Alignment of Data

Steven B Damelin[1] Charles Fefferman [2]

[1] Mathematical Reviews, Ann Arbor, MI

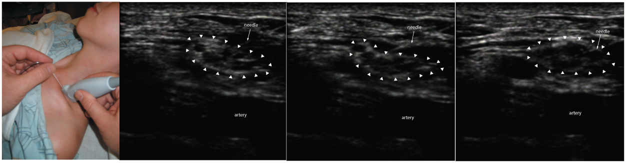
[2] Princeton, NJ

University of Michigan, January 2014

- **Basic question.**
- **"When do two sets of point clouds in some containing high dimensional Euclidean space describe approximately or exactly the 'same' manifold?"**

Example: Machine learning problem in medical imaging.

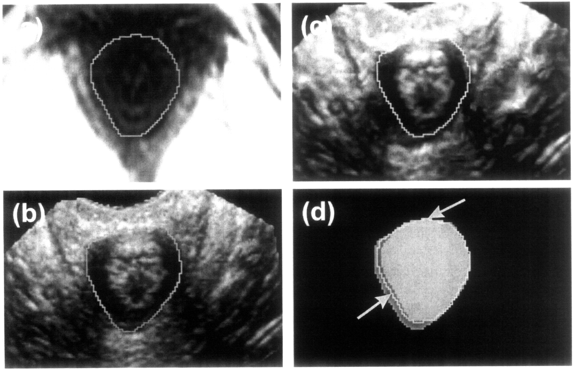
- **Several phantom training images of a certain part of the human body.**
- **Magnetic Resonance (MR) Scans.**
- **Reconstruction in some well defined sense say with high probability.**


needle
artery
needle
artery
needle
artery

Example: Recognition problem: We have multiple finite images of the pelvic floor indicating anatomical structures.

- **Anal region.**
- **Bladder.**
- **Coccyx.**
- **Lumbar vertebrae.**

Various views and you want to reconstruct the region with some high probability or exactly in some well defined sense.


(b)
(c)
(d)

Example: Identification of objects from a picture.

- **Variations in the position of the objector.**
- **Parameters of the camera.**
- **Variations in the image.**
- **Noise.**

- **Natural idea- old problem in geometry.**
- **Suppose we are given two sets of D dimensional data (in computer vision these are often called point clouds or point configurations), that is, sets of points in Euclidean D-space, where $D \geq 1$. The data sets are indexed by the same set (they are labeled) and we know that pairwise distances between corresponding points are equal in the two data sets.**
- **In other words, the sets are isometric.**
- **Can this correspondence be extended to an isometry of the ambient Euclidean space?**

- **The following statement essentially appears in Wells, J. H, Williams, L. R. Embeddings and extensions in analysis, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84. Springer-Verlag, New York-Heidelberg, 1975.**
- **Let** $D \geq 1$**. Let** $y_1, ...y_k$ **and** $z_1, ...z_k$ **be distinct points in** $\mathbb{R}^D$**. Let** $|;|$ **denote the Euclidean metric in** $\mathbb{R}^D$**. Suppose that**

$$|z_i - z_j| = |y_i - y_j|, \ 1 \leq i, j \leq k, \ i \neq j.$$

Then there exists a smooth isometry $\Psi : \mathbb{R}^D \to \mathbb{R}^D$**, ie**

$$\frac{|\Psi(x) - \Psi(y)|}{|x - y|} = 1, \ x, y \in \mathbb{R}^D, \ x \neq y$$

satisfying

$$\Psi(y_i) = z_i, \ 1 \leq i \leq k.$$

- The answer has been known for some time to be YES.

- **Equal distances used to reconstruct point configurations–not here.**
- **Reconstruction a point configuration from its distances or distributions of distances.**
- **Early 1970's in X-ray crystallography and in the mapping of sites of DNA, see for example the work of [Steflik (1978)] and [Dixa, Kieronska (1988)].**
- **Literature: (1) Turnpike and (2) Partial Digest.**
- $D \geq 1$.
- **Polynomial factorization and polynomial time algorithms [Rosenblatt-Seymour (1982); Lemke-Werman (1988)].**

- **Related Problem.**
- **Known distinct configurations**
- **Labeled: The configurations are indexed by the same set–say the natural numbers.**
- **Samples–larger–unknown sets say, manifolds in R^D.**
- **What can be said about the manifolds themselves?**
- **Difficult problem.**
- **How to even formulate?**

- **One formulation of the problem as a step to understanding.**
- **Determining whether two point configurations have an equivalent shape.**
- **Shape defined by a point configuration often called landmarks.**
- **More precisely: Does there exists a rotation and a translation or sometimes a reflection or scaling which maps the first configuration into the second.**

- **Labelled data.**
- **Rigid motions, ie rotations, translations and reflections in $\mathbb{R}^D$.**
- **Rigid motion: A pair (M, T), where M is an orthogonal m-m matrix and T is an D-dimensional column vector.**

We state the following result which is also somewhat old. We state it for labeled data. In the case of unlabeled data, a good reference is M. Boutin and G. Kemper (2007).

Let $D \geq 1$**. Let** $y_1, ...y_k$ **and** $z_1, ...z_k$ **be distinct points in** $\mathbb{R}^D$**. Let** $|;|$ **denote the Euclidean metric in** $\mathbb{R}^D$**. Suppose that**

$$|z_i - z_j| = |y_i - y_j|, \; 1 \leq i, j \leq k, \; i \neq j.$$

Then there exists a rigid motion (M, T) **such that** $My_i + T = z_i, \; i = 1, ..., k.$

- Expectation non equal distances.
- Noise.
- Demand that pairwise distances close in some reasonable metric.

- **Variations in the position of the objector.**
- **Parameters of the camera.**

- **Theoretical attempt at understanding how to extend the results to pairwise distances close in some reasonable metric.**

- Charles Fefferman, Steven.B.Damelin and William Glover, BMO Theorems for ε distorted diffeomorphisms on $\mathbb{R}^D$ and an application to comparing manifolds of speech and sound, Involve, a Journal of Mathematics 5-2 (2012), 159–172. DOI 10.2140/involve.2012.5.159.
- Charlie Fefferman and S.B.Damelin On the Approximate and Exact Alignment of δ distorted Data in Euclidean Space II, manuscript.
- Charlie Fefferman and S.B.Damelin, On the Approximate and Exact Alignment of δ distorted Data in Euclidean Space I, manuscript.

- **Throughout, we work in $\mathbb{R}^D$ where $D \geq 1$ is fixed.**
- **Whenever, we suppose we are given a positive number ε, we assume ε is less than equal to a small enough positive constant determined by D.**
- **Throughout, $O(D)$ denotes the group of orthogonal matrices of size d generated by rotations and reflections.**
- **$SO(D)$ denotes the elements of $O(D)$ of determinant 1 or the subgroup of $O(d)$ generated by the rotations.**

- **We are interested in the following question.**
- **Suppose we are given sets of distinct labeled points of size** k**,** $y_1, ..., y_k$ **and** $z_1, ..., z_k$ **in** $\mathbb{R}^D$**. Assume that the points are distorted by a aprori fixed amount, ie there exists** $\delta > 0$ **such that for all** i, j
$$(1-\delta) \leq \frac{|y_i - y_j|}{|z_i - z_j|} \leq (1+\delta).$$
- **Do there exist rigid motions** Φ **which approximately or exactly align the two sets of labeled points?**
- **Henceforth** $y_1, ..., y_k$ **and** $z_1, ..., z_k$ **in** $\mathbb{R}^D$**, we will suppose that the** y_i **as well as the** z_i **are all distinct.**

- **Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be a diffeomorphism. (In particular, Φ is one to one and onto).**
- **We say that Φ is "$\varepsilon > 0$-distorted" provided**

$$(1+\varepsilon)^{-1} I \leq [\Phi'(x)]^T [\Phi'(x)] \leq (1+\varepsilon) I$$

as matrices, for all $x \in \mathbb{R}^D$.
- **I denotes the identity matrix in $\mathbb{R}^D$.**
- **A is a self adjoint matrix, then by $A \geq 0$, we mean that A is semi-positive definite.**

- **Suppose Φ is ε-distorted. If τ is a piecewise smooth curve in $\mathbb{R}^D$, then the length of $\Phi(\tau)$ differs from that of τ by at most a factor of $(1+\varepsilon)$.**
- **Consequently, if $x, x' \in \mathbb{R}^D$, then $|x - x'|$ and $|\Phi(x) - \Phi(x')|$ differ by at most a factor $(1+\varepsilon)$.**

- **A "'Euclidean motion"' is a map $x \to Tx + x_0$ from $\mathbb{R}^D \to \mathbb{R}^D$ with $T \in O(D)$.**
- **The Euclidean motion is proper"' if $T \in SO(D)$. Otherwise, the motion is "'improper".**

- Construction of required matching ε-distorted diffeomorphisims of $\mathbb{R}^D$.
- Slow twists.
- Slides.

- **Boutin-Kemper generalization.**

Theorem (Damelin-Fefferman)

Given $\varepsilon > 0$ and $k \geq 1$, there exists $\delta > 0$ such that the following holds. Let $y_1, ...y_k$ and $z_1, ..., z_k$ be points in $\mathbb{R}^D$. Suppose

$$(1+\delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq 1 + \delta,\ i \neq j.$$

Then, there exists a Euclidean motion $\Phi_0 : x \to Tx + x_0$ such that

$$|z_i - \Phi_0(y_i)| \leq \varepsilon \mathrm{diam}\,\{y_1, ..., y_k\}$$

for each i. If $k \leq D$, then we can take Φ_0 to be a proper Euclidean motion on $\mathbb{R}^D$.

Theorem (Damelin-Fefferman)

Let $k \geq 1$. There exist constants $\alpha, \alpha' > 0$ depending on k and D such that the following holds. Let $\delta > 0$. Let $y_1, ...y_k$ and $z_1, ..., z_k$ be points in $\mathbb{R}^D$ scaled so that

$$\sum_{i \neq j} |y_i - y_j|^2 + \sum_{i \neq j} |z_i - z_j|^2 = 1.$$

Suppose

$$||z_i - z_j| - |y_i - y_j|| < \delta.$$

Then, there exists a Euclidean motion $\Phi_0 : x \rightarrow Tx + x_0$ such that

$$|z_i - \Phi_0(y_i)| \leq \alpha' \delta^\alpha$$

for each i. If $k \leq D$, then we can take Φ_0 to be a proper Euclidean motion on $\mathbb{R}^D$.

- **Wells-Williams generalization.**

Theorem (Damelin-Fefferman)

Let $\varepsilon > 0$, $D \geq 1$ and $1 \leq k \leq D$. Then there exists $\delta > 0$ such that the following holds: Let $y_1, ...y_k$ and $z_1, ...z_k$ be distinct points in $\mathbb{R}^D$. Suppose that

$$(1+\delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1+\delta),\ 1 \leq i,j \leq k,\ i \neq j.$$

Then there exists a diffeomorphism, $\Psi : \mathbb{R}^D \to \mathbb{R}^D$ with

$$(1+\varepsilon)^{-1} \leq \frac{|\Psi(x) - \Psi(y)|}{|x - y|} \leq (1+\varepsilon),\ x, y \in \mathbb{R}^D,\ x \neq y$$

satisfying

$$\Psi(y_i) = z_i,\ 1 \leq i \leq k.$$

- **Let $\varepsilon > 0$, a dimension D and $k \leq D$ be given. There exists $\delta > 0$, $\delta = \delta(\varepsilon; k)$, so that any $(1 + \delta)$-bilipschitz mapping ϕ of k points from $\mathbb{R}^D$ into $\mathbb{R}^D$ can be extended to a $(1 + \varepsilon)$ bilipchitz diffeomorphism of $\mathbb{R}^D$ onto $\mathbb{R}^D$.**
- **An example showing that for $2D + 1$ points in $\mathbb{R}^D$ such an extension result does not hold in general exists.**

- **Roughly: the number of points still has to be finite but no longer bounded by D.**
- **Instead, we understand now that , very roughly speaking, what is required is that on any $D+1$ of the k points which form vertices of a relatively volu-minous simplex, the mapping ϕ is orientation preserving.**
- **A complementary statement about when the extension does not exist is also fairly well understood by us.**

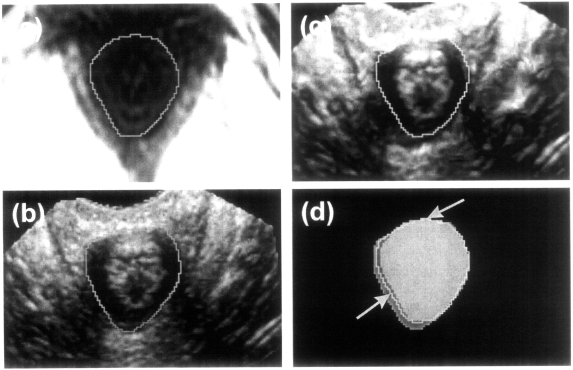
- **Fine analysis between δ and ε.**
- **There exist positive constants C and C' so that in many situations we investigate we have exact alignment up to rigid/proper rigid transformations for $\delta = \exp(-C/\epsilon)$ and we do not for $\delta = \exp(-C'/\epsilon)$.**

Ongoing Program of Research.

- **Enormous difficulty in applications in trying to identify point configurations up to rigid motions is the absence of labels for the points, one often does not know which point is mapped to which.**
- **Distances between the points are replaced by a suitable distribution and unlabeling is done via permutation groups.**

- **Metrics sensitive to bending, preserving area and volume.**
- **Projective and camera rotations, flows, depth.**
- **Ongoing work on Brachial Plexus and Pelvic Floor**

- **Numerical implementation.**
- **Dimension reduction.**


(b)
(d)