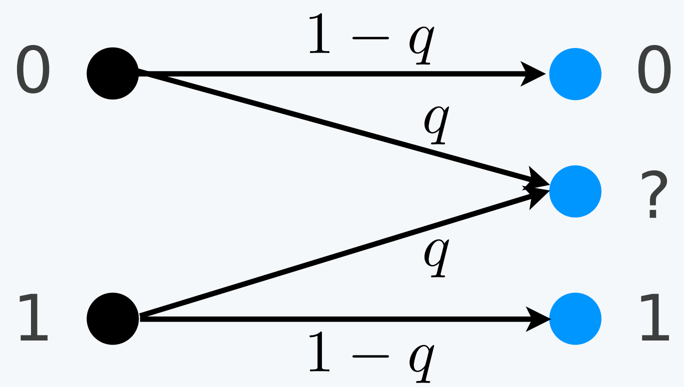

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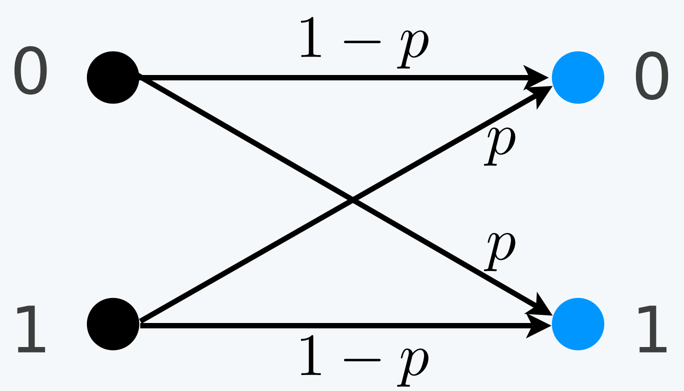

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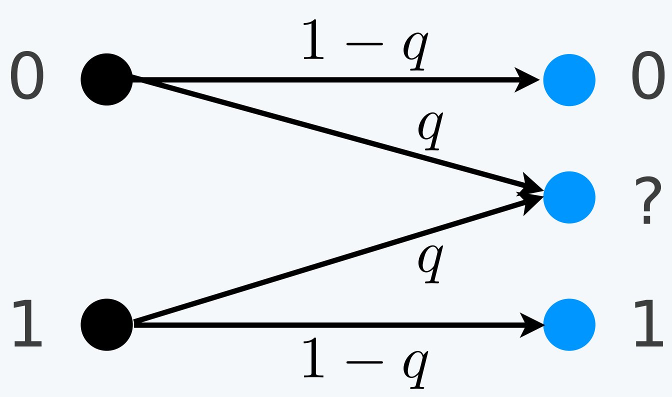

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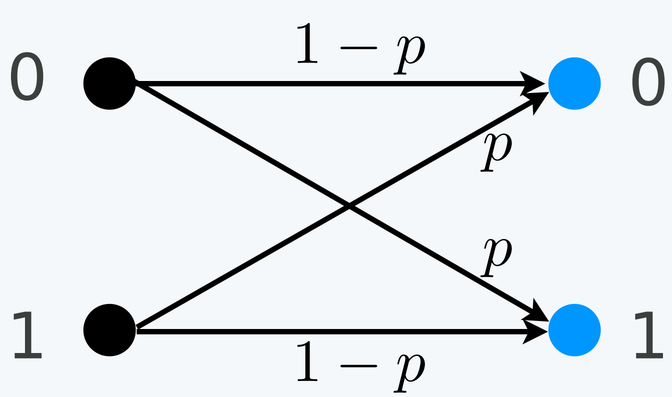

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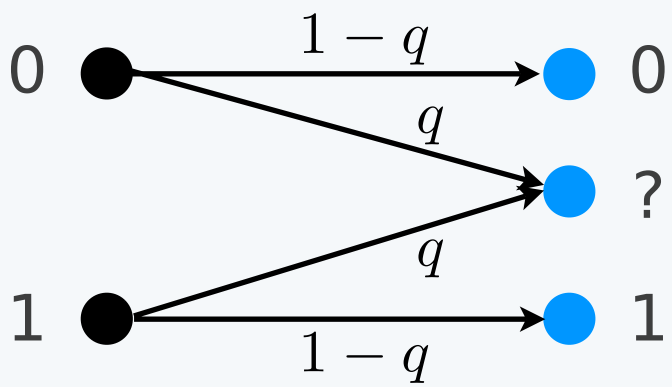

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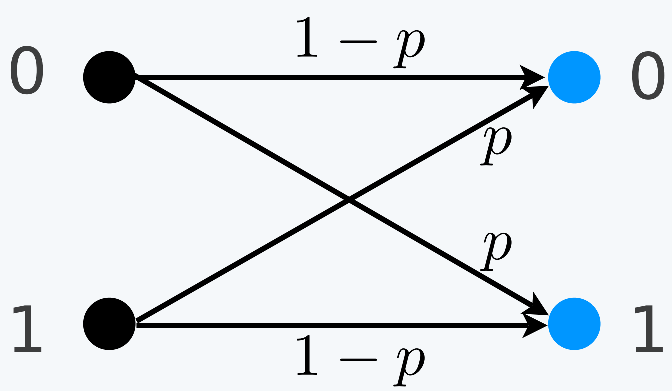

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Is it a "zoo?"

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General Channels

Theorem [Altuğ-Wagner '12]:
For any *(N,R)* constant composition code and any $\epsilon > 0$,

$$\mathrm{P}_e \geq \frac{K_3(R,\epsilon)}{N^{0.5(1+|\mathrm{E}'(R-)|+\epsilon)}} 2^{-N\mathrm{E}(R)}$$

Summary of Results

Pre-factor Bounds	Symmetric	Asymmetric
Regular	$O(N^{0.5(1+\lvert\mathrm{E}'(R)\rvert)})$ $\Omega(N^{0.5(1+\lvert\mathrm{E}'(R)\rvert)})$	$O(N^{0.5(1+\lvert\mathrm{E}'(R-)\rvert)})$ $\Omega(N^{0.5(1+\lvert\mathrm{E}'(R-)\rvert)})$
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Red: constant composition codes only

Summary of Results

What is symmetric?

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Why this?

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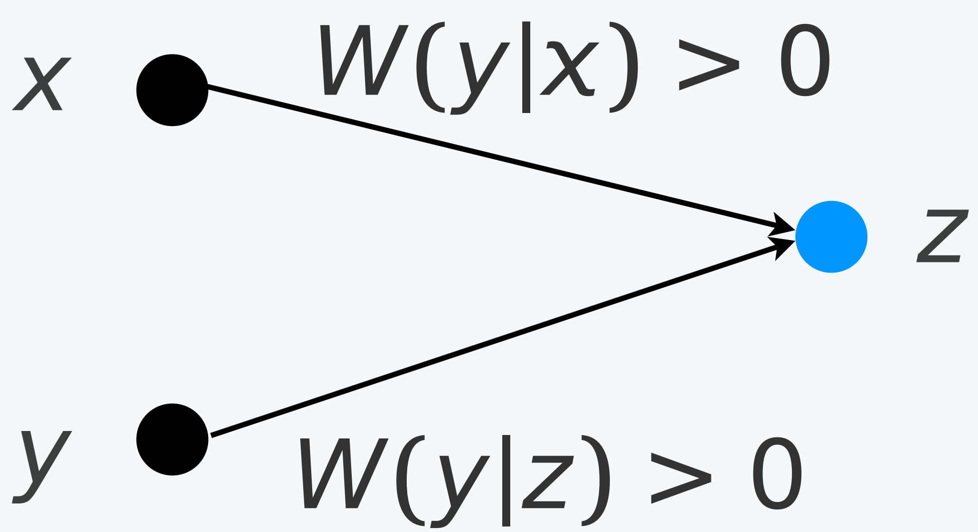
Not symmetric

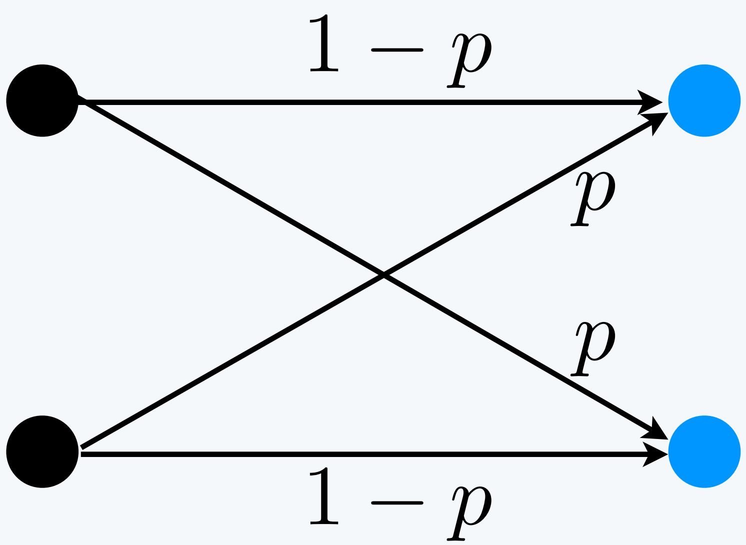
Regular Channels

Definition: A symmetric channel W is *regular* if for some x, y, and z:



$$W(y|x) \neq W(y|z)$$

Regular Channels



Regular

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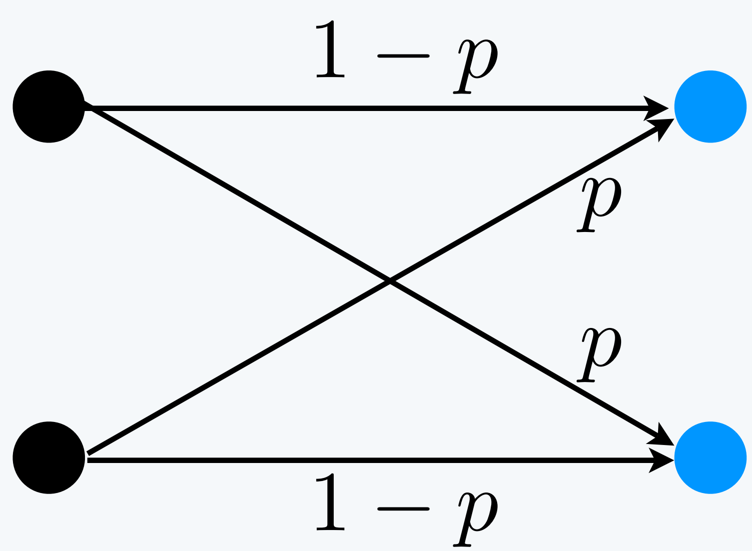

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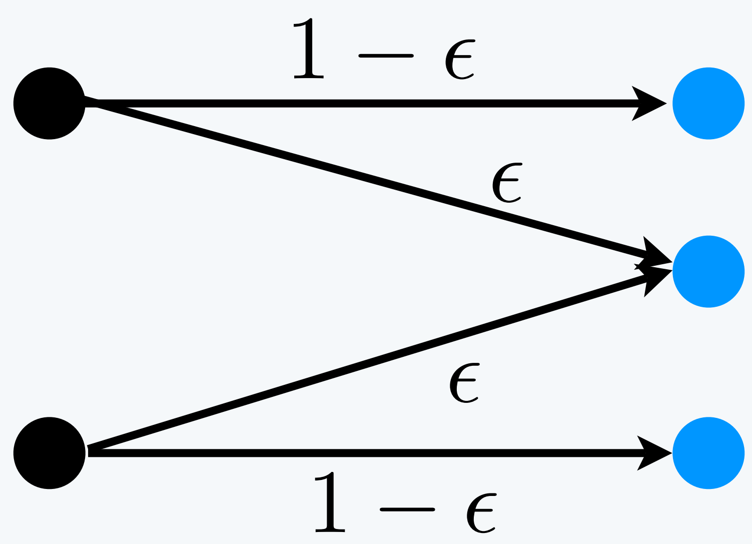

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Why this?

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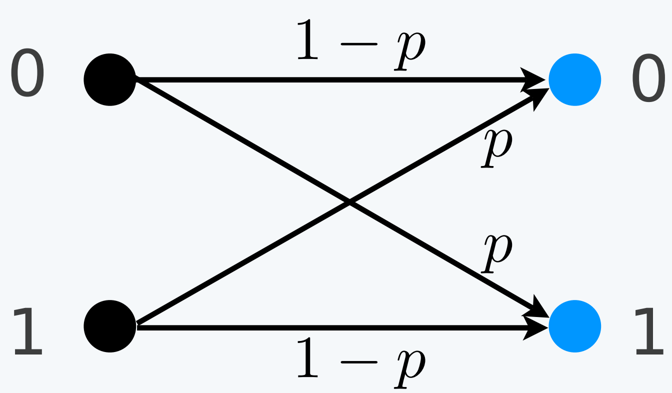
Why the Slope Thingy?

Binary Symmetric Channel (BSC):



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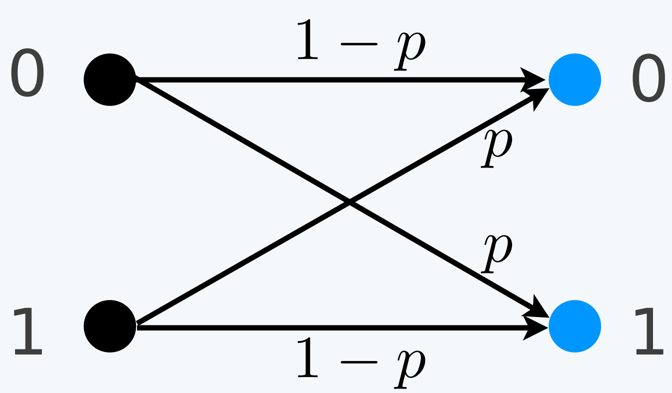
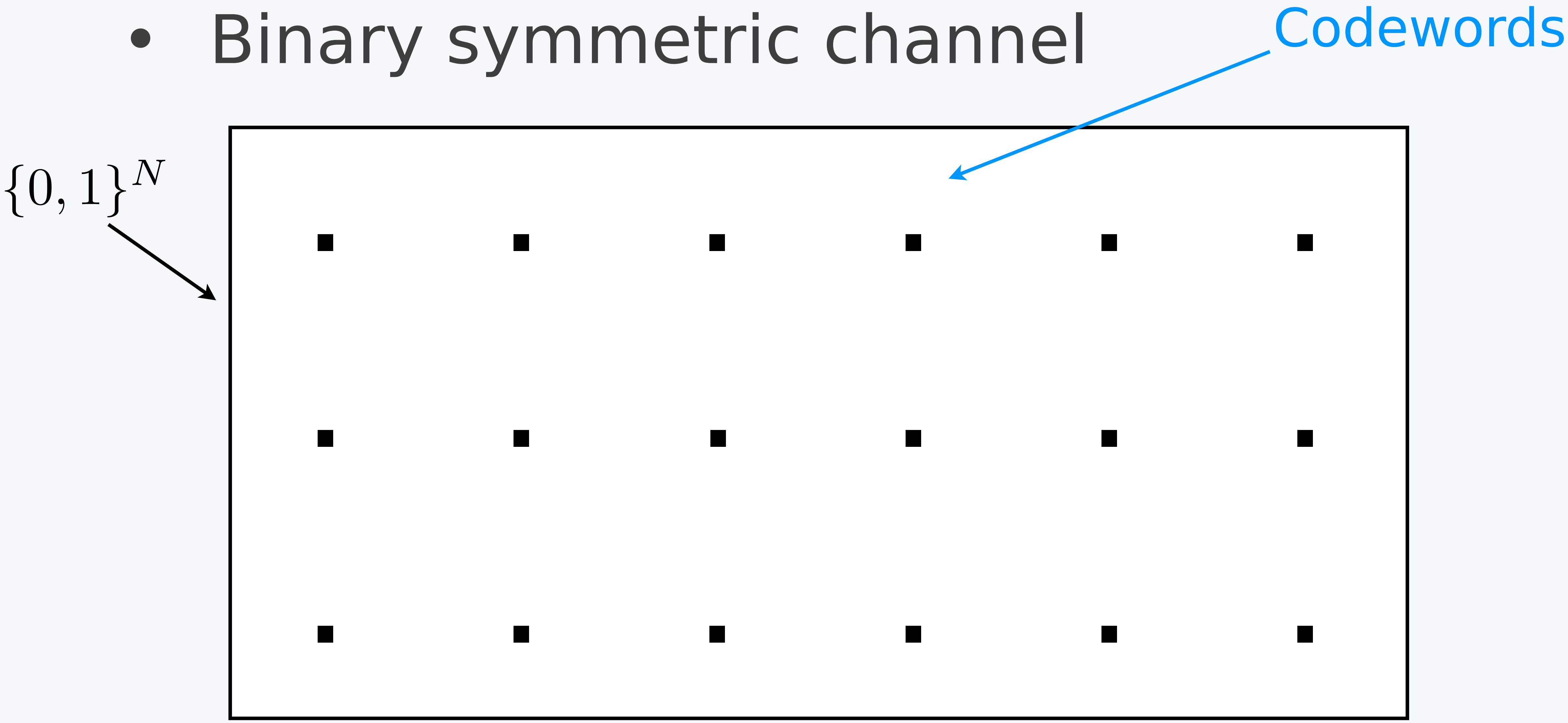


$$\mathrm{P_e}(N,R) = \Theta\left(N^{-0.5(1+|\mathrm{E}'(R)|)}\right) 2^{-N\mathrm{E}(R)}$$

Packing Spheres

- Binary symmetric channel



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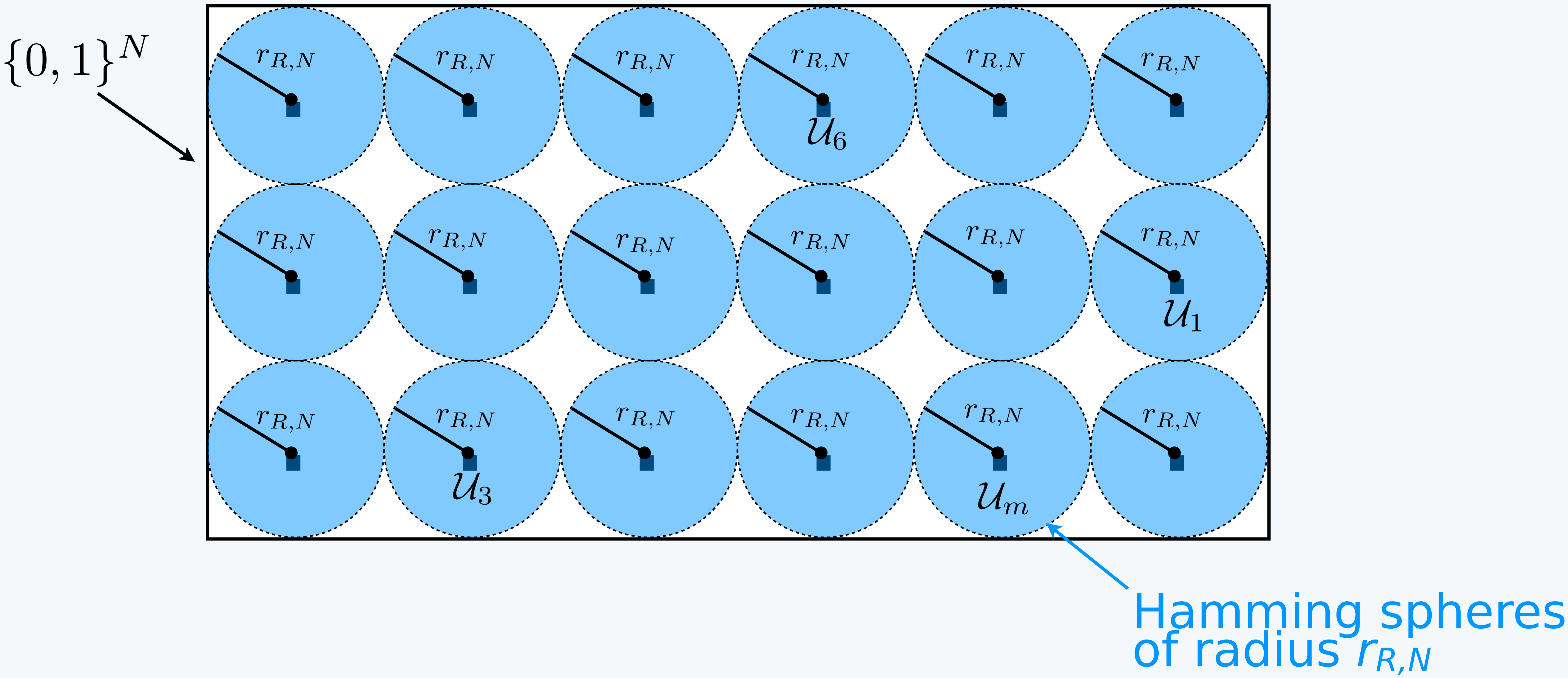

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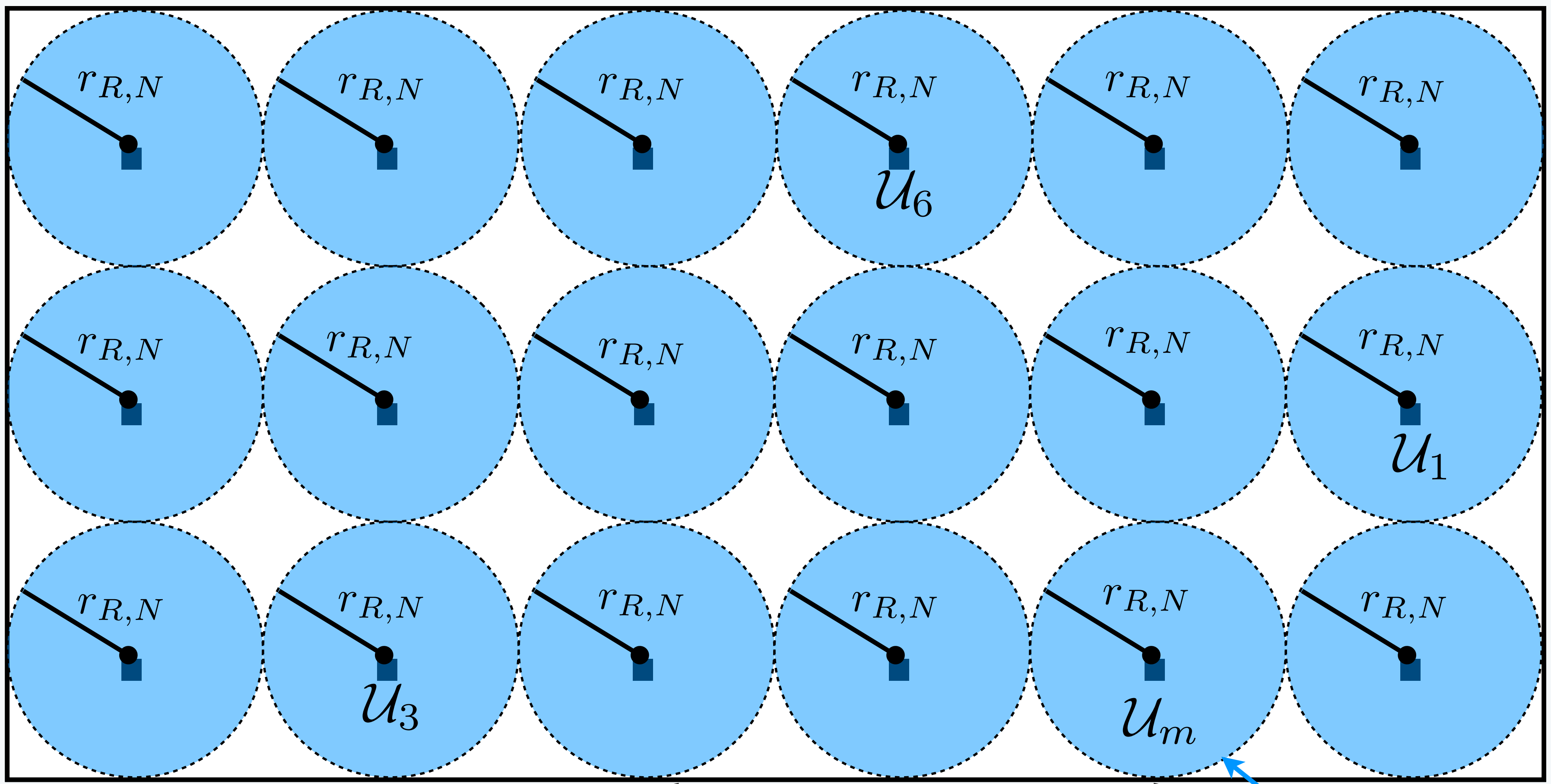
$\{0,1\}^N$



Hamming spheres of radius $r_{R,N}$

$$P_e(N,R) \approx \Pr\left(\sum_{i=1}^{N} T_i > r_{R,N}\right)$$

i.i.d. channel bit flips

$$\approx \frac{c}{\sqrt{N}} 2^{-N\cdot E(R)}?$$

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What about constants/lower order behavior?

Red: constant composition codes only

A Criticism of Error Exponents

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Sergio Verdú (2007 Shannon lecture)

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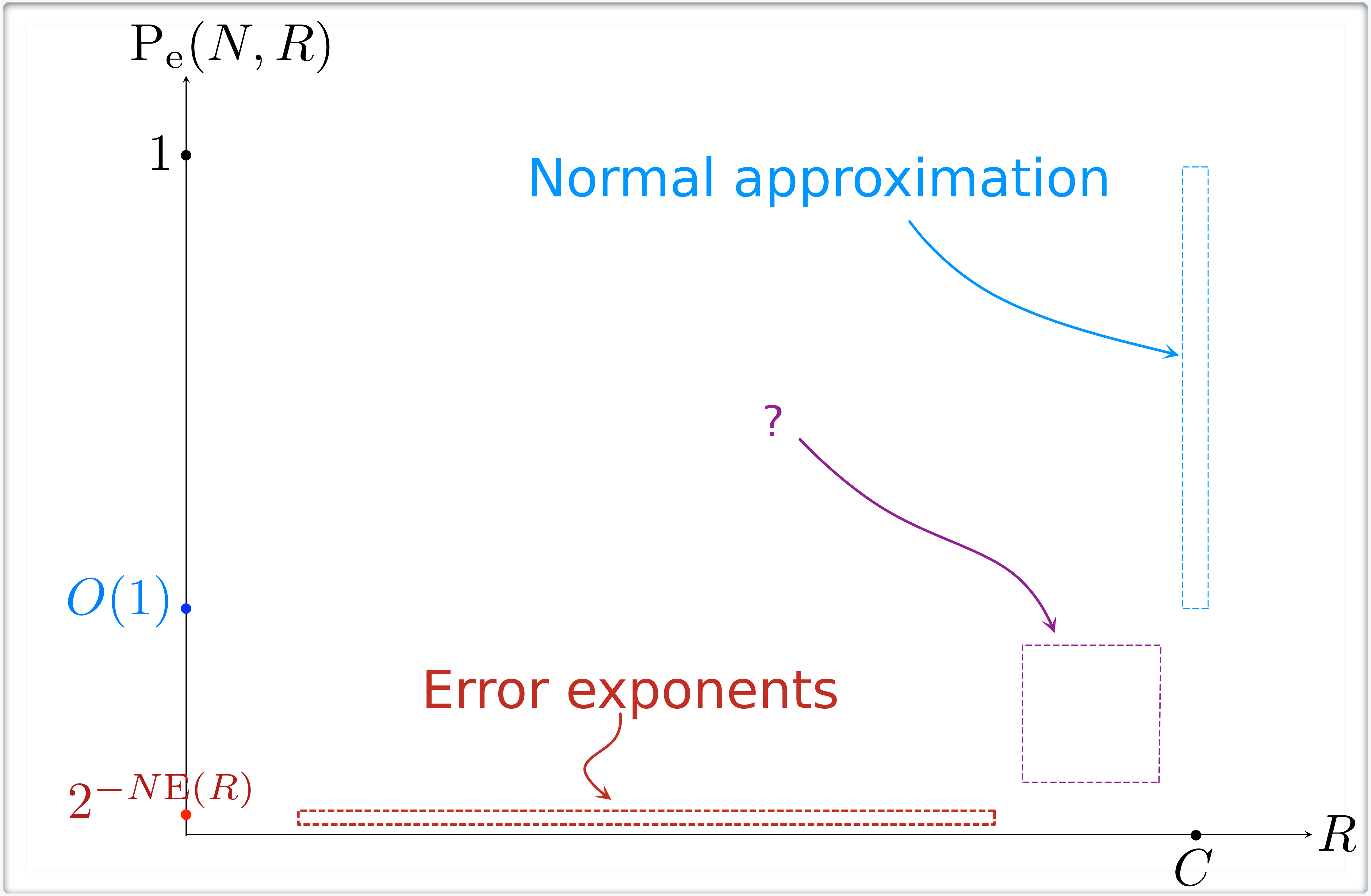
Channel dispersion

Analogous to CLT. Is it a better approach?

Fix some large *N*.



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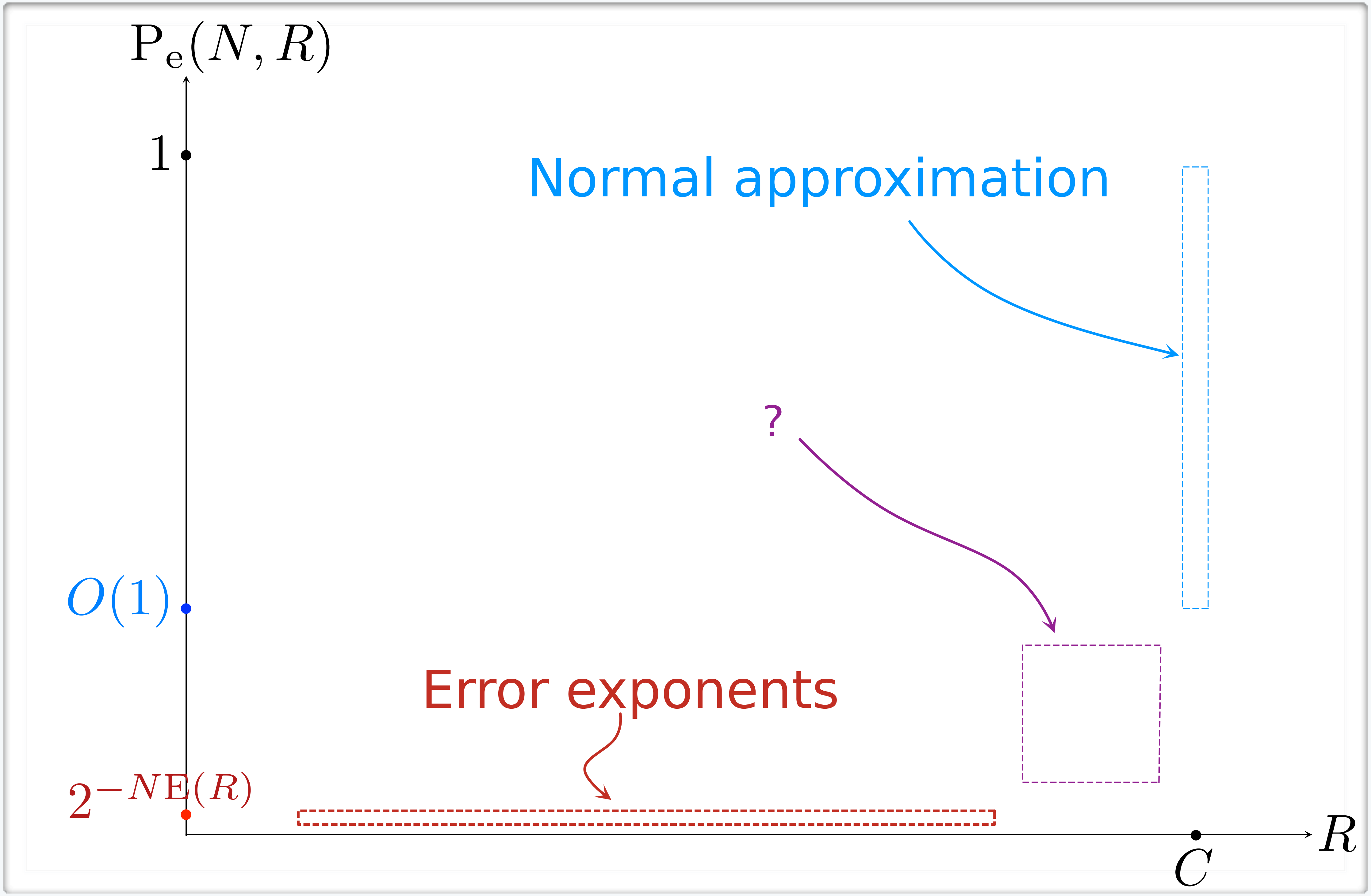

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- What is in between?

Moderate Deviations

Theorem: [Altuğ-Wagner ’10]

- Let $R_N = C$ - ϵ_N and assume

$$\lim_{N\to\infty} \epsilon_N = 0 \qquad \lim_{N\to\infty} \epsilon_N \sqrt{N} = \infty$$

- Then

$$\lim_{N\to\infty} -\frac{\log \mathrm{P_e}(N, R_N)}{\epsilon_N^2 N} = \frac{1}{2V}$$

$$\left(\mathrm{P_e}(N, R_N) \approx 2^{-\epsilon_N^2 \cdot N/(2V)}\right)$$

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Characterizing the lower-order factors would unify error exponents and moderate deviations

Analogies

I.I.D. Sums		**Channel Coding**
1. WLLN	$\longleftrightarrow$	1. Channel Coding Theorem
2. LDP	$\longleftrightarrow$	2. Error Exponents
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Too many results!

Can we unify them?

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- Lower-order results could unify regimes

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- Moderate Deviations: if β is in (1/2, 1):

$$\lim_{N \to 0} \frac{1}{N^{2\beta-1}} \log \Pr\left(\sum_{i=1}^{N} X_i > \epsilon N^{\beta}\right) = \Lambda^*_{\mathcal{N}}(\epsilon)$$