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(JOINT WORK WITH ELAD HAZAN — PRINCETON)

FASTER CONVEX OPTIMIZATION

SIMULATED ANNEALING WITH AN EFFICIENT UNIVERSAL BARRIER

THIS TALK — OUTLINE

1. The goal of Convex Optimization
2. Interior Point Methods and Path following
3. Hit-and-Run and Simulated Annealing
4. The Annealing-IPM Connection
5. Faster Optimization

GENERAL CONVEX OPTIMIZATION PROBLEM

- Let K be a bounded convex set, we want to solve

$$\min_{x \in K} \theta^\top x$$

- Can always convert non-linear objective into a linear one

$$\min_{x \in K} f(x) \quad \to \quad \min_{\substack{(x,c) \in K \times \mathbb{R} \\ f(x) \leq c}} c$$

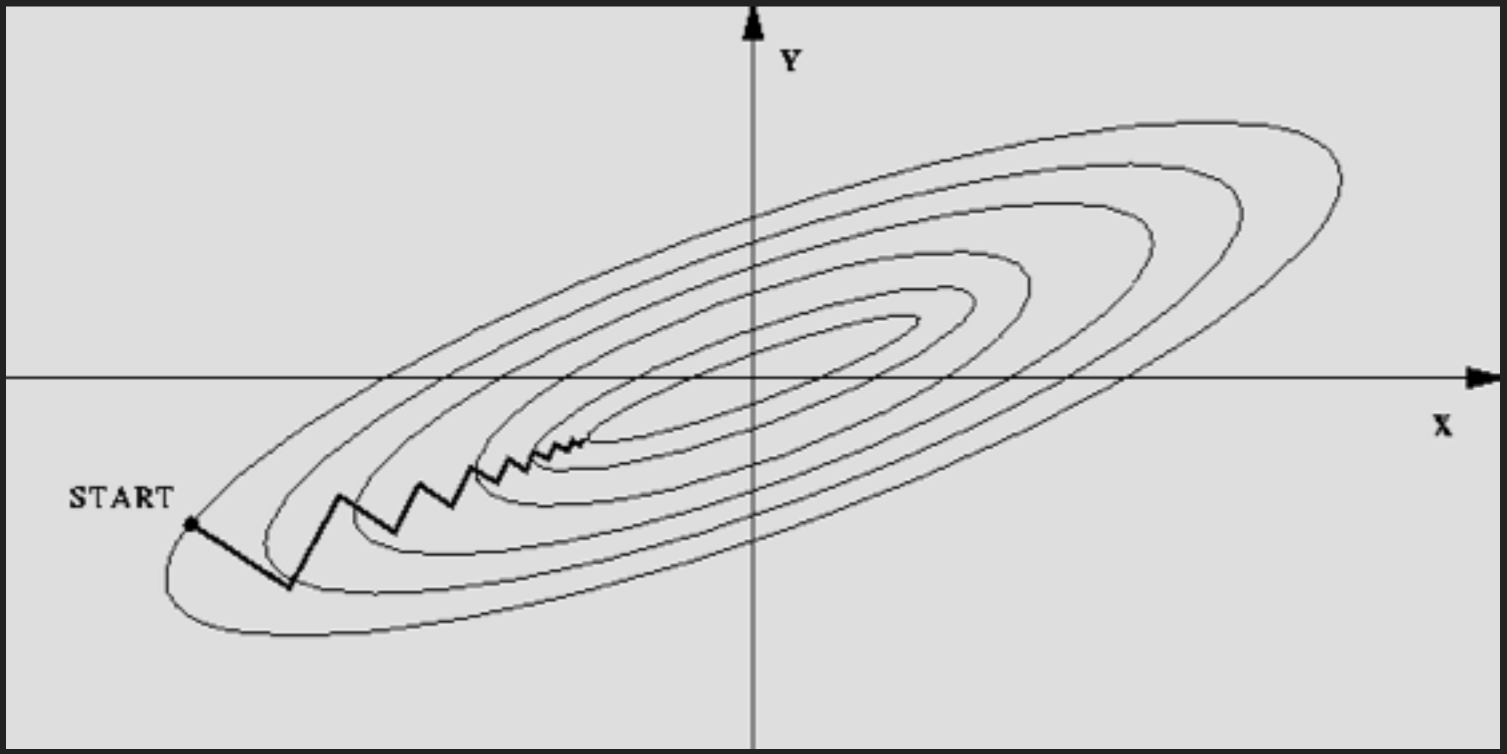
THE GRADIENT DESCENT ALGORITHM

- The gradient descent algorithm:

$$\begin{aligned}&\text{For } t = 1, 2, \ldots :\\ &\qquad \tilde{x}_t = x_{t-1} - \eta \nabla f(x_{t-1})\\ &\qquad x_t = \mathrm{Proj}_K(\tilde{x}_t)\end{aligned}$$

- Challenge: the Projection step can often be just as hard as the original optimization

GRADIENT DESCENT NOT IDEAL WITH LOTS OF CURVATURE



- The gradient descent algorithm doesn't use any knowledge of the curvature of objective function

USE THE CURVATURE: NEWTON'S METHOD

- Newton's Method is a "smarter" version of gradient descent, moves along the gradient after a transformation

BAD: Need to invert NxN mtx requires possible O(n^2.373…)

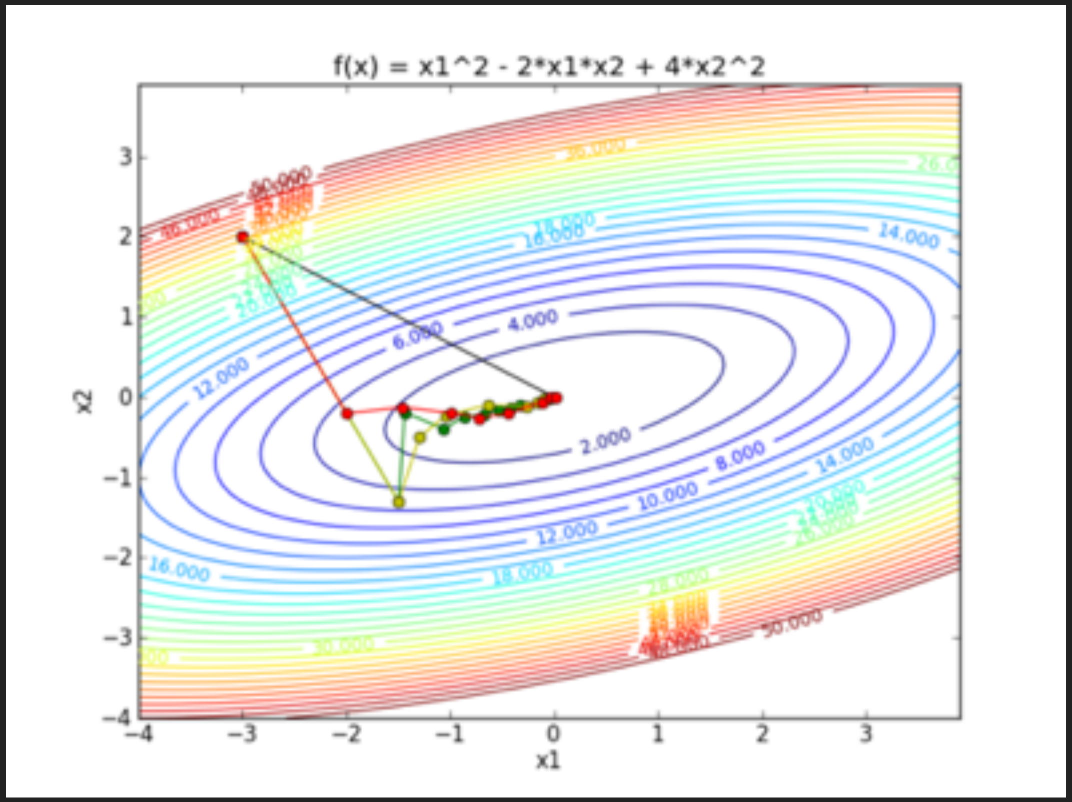
$$
\begin{aligned}
&\text{For } t = 1, 2, \ldots : \\
&\qquad \tilde{x}_t = x_{t-1} - \nabla^{-2} f(x_{t-1}) \nabla f(x_{t-1}) \\
&\qquad x_t = \mathrm{Proj}_K(\tilde{x}_t)
\end{aligned}
$$

GOOD: Typically this step is not required

NEWTON'S METHOD VERSUS GRADIENT DESCENT

- For a quadratic function, one only needs a **single newton step** to reach the global minimum

$$\text{For } t = 1, 2, \ldots :$$
$$\tilde{x}_t = x_{t-1} - \nabla^{-2} f(x_{t-1}) \nabla f(x_{t-1})$$
$$x_t = \text{Proj}_K(\tilde{x}_t)$$



WAIT! OUR ORIGINAL OBJECTIVE ISN'T CURVED…

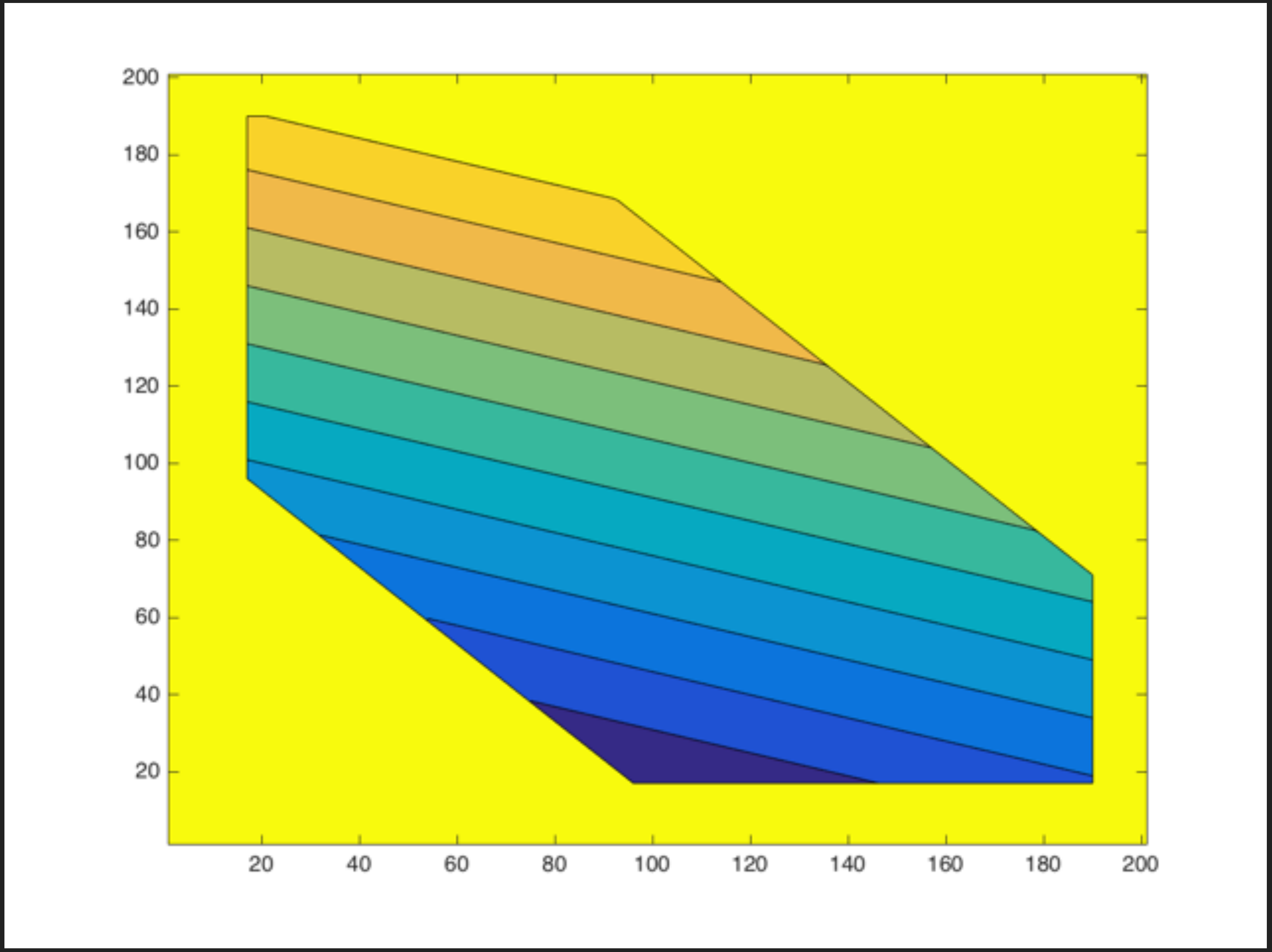
- How does this help us with linear optimization?

$$\min_{x \in K} \theta^\top x + \phi(x)$$

- Add a curved function $\phi()$ to the objective!
- $\phi()$ should be "super-smooth" (more on this later)
- $\phi()$ should be a "barrier", i.e. goes to ∞ on the boundary, but not too quickly!

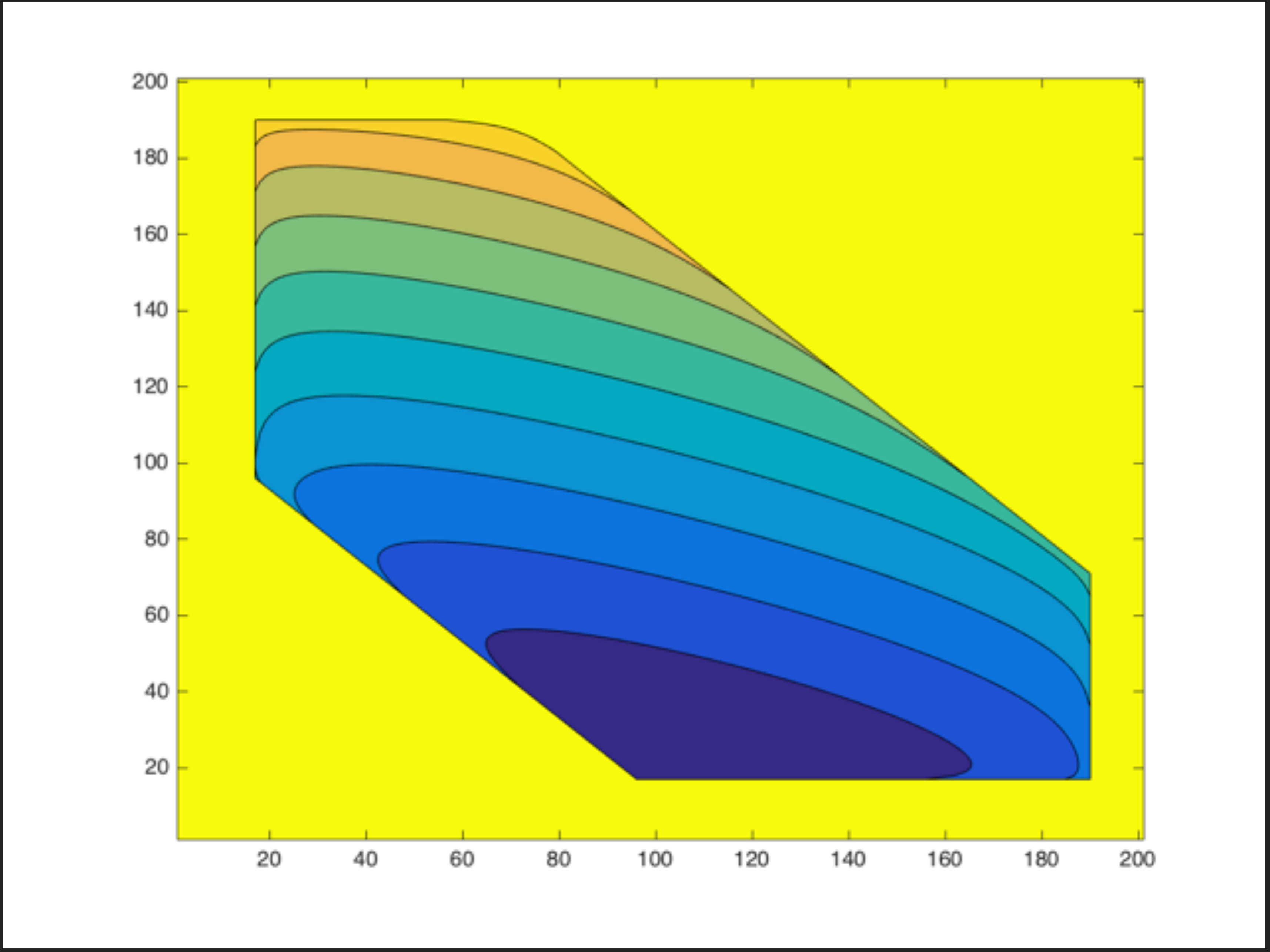
OPTIMIZATION WITHOUT A BARRIER

$$\min_{x \in K} \theta^\top x$$



OPTIMIZATION WITH A BARRIER

$$\min_{x \in K} \theta^\top x + \phi(x)$$



WHAT IS A GOOD BARRIER?

- What is needed for this "barrier func." $\phi()$?
- Canonical example: if set is a polytope K = {x : Ax ≤ b} then the *logarithmic barrier* suffices: $\phi(x) = -\sum_i \log(b_i - A_i x)$
- In general, Nesterov and Nemirovski proved that the following two conditions are sufficient. Any function satisfying these conditions is a ***self-concordant barrier:***

$$\begin{aligned} \nabla^3\phi[h,h,h] &\leq 2(\nabla^2\phi[h,h])^{3/2}, \text{ and} \\ \nabla\phi[h] &\leq \sqrt{\nu\nabla^2\phi[h,h]}, \end{aligned}$$

- ν is the *barrier parameter* which will be important later

ALGORITHM: INTERIOR POINT PATH FOLLOWING METHOD

- Nesterov and Nemirovski developed the sequential "path following" method, described as follows:

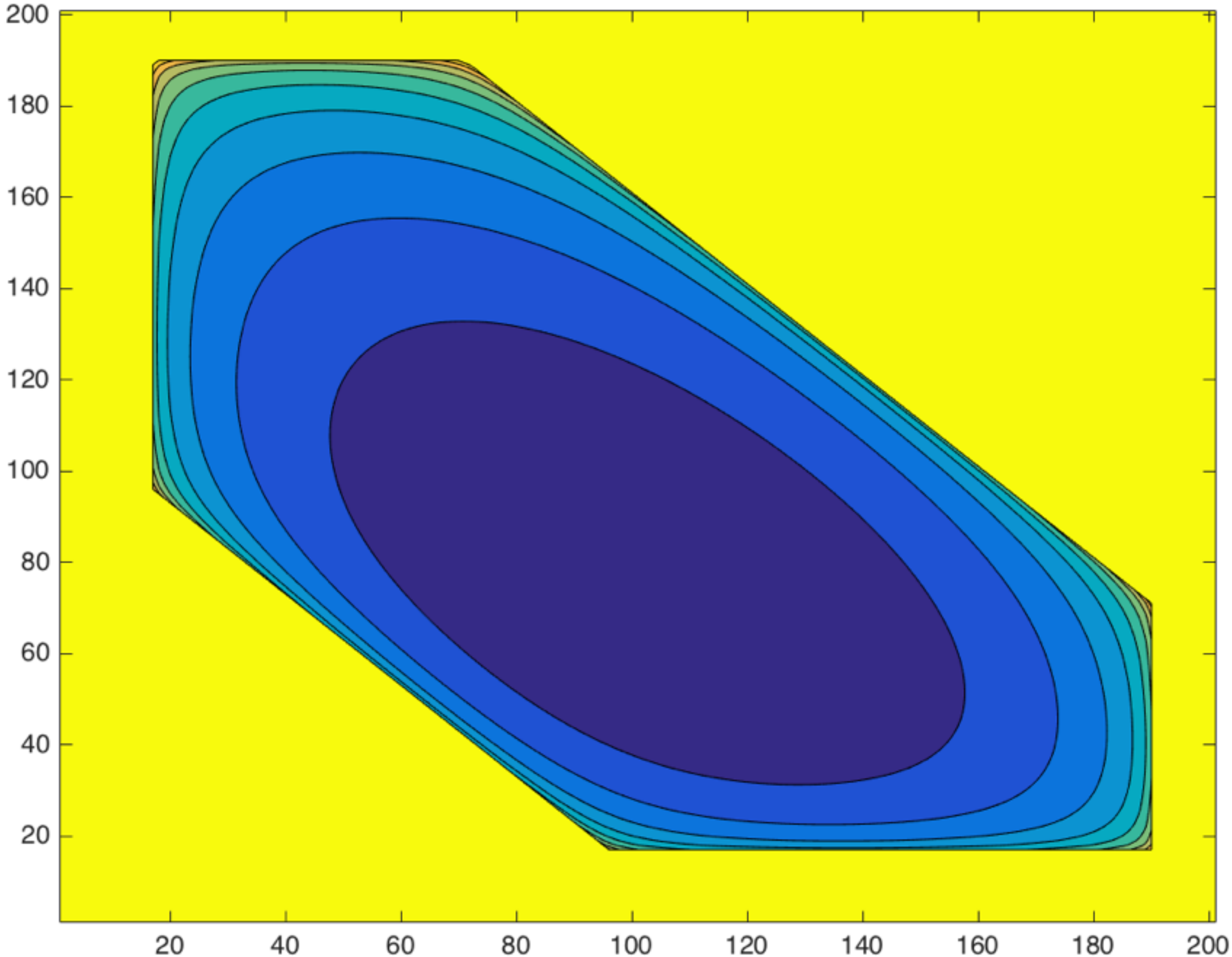
- Let $\alpha = (1 + 1/\sqrt{\nu}) \quad \text{the ``inflation'' rate}$
- For t=1,2,...

1. Update temperature: $f_k(x) := \alpha^k(\theta^\top x) + \phi(x)$
2. Newton update: $\hat{x} \leftarrow \hat{x} - \frac{1}{1+c_k}\nabla^{-2} f_k(\hat{x}) \nabla f_k(\hat{x})$

WHAT DOES THE SEQUENCE OF OBJECTIVES LOOK LIKE?

$$f_k(x) := \alpha^k(\theta^\top x) + \phi(x)$$

- Let's show these objective function as we increase k!!



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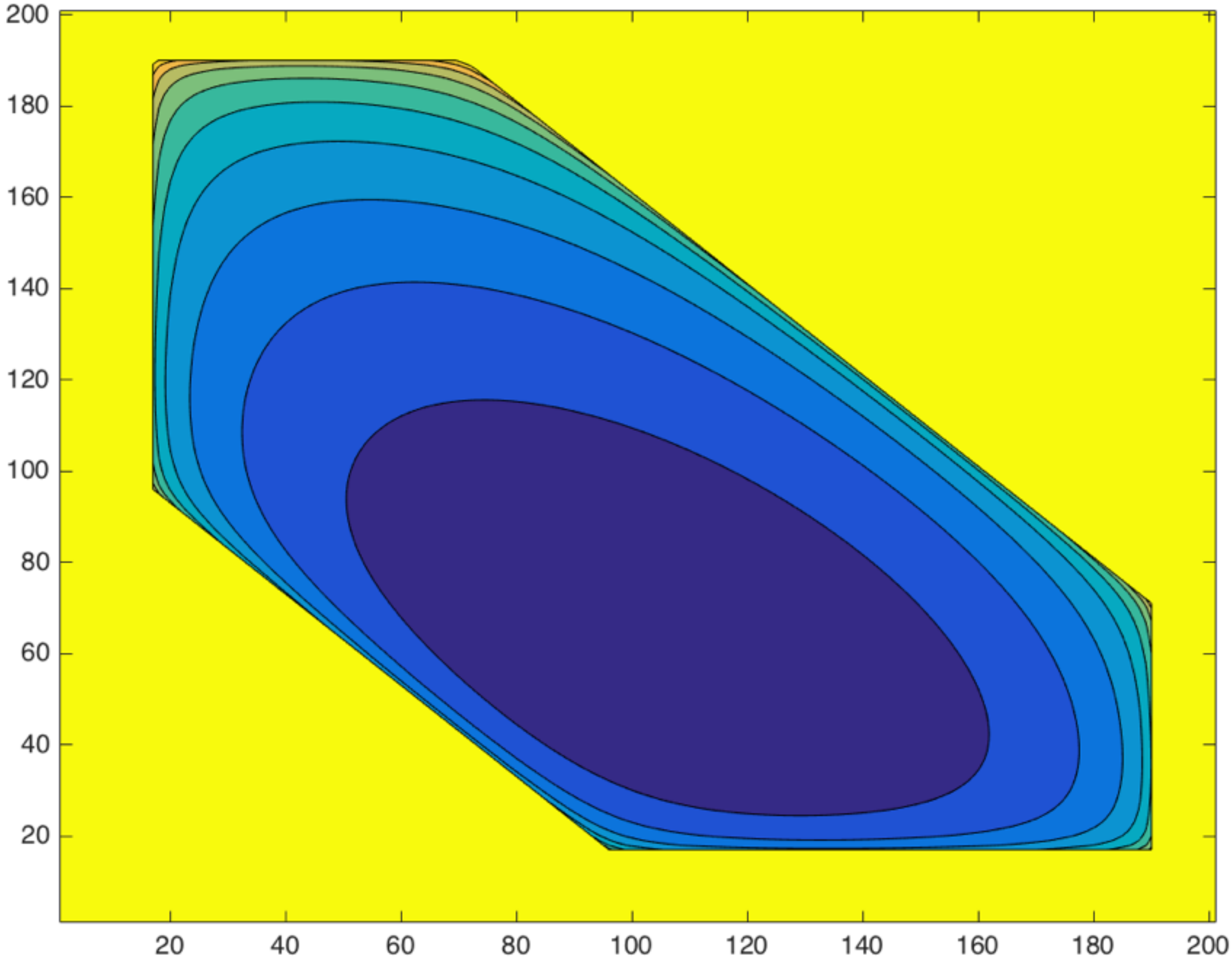
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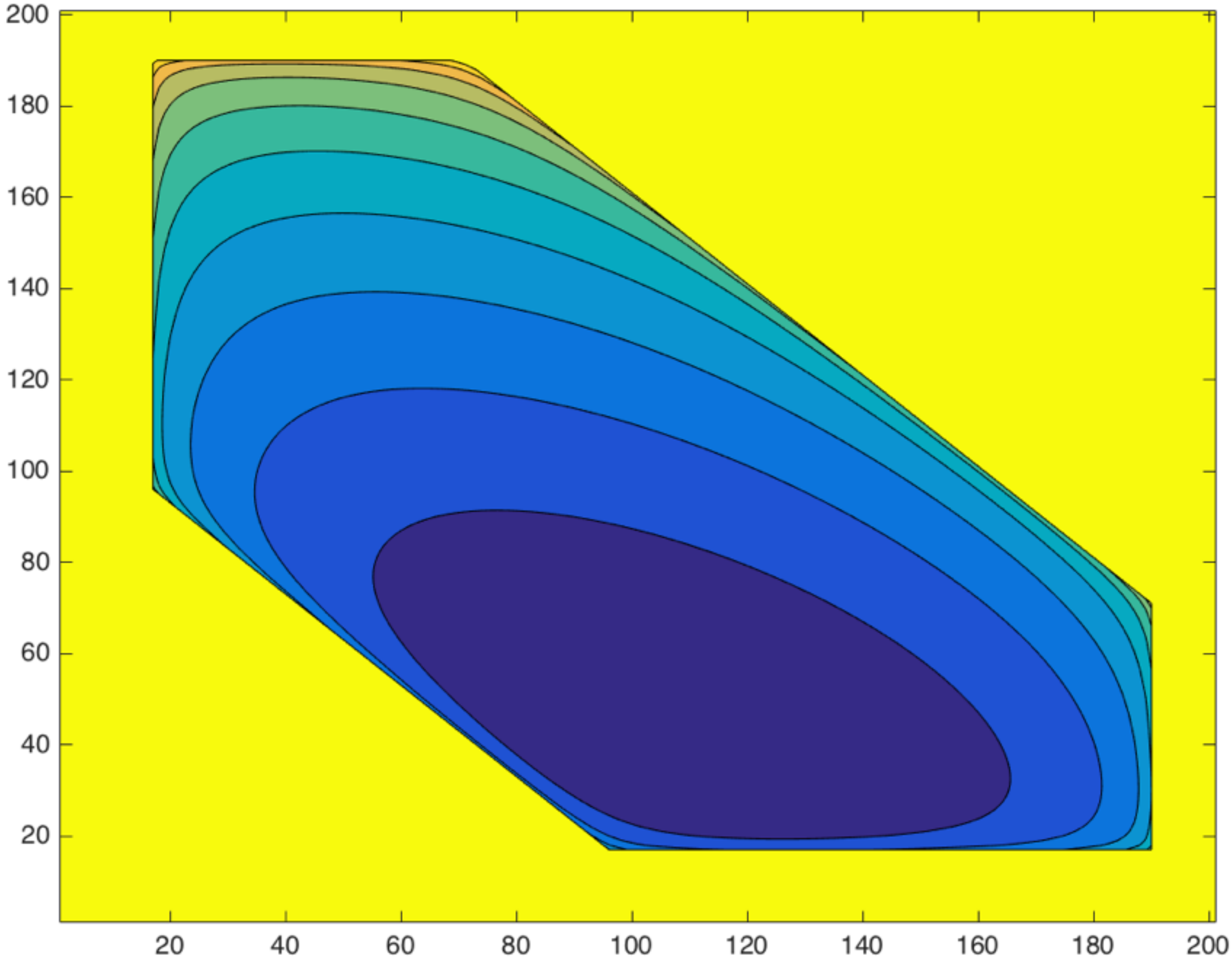

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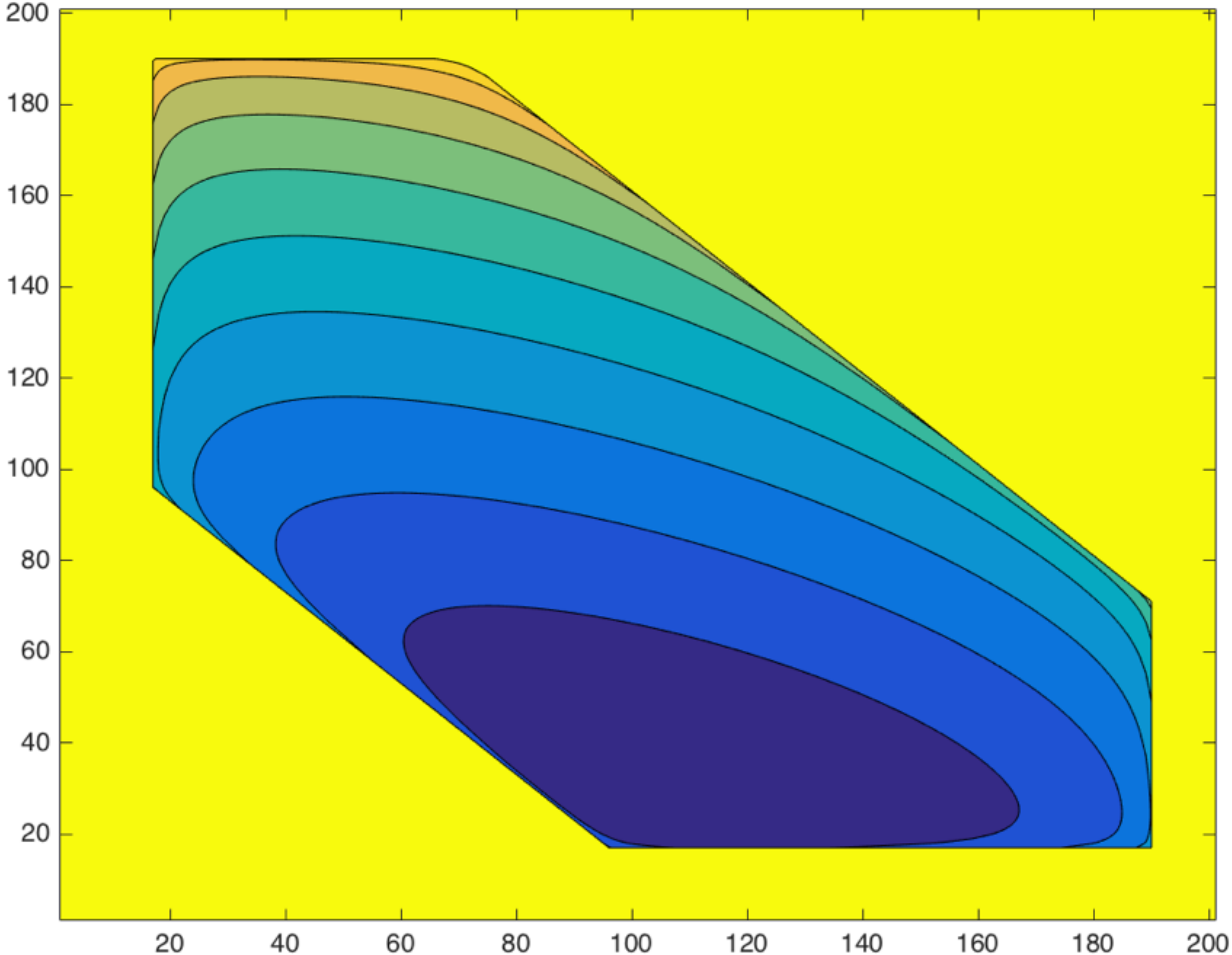
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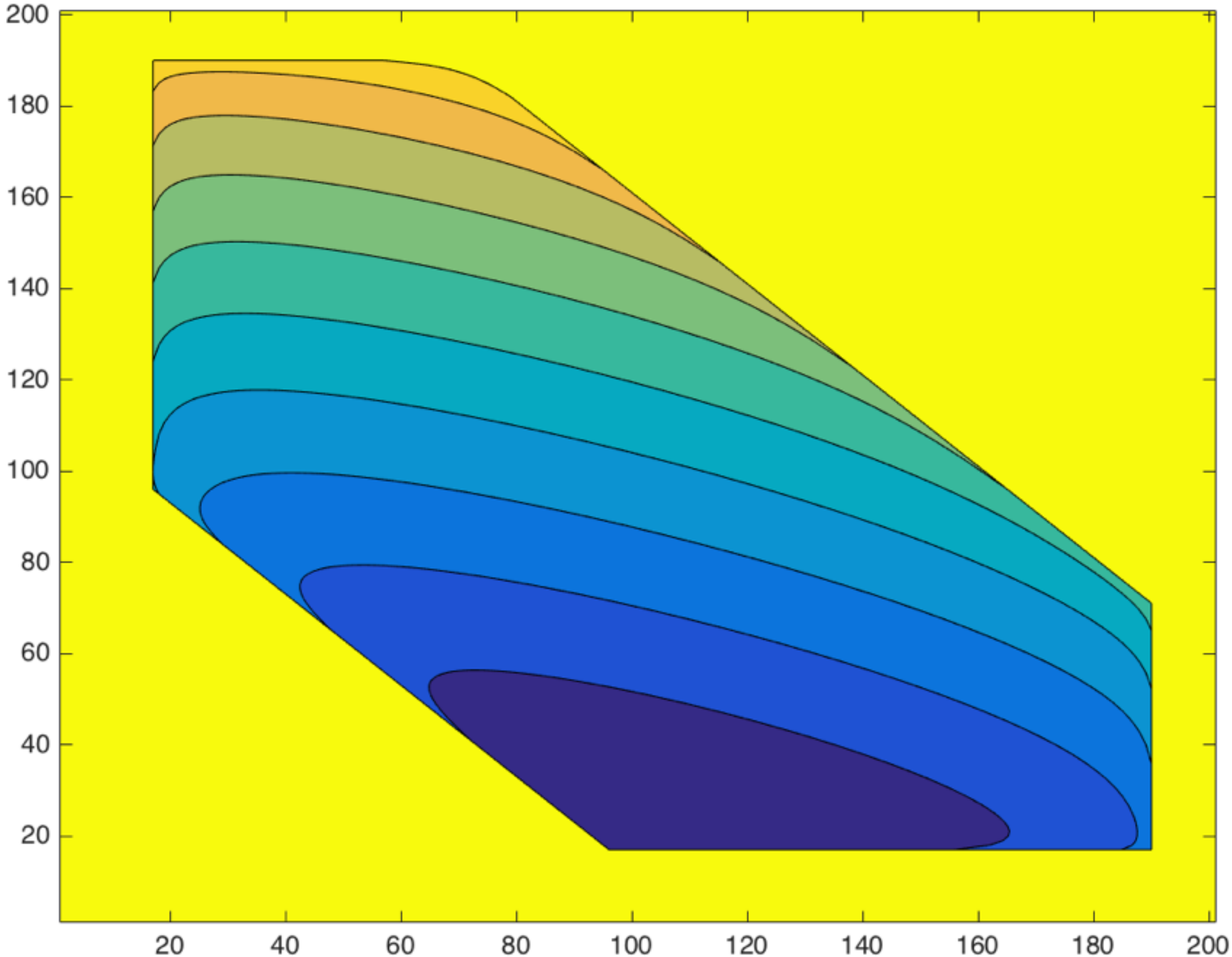
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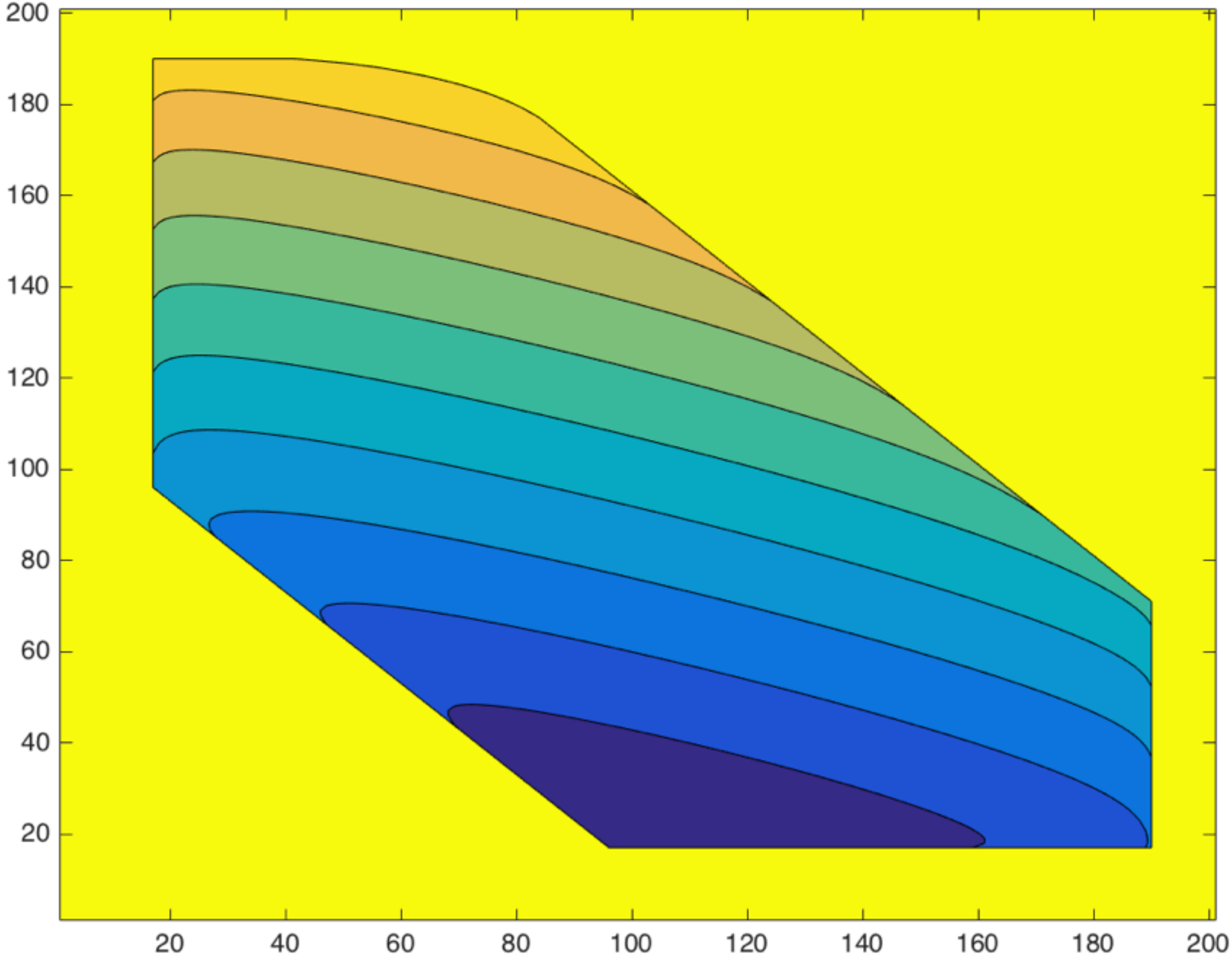
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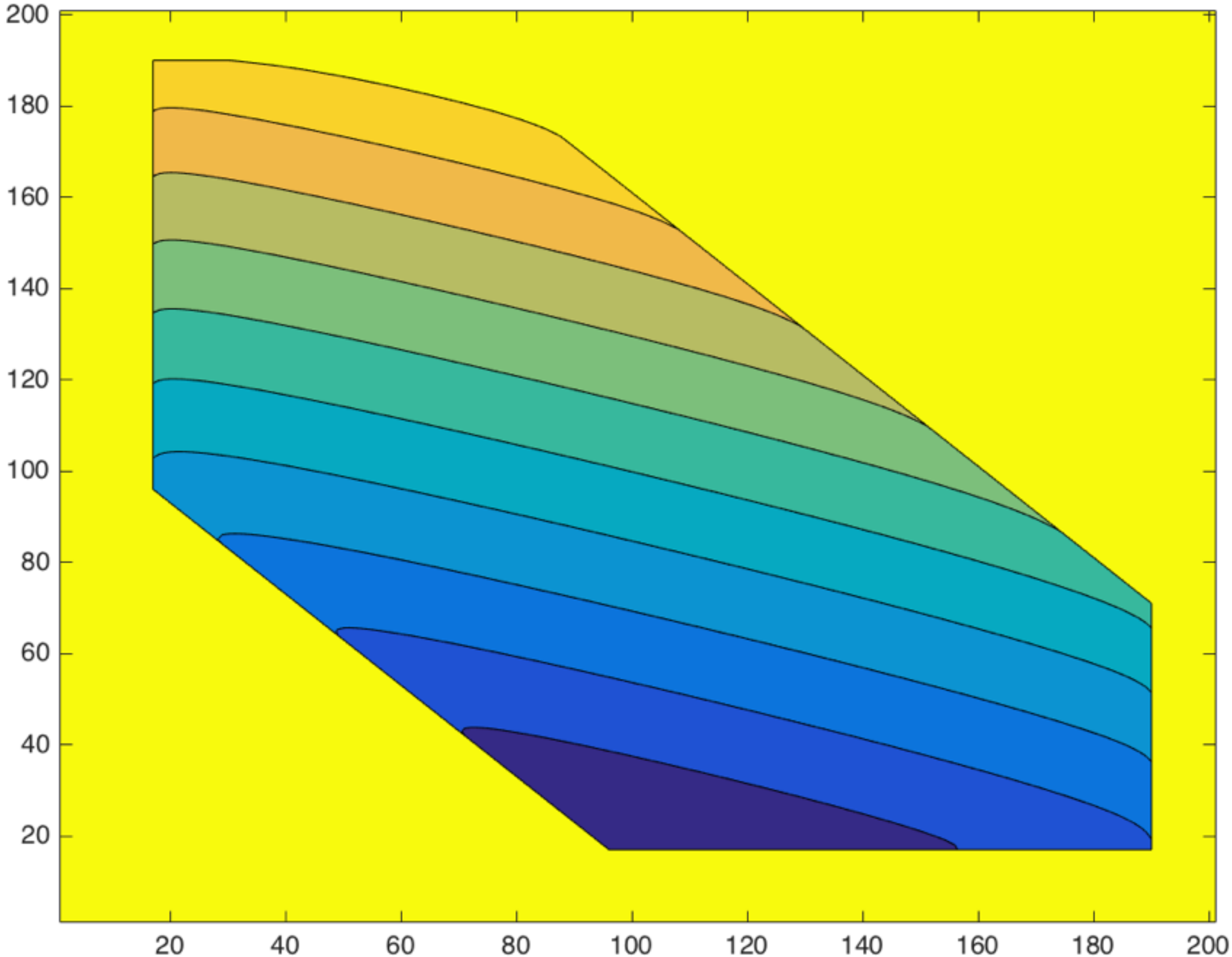
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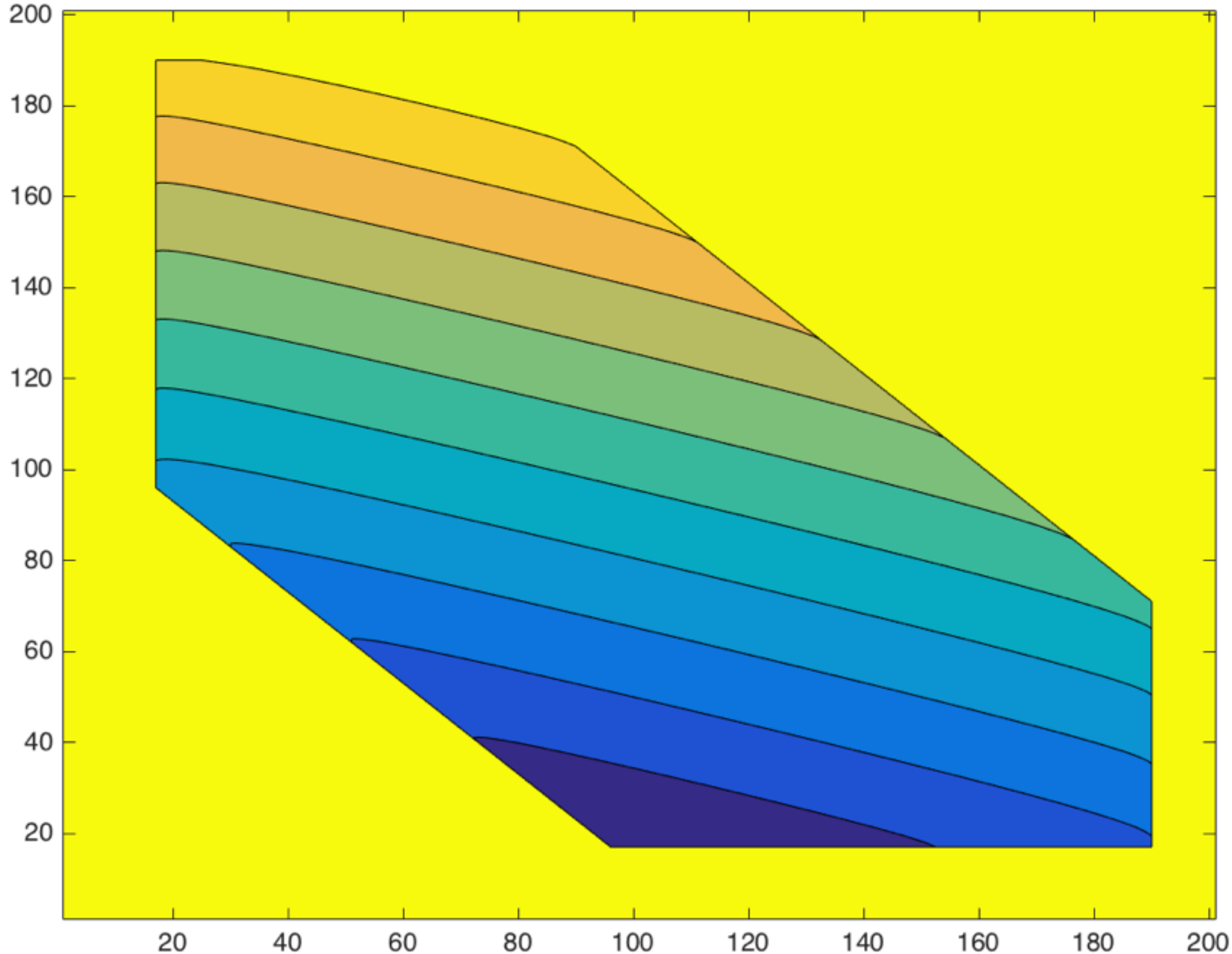
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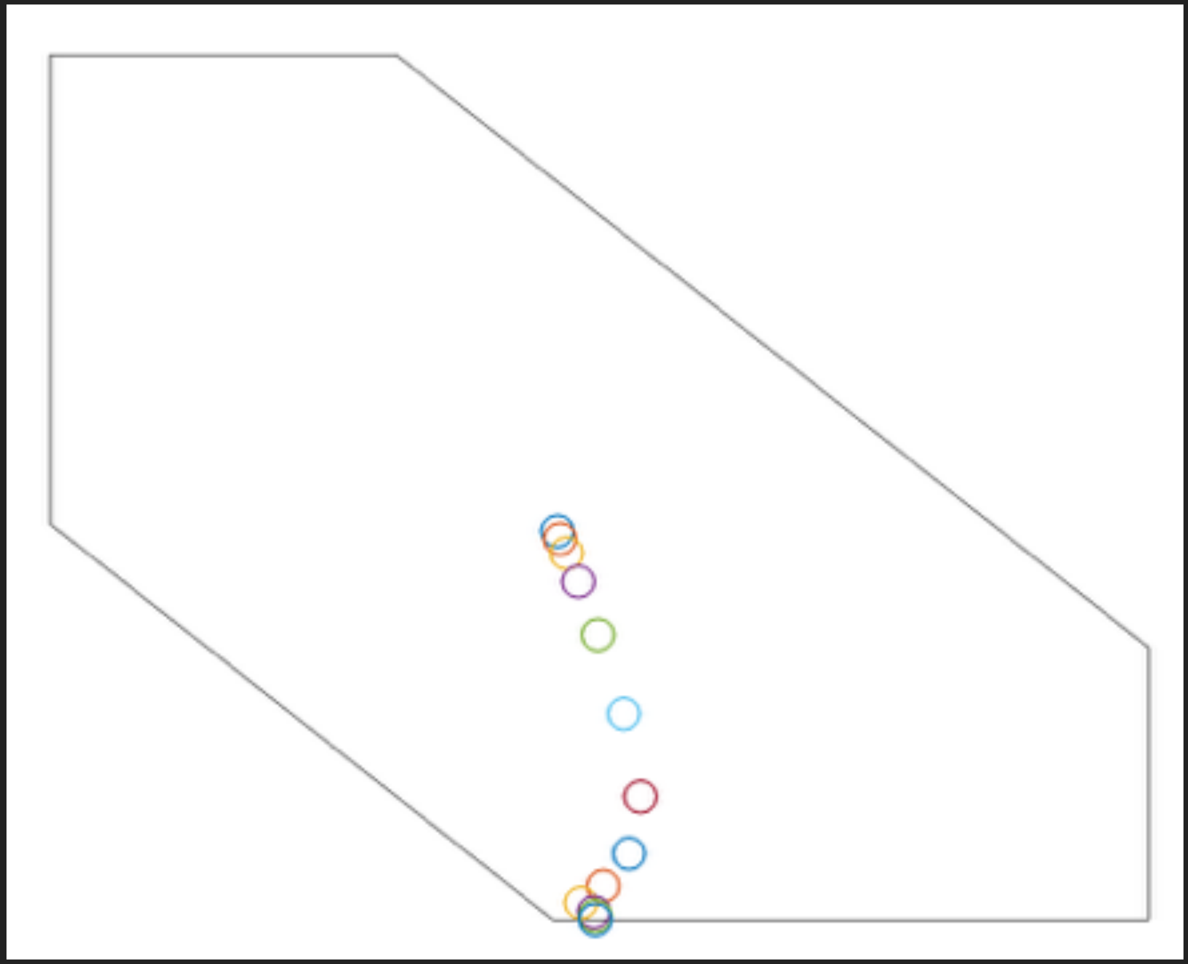
WHY IS THIS CALLED "PATH FOLLOWING"?

$$\Phi(\alpha) := \arg\min_{x \in K} \alpha(\theta^\top x) + \phi(x)$$

- As we increase inflation, the minimizer moves closer to the true desired minimum. We can plot this minimizer as α increases. This is known as the **Central Path.**



CONVERGENCE RATE OF PATH FOLLOWING

- Nesterov and Nemirovski showed:

  1. Best inflation rate is $\alpha_k = (1 + 1/\sqrt{\nu})^k$
  2. Approx error after k iter is $\epsilon = \frac{\nu}{(1+1/\sqrt{\nu})^k}$
  3. Hence, to achieve ϵ error, need $k = O(\sqrt{\nu} \cdot \log(\nu/\epsilon))$

- The barrier parameter ν is pretty important. Nesterov and Nemirovski showed that every set has a self-concordant barrier with barrier parameter ν = O(n)

THE PROBLEM: EFFICIENT SELF-CONCORDANT BARRIER IN GENERAL?

- Given any convex set K, how can we construct a self-concordant barrier for K?
- Polytopes are easy. So are L2-balls. We have barriers for some other sets also, e.g. the PSD cone.
- PROBLEM: Find an efficient universal barrier construction?
- Open problem for some time.

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3. *Hit-and-Run and Simulated Annealing*
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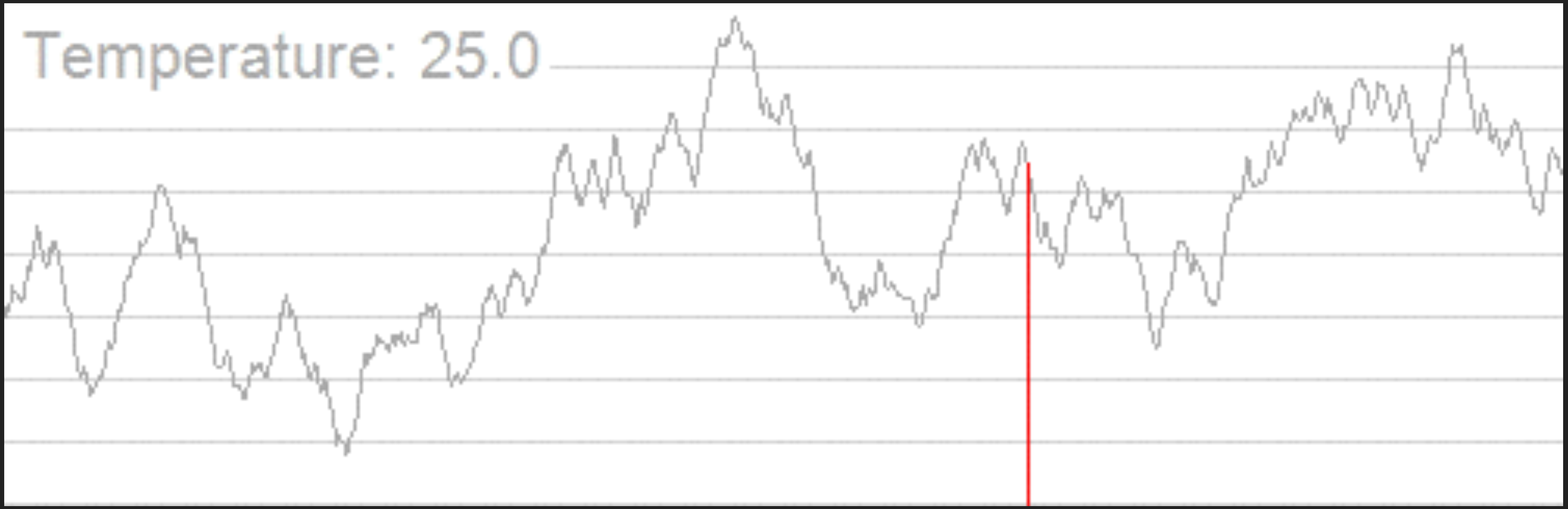
SIMULATED ANNEALING FOR OPTIMIZATION



- From Wikipedia: Optimization of a 1-dimensional function

INTRODUCTION TO SIMULATED ANNEALING

- Your goal is to solve the optimization

$$\min_{x \in K} f(x)$$

- Maybe it is easier to *sample from* the distribution

$$P_t(x) = \frac{\exp(-f(x)/t)}{\int_K \exp(-f(x')/t) dx'}$$

for a *temperature parameter* t

INTUITION BEHIND SIMULATED ANNEALING HEURISTIC

$$P_t(x) = \frac{\exp(-f(x)/t)}{\int_K \exp(-f(x')/t)dx'}$$

- Why is sampling easier? And why would it help anyway?
- First, when t is very large, sampling from $P_t(\theta)$ is equivalent to sampling from the uniform distribution on K. Easy(ish)!
- Second, when t is very small, all mass of $P_t(\theta)$ is concentrated around minimizer of f(x). That's what we want!
- Third, the successive distributions $P_t(\theta)$ and $P_{t+1}(\theta)$ are all very close, so we can "warm start" from previous samples

HIT-AND-RUN FOR LOG-CONCAVE DISTRIBUTIONS

$$P_t(x) = \frac{\exp(-f(x)/t)}{\int_K \exp(-f(x')/t)dx'}$$

- Notice that f() convex in x ==> log P_t is *concave* in x
- Lovasz/Vempala showed that problem of sampling log-concave dists is **poly-time** using **Hit-And-Run random walk** ………. IF you have a warm start (more on this later)
- **Hit-And-Run** is an interesting randomization procedure to sample from a convex body, with an interesting history

WHO INVENTED HIT-AND-RUN?

HIT-AND-RUN

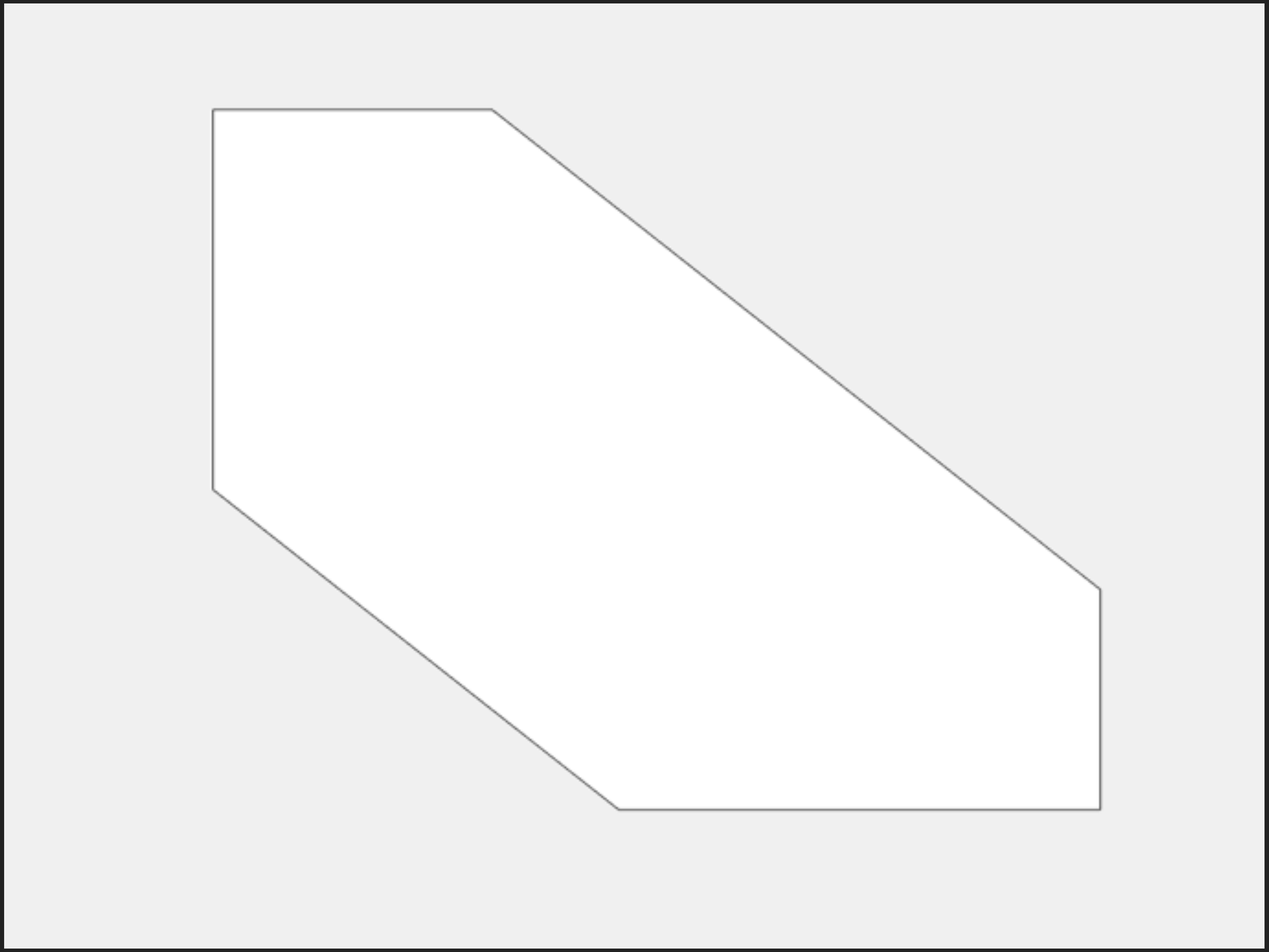
Inputs: distribution P, #iter N, initial $X_0 \in K$.
For $i = 1, 2, \ldots, N$

1. Sample random direction $u \sim N(0, I)$
2. Compute line segment $R = \{X_{i-1} + \rho u \ : \ \rho \in \mathbb{R}\} \cap K$
3. Sample X_i from P restricted to R

Return X_N

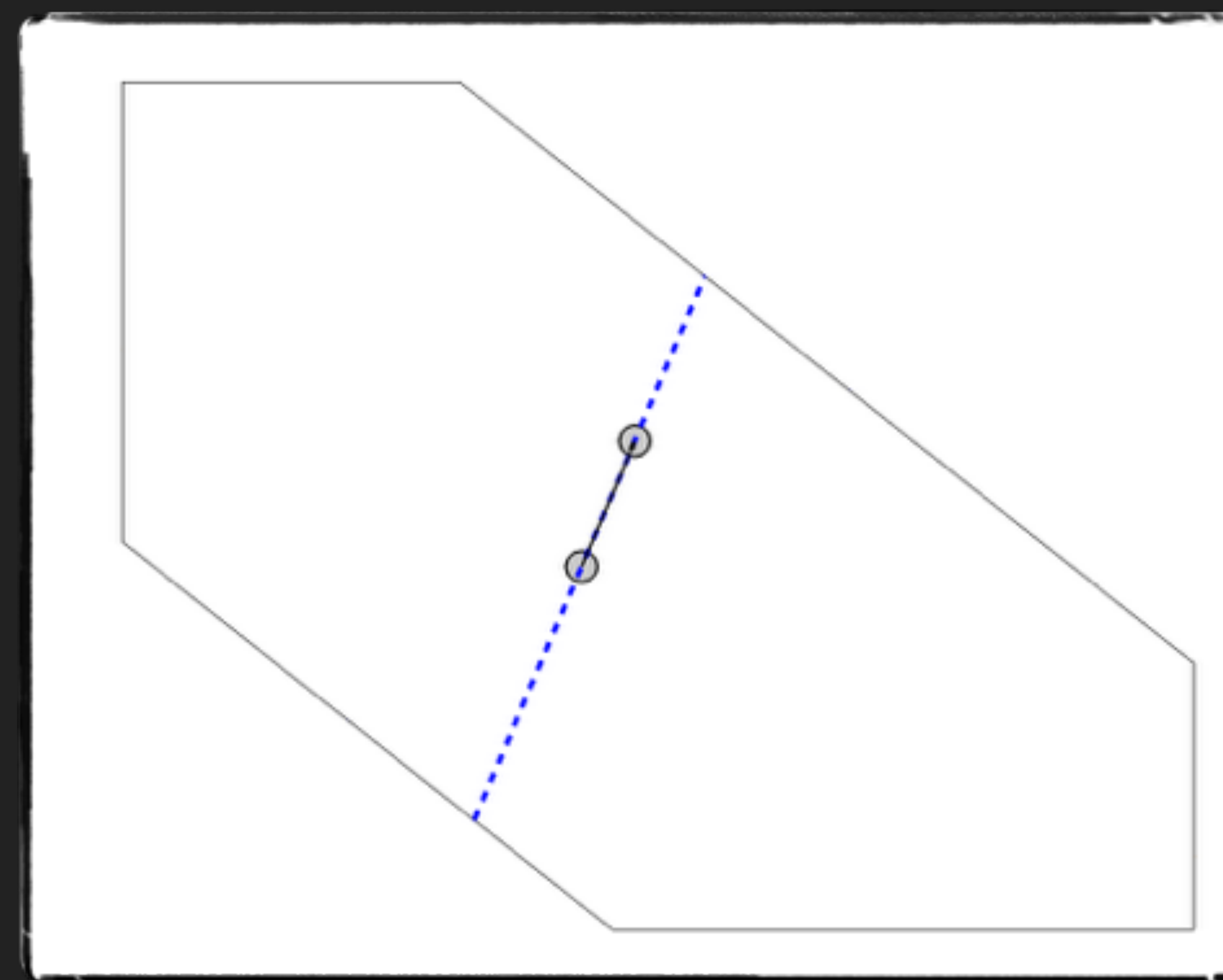
- Claim: Hit-And-Run walk has stationary distribution P
- Question: In what way does K enter into this random walk?

HIT-AND-RUN



HIT-AND-RUN REQUIRES ONLY A MEMBERSHIP ORACLE

- Notice: a single update of Hit-And-Run required only computing the endpoints of a line segment.
- Can be accomplished using binary search with a *membership oracle*

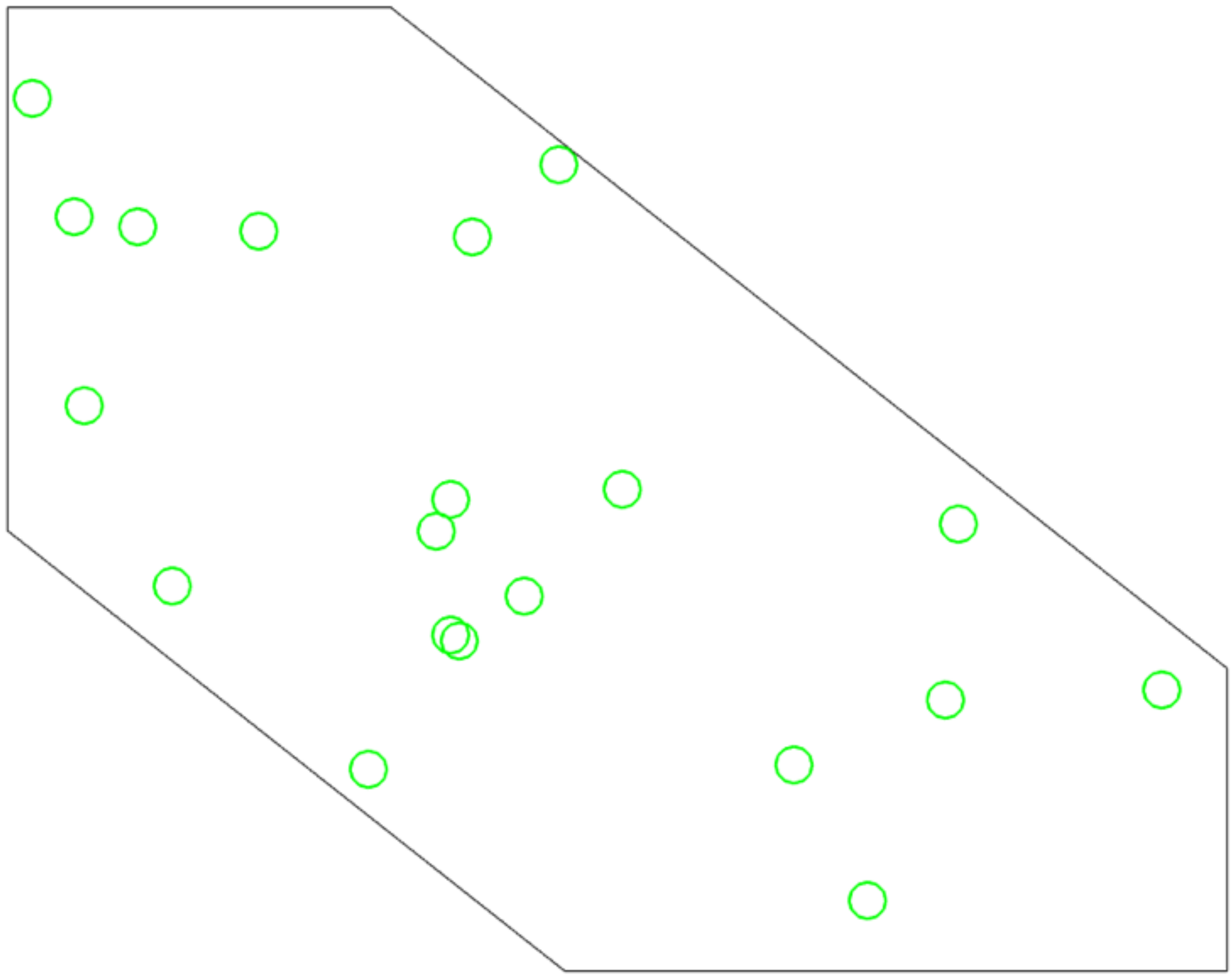


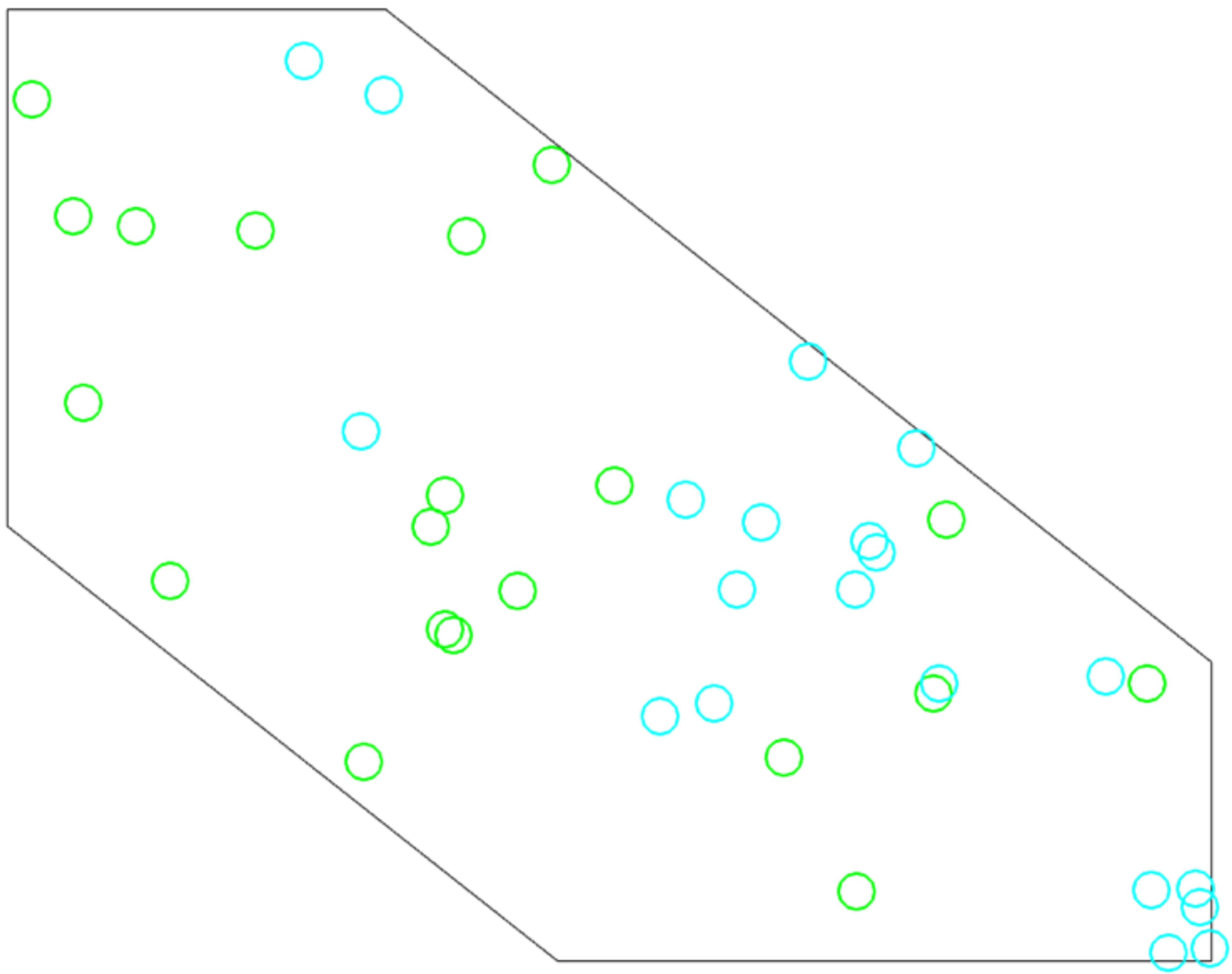
POLYTIME SIMULATED ANNEALING CONVERGENCE RESULT

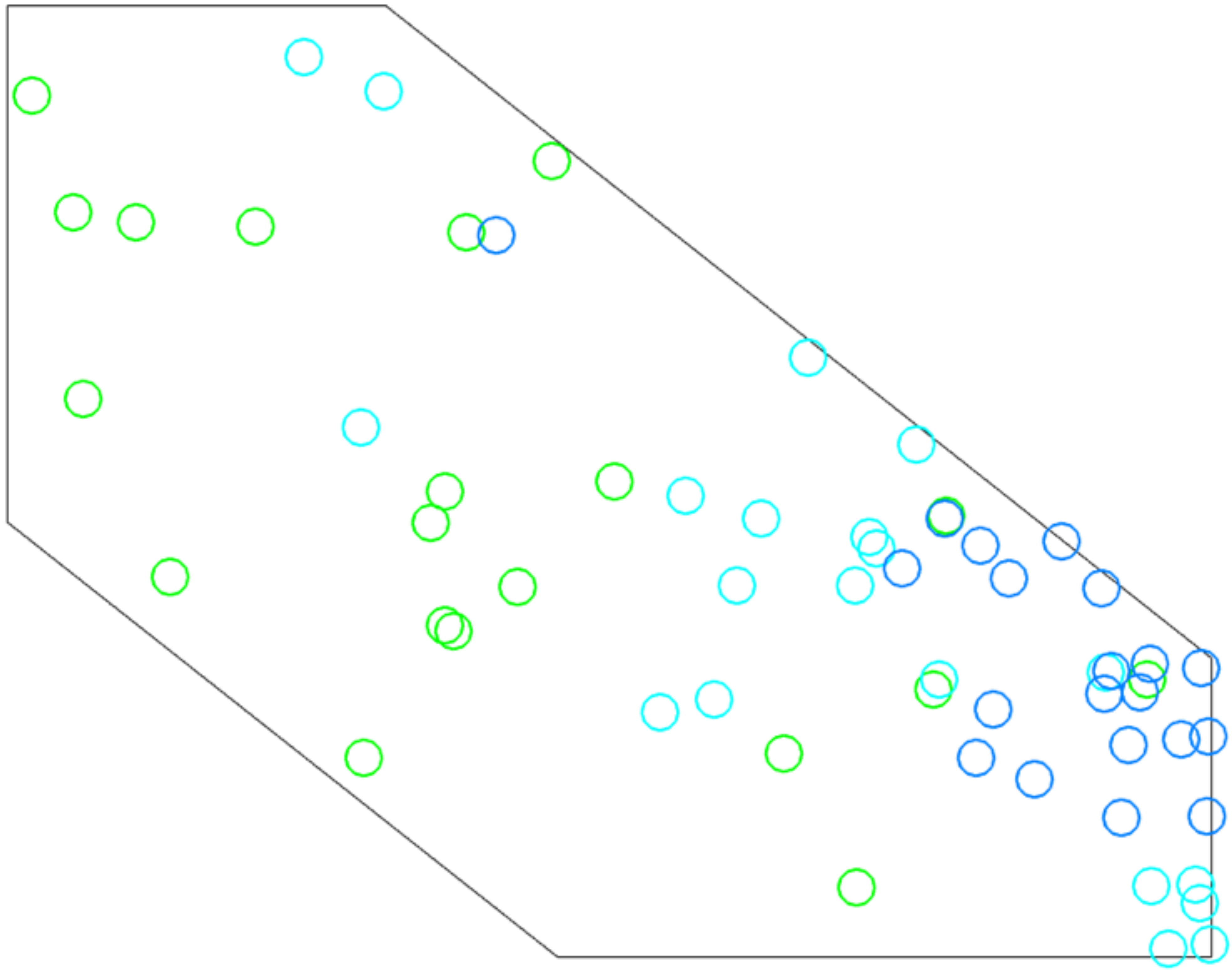
- Kalai and Vempala (2006) gave a poly-time guarantee for annealing using Hit-and-Run (membership oracle only!)

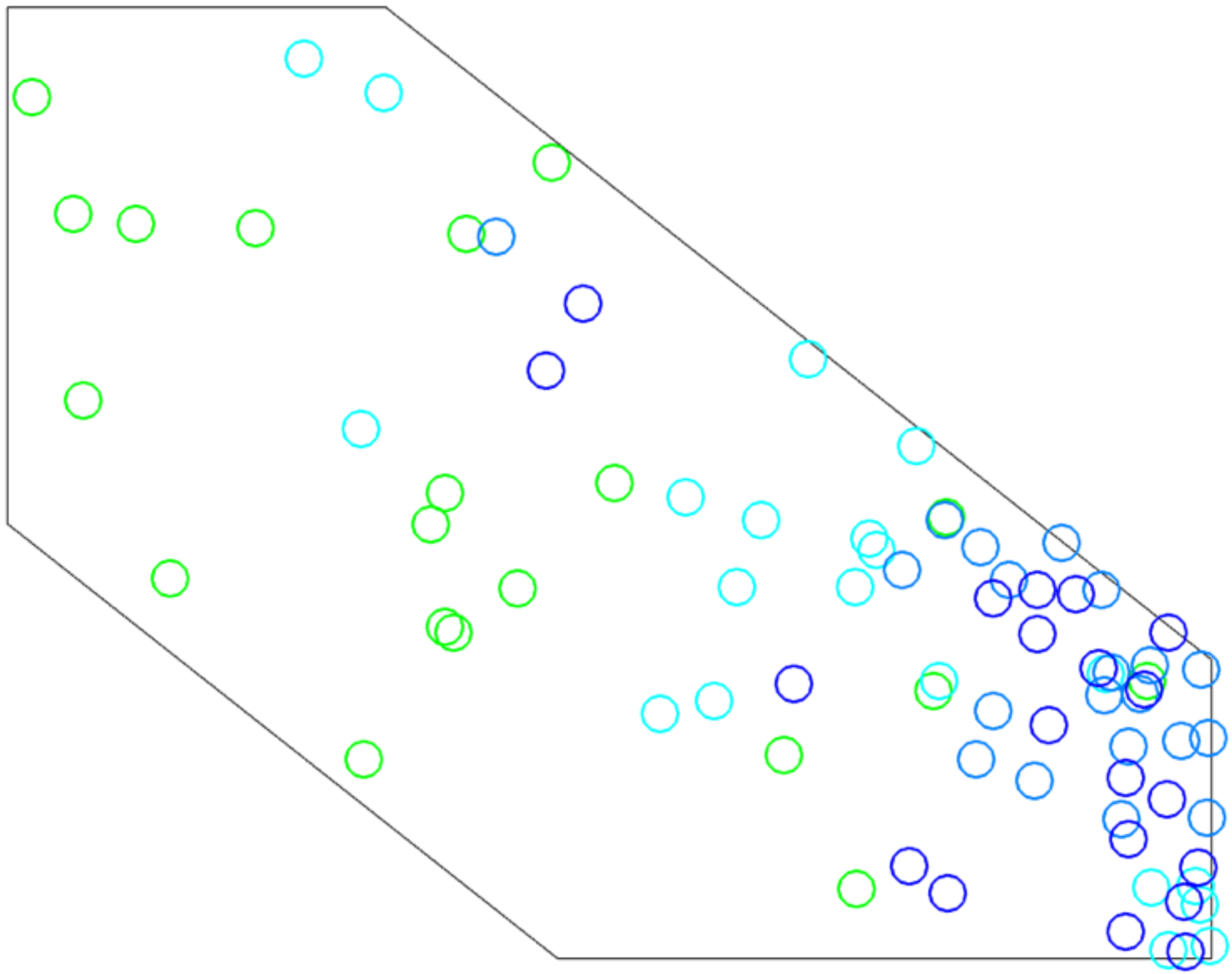
1. Sample from $P_k(x) \propto \exp(-\theta^\top x / t_k)$
2. Successive dists are "close enough" if $\mathrm{KL}(P_{k+1}(x) || P_k(x)) \leq 1/2$
3. The closeness is guaranteed as long as $t_k \approx (1 - 1/\sqrt{n})^k$
4. Roughly $O(\sqrt{n} \log 1/\epsilon)$ phases needed, $O(n^3)$ Hit-and-Run steps needed for mixing, and $O(n)$ samples needed per phase

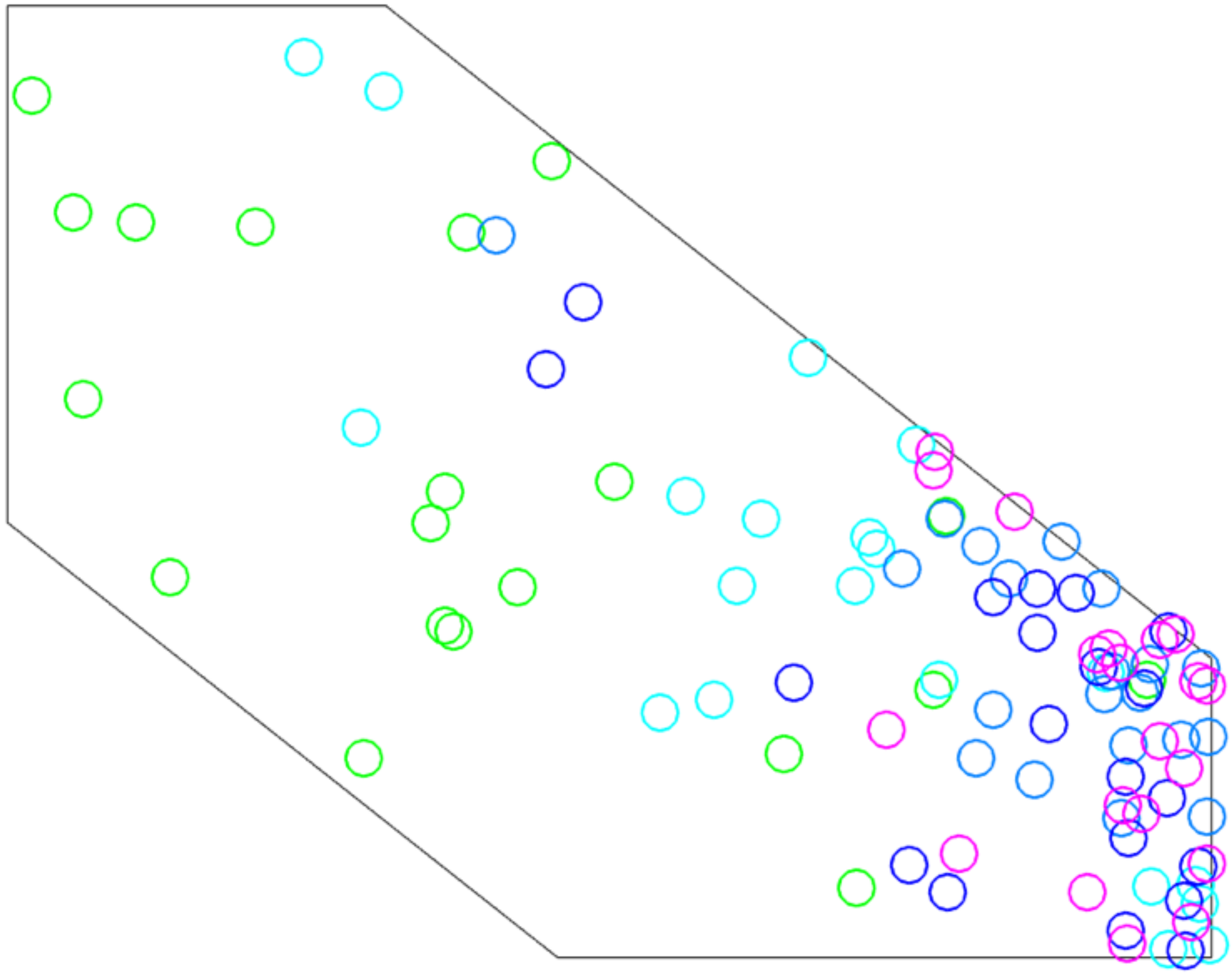
- Total running time is about $O(n^{4.5})$

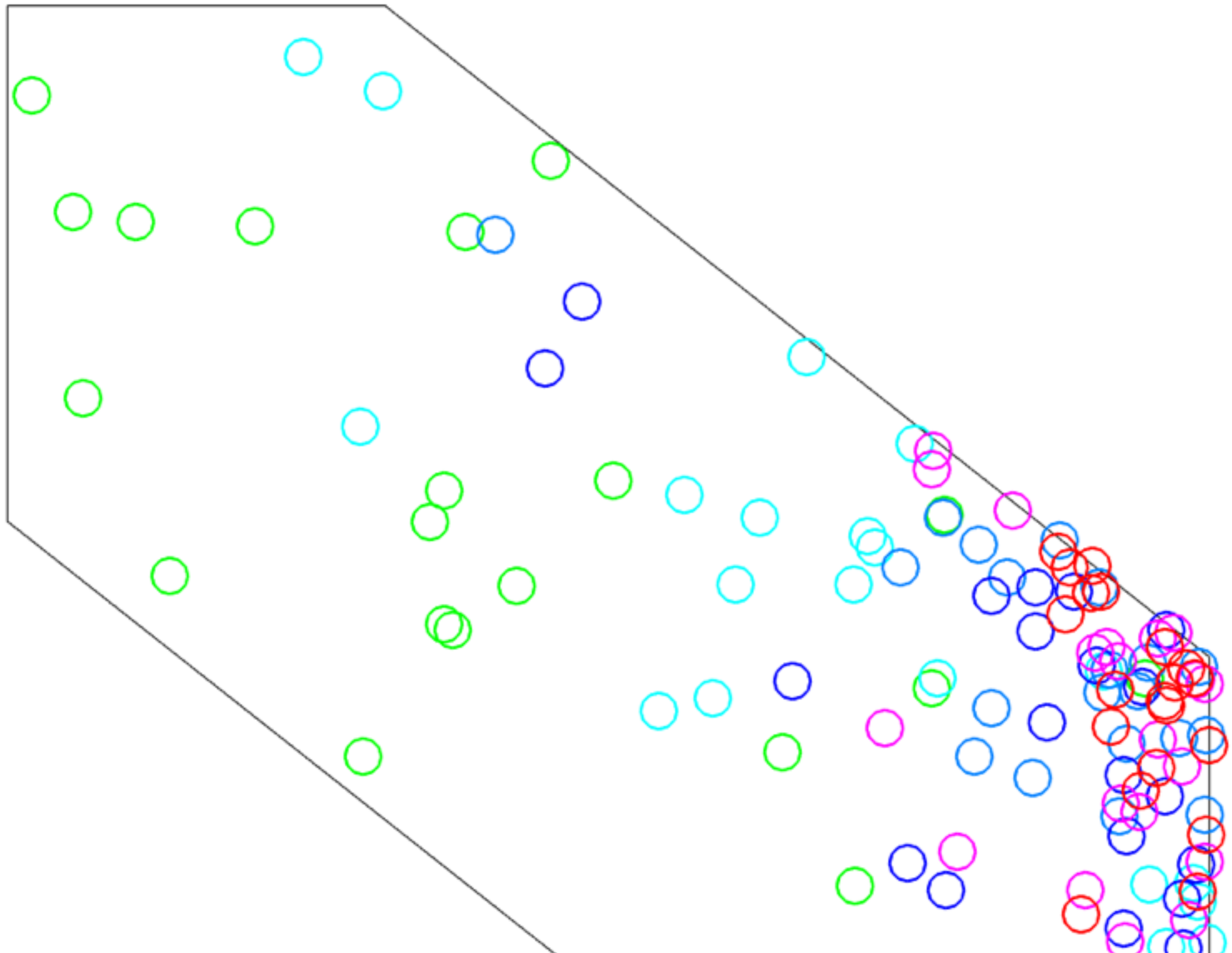


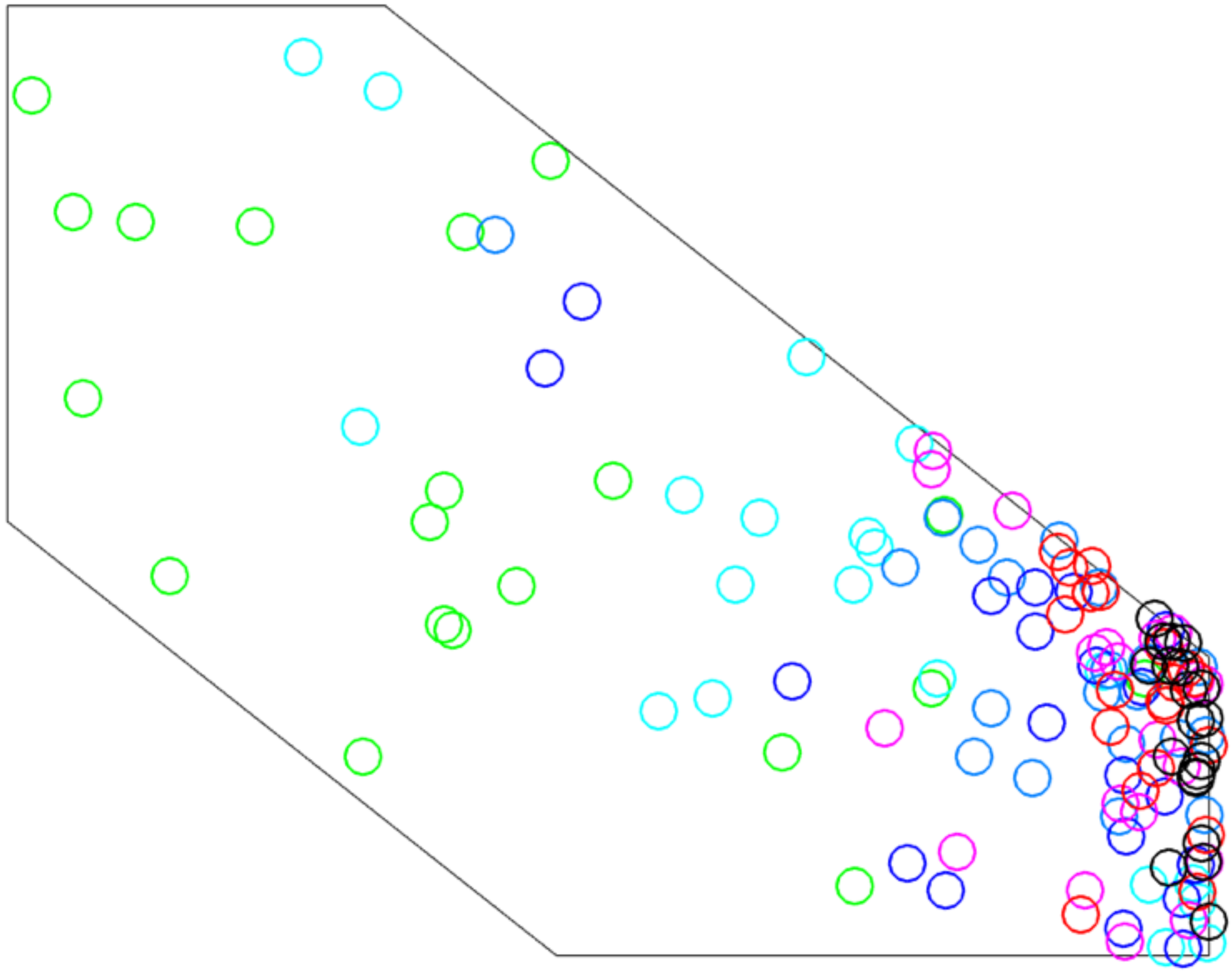










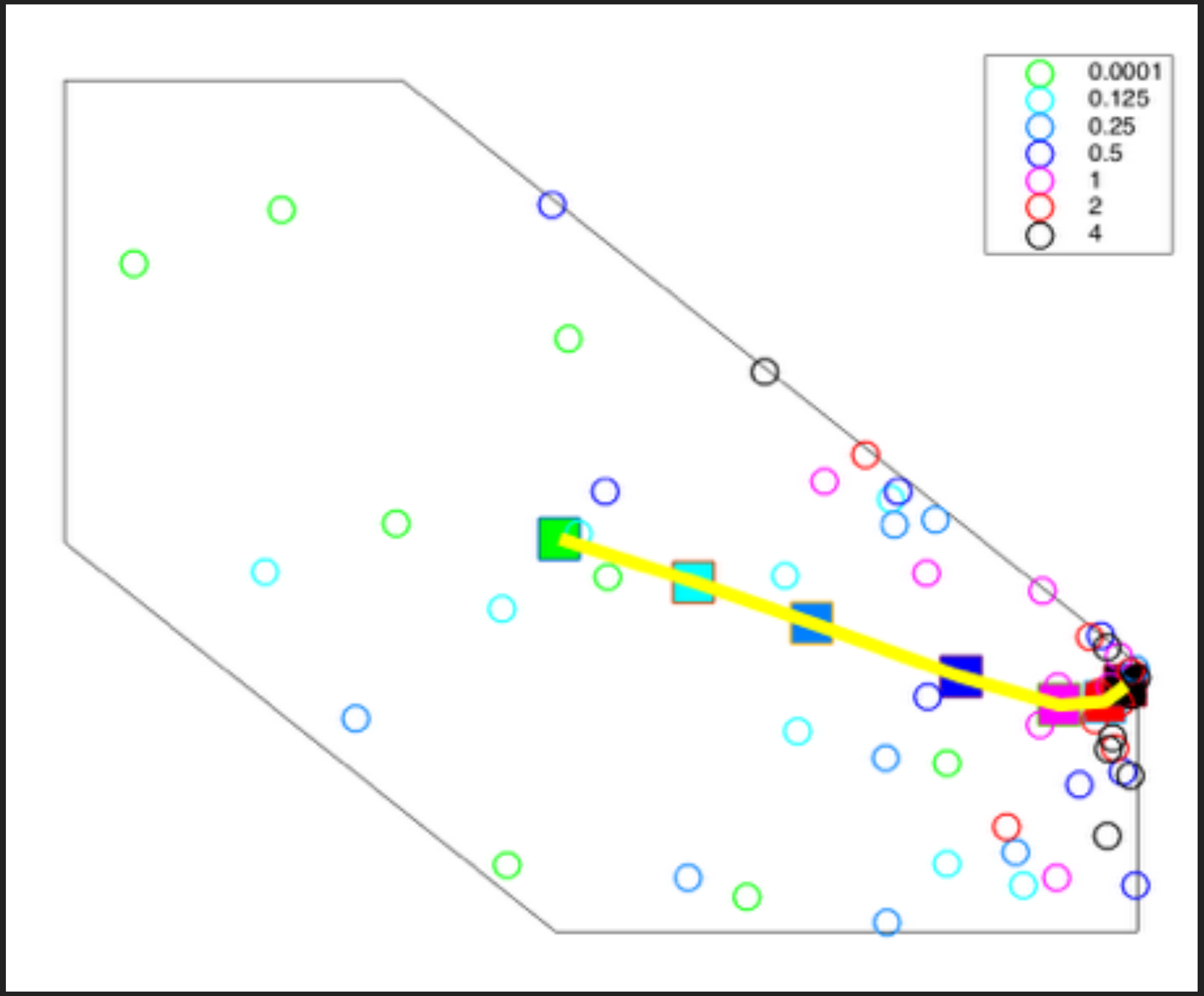


THE HEATPATH

- We can define a path according to the sequence of means one obtains as we turn down the temperature. Let

$$\chi(t) := \mathop{\mathbb{E}}_{X \sim \exp(-\theta^\top x/t)/Z} [X]$$

be the **HeatPath.**



TWO DIFFERENT CONVEX OPTIMIZATION TECHNIQUES

Not Really Different

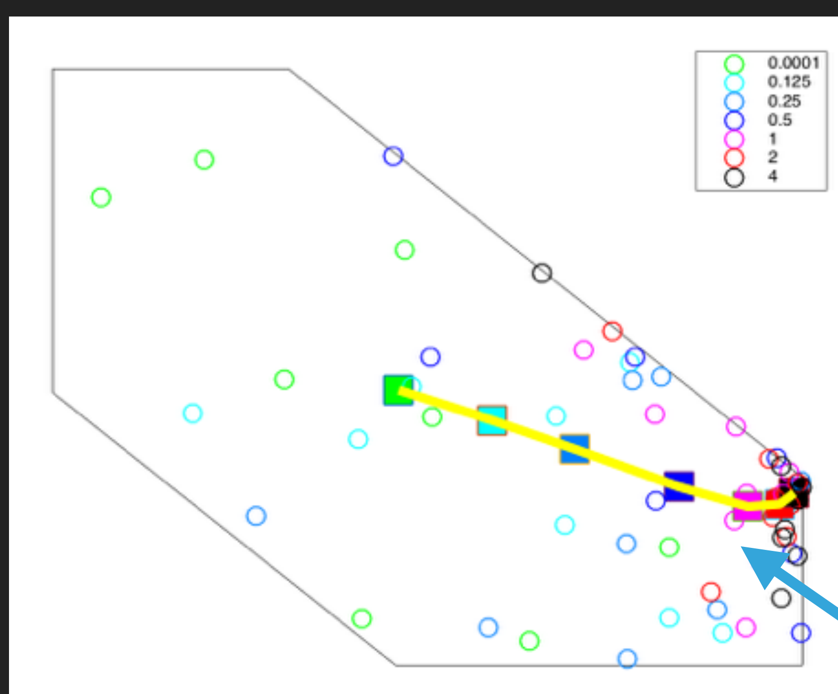
Simulated Annealing via Hit-and-Run

Interior Point Methods via Path Following

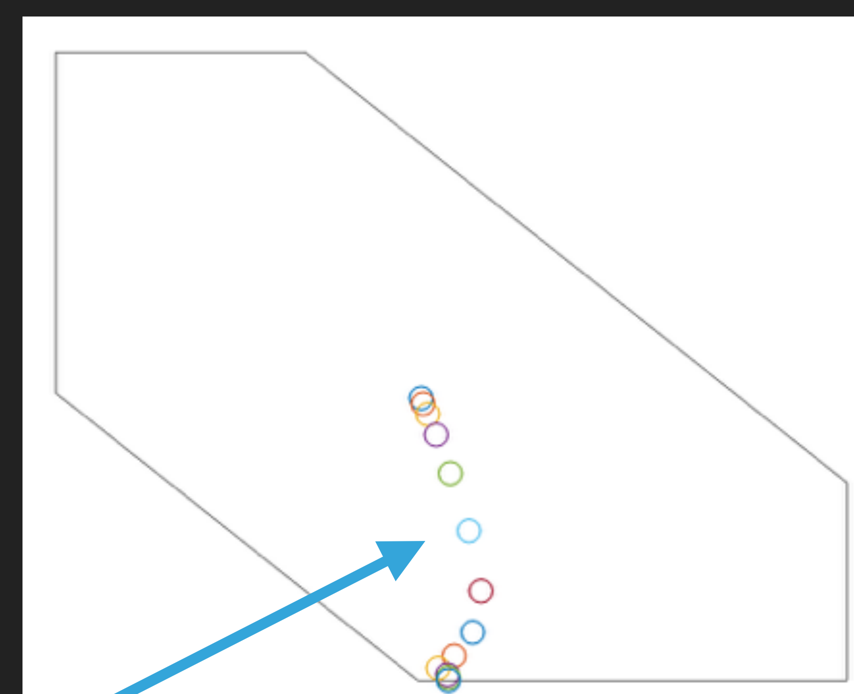
THE EQUIVALENCE OF THE CENTRAL PATH AND THE HEAT PATH

- Key result of A./Hazan 2015: there exists a barrier function $\phi()$ such that the CentralPath (for $\phi()$) is *identically* the HeatPath for the sequence of annealing distributions

HeatPath



CentralPath



These are the same object

WHAT IS THE SPECIAL BARRIER?

- The barrier $\phi()$ corresponds to the "differential entropy" of the exponential family distribution. Equivalently, it's the Fenchel conjugate of the log-partition function.
  - Let $A(\theta) = \log \int_K \exp(\theta^\top x)dx$
  - Let $A^*(x) = \sup_\theta \theta^\top x - A(\theta)$
  - A fact about exponential families: $\nabla A(\theta) = \mathbb{E}_{X \sim P_\theta}[X]$
  - A fact about Fenchel duality: $\nabla A(\theta) = \arg\max_{x \in K} \theta^\top x - A^*(x)$
- Guler 1996 showed this function is a barrier for cones. Bubeck and Eldan 2015 showed this in general, and gave an optimal parameter bound of n(1 + o(1)).

WHAT IS THE BENEFIT OF THIS CONNECTION?

- Benefit 1: This observation unifies to big areas of literature, and lets you borrow tricks from barrier methods to understand annealing, and vice versa
- Benefit 2: We were able to get a speedup on annealing using barrier methods, improving Kalai/Vempala's rate of $O(n^{4.5})$ to $O(\nu^{1/2}n^4)$

FIN