SHARED INFORMATION

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with

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Outline

Two-terminal model: Mutual information

Operational meaning in:

- Channel coding: channel capacity
- Lossy source coding: rate distortion function
- Binary hypothesis testing: Stein's lemma

Interactive communication and common randomness

- *Two-terminal model: Mutual information*
- *Multiterminal model: Shared information*

Applications

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Mutual Information

Mutual information is a measure of mutual dependence between two rvs.

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Mutual information is a measure of mutual dependence between two rvs.

Let X_1 and X_2 be $\mathbb{R}$-valued rvs with joint probability distribution $P_{X_1 X_2}$.

The **mutual information** between X_1 and X_2 is

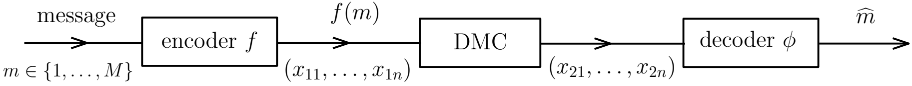
$$\begin{aligned} I\left(X_1 \wedge X_2\right) &= \begin{cases} \mathbb{E}_{P_{X_1 X_2}}\left[\log \frac{dP_{X_1 X_2}}{dP_{X_1} \times P_{X_2}}\left(X_1, X_2\right)\right], & \text{if } P_{X_1 X_2} \prec P_{X_1} \times P_{X_2} \\ \infty, & \text{if } P_{X_1 X_2} \not\prec P_{X_1} \times P_{X_2} \end{cases} \\ &= D\Big(P_{X_1 X_2} \,\|\, P_{X_1} \times P_{X_2}\Big). \quad (\text{Kullback} - \text{Leibler divergence}) \end{aligned}$$

When X_1 and X_2 are *finite-valued*,

$$\begin{aligned} I\left(X_1 \wedge X_2\right) &= H\left(X_1\right) + H\left(X_2\right) - H\left(X_1, X_2\right) \\ &= H\left(X_1\right) - H\left(X_1 \mid X_2\right) = H\left(X_2\right) - H\left(X_2 \mid X_1\right) \\ &= H\left(X_1, X_2\right) - \Big[H\left(X_1 \mid X_2\right) + H\left(X_2 \mid X_1\right)\Big]. \end{aligned}$$

Channel Coding

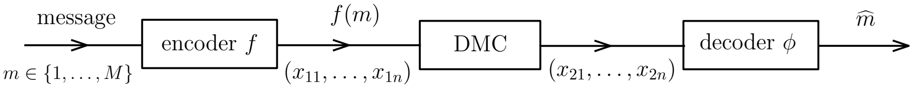
Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be finite alphabets, and $W : \mathcal{X}_1 \to \mathcal{X}_2$ be a stochastic matrix.



Discrete memoryless channel (DMC):

$$W^{(n)}\left(x_{21}, \ldots, x_{2n} \mid x_{11}, \ldots, x_{1n}\right) = \prod_{i=1}^{n} W\left(x_{2i} \mid x_{1i}\right).$$

Channel Capacity



Goal: Make code rate $\frac{1}{n} \log M$ as large as possible while keeping

$$\max_m P\big(\phi\,(X_{21}, \ldots, X_{2n}) \neq m \mid f(m)\big)$$

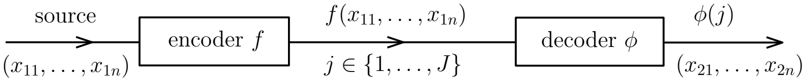
to be small, in the asymptotic sense as $n \to \infty$.

[C.E. Shannon, 1948]

$$\text{Channel capacity } C \;=\; \max_{P_{X_1} : P_{X_2 \mid X_1} = W} \; I\,(X_1 \wedge X_2)\,.$$

Lossy Source Coding

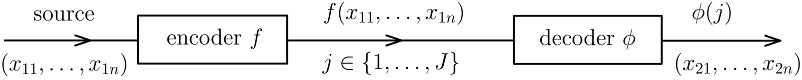
Let $\{X_{1t}\}_{t=1}^{\infty}$ be an $\mathcal{X}_1$-valued i.i.d. source.



Distortion measure:

$$d\big((x_{11}, \ldots, x_{1n}), (x_{21}, \ldots, x_{2n})\big) \;=\; \frac{1}{n}\sum_{i=1}^{n} d\big(x_{1i}, x_{2i}\big).$$

Rate Distortion Function



Goal: Make (compression) code rate $\frac{1}{n} \log J$ as small as possible while keeping

$$P\left(\frac{1}{n} \sum_{i=1}^{n} d\left(X_{1i}, X_{2i}\right) \;\leq\; \Delta\right)$$

to be large, in the asymptotic sense as $n \to \infty$.

[Shannon, 1948, 1959]

$$\text{Rate distortion function } R\left(\Delta\right) \;=\; \min_{P_{X_2|X_1}:\ \mathbb{E}[d(X_1, X_2)] \leq \Delta} I\left(X_1 \wedge X_2\right).$$

Simple Binary Hypothesis Testing

Let $\{(X_{1t}, X_{2t})\}_{t=1}^{\infty}$ be an $\mathcal{X}_1 \times \mathcal{X}_2$-valued i.i.d. process generated according to

$$H_0 : \ P_{X_1 X_2} \qquad \text{or} \qquad H_1 : \ P_{X_1} \times P_{X_2}.$$

Test:

$$\begin{aligned} \text{Decides } H_0 \text{ w.p. } & T\left(0 \mid x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}\right), \\ H_1 \text{ w.p. } & T\left(1 \mid x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}\right) = 1 - T\left(0 \mid \ldots\right). \end{aligned}$$

Stein's lemma [H. Chernoff, 1956]: For *every* $0 < \epsilon < 1$,

$$\lim_n -\frac{1}{n} \log \inf_{T:\ P_{H_0}(T \text{ says } H_0) \geq 1-\epsilon} P_{H_1}\left(T \text{ says } H_0\right)$$

$$= \ D\left(P_{X_1 X_2} \,\|\, P_{X_1} \times P_{X_2}\right) \ = \ I\left(X_1 \wedge X_2\right).$$

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Two-terminal model: Mutual information

Interactive communication and common randomness

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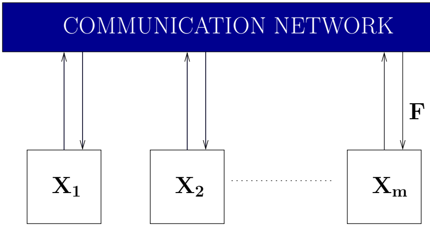
- *Multiterminal model: Shared information*

Applications

Multiterminal Model



- Set of terminals $= \mathcal{M} = \{1, \ldots, m\}$.
- $X_1, \ldots, X_m$ are finite-valued rvs with known joint distribution $P_{X_1 \ldots X_m}$ on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_m$.
- Terminal $i \in \mathcal{M}$ observes data X_i.
- Multiple rounds of *interactive communication* on a *noiseless channel* of *unlimited capacity*; all terminals hear *all communication*.

Interactive Communication

Interactive communication

- Assume: Communication occurs in consecutive time slots in r rounds.
- The corresponding rvs representing the communication are

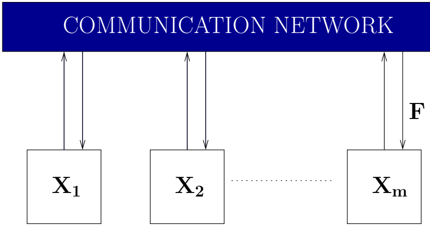
$$\mathbf{F} = \mathbf{F}(X_1, \ldots, X_m) = \left(F_{11}, \ldots, F_{1m},\ F_{21}, \ldots, F_{2m},\ \ldots,\ F_{r1}, \ldots, F_{rm}\right)$$

  - $F_{11} = f_{11}(X_1),\ F_{12} = f_{12}(X_2, F_{11}),\ \ldots$
  - $F_{ji} = f_{ji}(X_i;\ \text{all previous communication}).$

Simple communication: $\mathbf{F} = (F_1, \ldots, F_m), \quad F_i = f_i(X_i),\ 1 \le i \le m.$

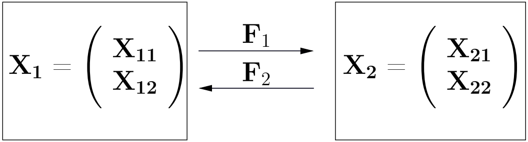
A. Yao, "Some complexity questions related to distributive computing," Proc. Annual Symposium on Theory of Computing, 1979.

Applications



- *Data exchange: Omniscience*
- *Signal recovery: Data compression*
- *Function computation*
- *Cryptography: Secret key generation*

WatanExample: Function Computation



[S. Watanabe]

- $X_{11}, X_{12}, X_{21}, X_{22}$ are mutually independent $(0.5, 0.5)$ bits.
- Terminals 1 and 2 wish to compute:

$$G = g(X_1, X_2) = \mathbb{1}\Big((X_{11}, X_{12}) = (X_{21}, X_{22})\Big).$$

- *Simple communication*: $\mathbf{F} = \Big(F_1 = (X_{11}, X_{12}),\ F_2 = (X_{21}, X_{22})\Big)$.
  - Communication complexity: $H(\mathbf{F}) = 4$ bits.
  - No privacy: Terminal 1 or 2, or an observer of $\mathbf{F}$, learns all the data X_1, X_2.

WatanExample: Function Computation

$$\boxed{\mathbf{X_1} = \begin{pmatrix} \mathbf{X_{11}} \\ \mathbf{X_{12}} \end{pmatrix}} \;\; \begin{array}{c} \xrightarrow{\mathbf{F}_{11}} \\ \xleftarrow{\mathbf{F}_{12}} \end{array} \;\; \boxed{\mathbf{X_2} = \begin{pmatrix} \mathbf{X_{21}} \\ \mathbf{X_{22}} \end{pmatrix}}$$

- *An interactive communication protocol*:
  - $\mathbf{F} = \Big(F_{11} = (X_{11}, X_{12}),\ F_{12} = G\Big)$.
  - Complexity: $H(\mathbf{F}) = 2.81$ bits.
  - Some privacy: Terminal 2, or an observer of $\mathbf{F}$, learns X_1; Terminal 1, or an observer of $\mathbf{F}$, either learns X_2 w.p. 0.25 or w.p. 0.75 that X_2 differs from X_1.

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¿ Can a communication complexity of 2.81 bits be bettered ?

Related Work

- Exact function computation
  - Yao '79: Communication complexity.
  - Gallager '88: Algorithm for parity computation in a network.
  - Giridhar-Kumar '05: Algorithms for computing functions over sensor networks.
  - Freris-Kowshik-Kumar '10: Survey: Connectivity, capacity, clocks, computation in large sensor networks.
  - Orlitsky-El Gamal '84: Communication complexity with secrecy.
- Information theoretic function computation
  - Körner-Marton '79: Minimum rate for computing parity.
  - Orlitsky-Roche '01: Two terminal function computation.
  - Nazer-Gastpar '07: Computation over noisy channels.
  - Ma-Ishwar '08: Distributed source coding for interactive computing.
  - Ma-Ishwar-Gupta '09: Multiround function computation in colocated networks.
  - Tyagi-Gupta-Narayan '11: Secure function computation.
  - Tyagi-Watanabe '13, '14 Secrecy generation, secure computing.
- Compressing interactive communication
  - Schulman '92: Coding for interactive communication.
  - Braverman-Rao '10: Information complexity of communication.
  - Kol-Raz '13, Heupler '14: Interactive communication over noisy channels.

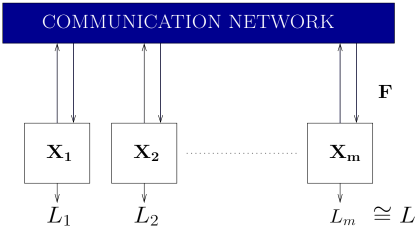
Mathematical Economics: Mechanism Design

– Thomas Marschak and Stefan Reichelstein,
"Communication requirements for individual agents in networks and hierarchies,"
in *The Economics of Informational Decentralization: Complexity, Efficiency and Stability: Essays in Honor of Stanley Reiter*, John O. Ledyard, Ed., Springer, 1994.

– Kenneth R. Mount and Stanley Reiter,
Computation and Complexity in Economic Behavior and Organization, Cambridge U. Press, 2002.

Courtesy: Demos Teneketzis

Common Randomness



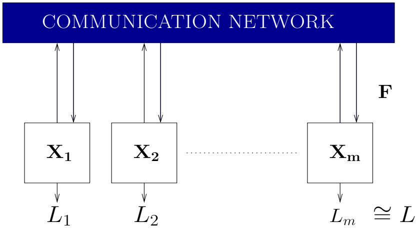
For $0 \leq \epsilon < 1$, given interactive communication $\mathbf{F}$, a rv $L = L(X_1, \ldots, X_m)$ is ϵ-CR for the terminals in $\mathcal{M}$ using $\mathbf{F}$, if there exist *local estimates*

$$L_i = L_i\big(X_i, \mathbf{F}\big), \quad i \in \mathcal{M},$$

of L satisfying

$$P\Big(L_i = L, \quad i \in \mathcal{M}\Big) \geq 1 - \epsilon.$$

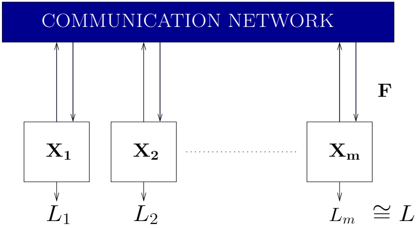
Common Randomness



Examples:

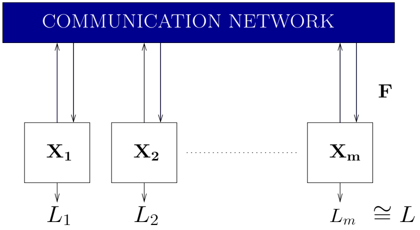
- *Data exchange: Omniscience:* $L = \big(X_1, \ldots, X_m\big)$.
- *Signal recovery: Data compression:* $L \supseteq X_{i^*}$, for some fixed $i^* \in \mathcal{M}$.
- *Function computation:* $L \supseteq g\big(X_1, \ldots, X_m\big)$ for a given g.
- *Cryptography: Secret CR, i.e., secret key:* L with $I(L \wedge \mathbf{F}) \cong 0$.

A Basic Operational Question



¿ What is the *maximal* CR, as measured by $H\big(L|\mathbf{F}\big)$, that can be generated by a *given* interactive communication $\mathbf{F}$ for a distributed processing task **?**

A Basic Operational Question



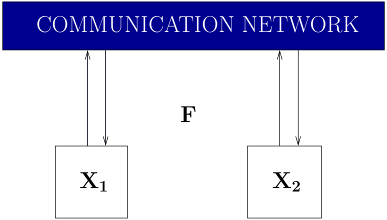
¿ What is the *maximal* CR, as measured by $H(L|\mathbf{F})$, that can be generated by a *given* interactive communication $\mathbf{F}$ for a distributed processing task **?**

Answer in two steps:

- Fundamental structural property of interactive communication
- Upper bound on amount of CR achievable with interactive communication.

Shall start with the case of $m = 2$ terminals.

Fundamental Property of Interactive Communication



Lemma: [U. Maurer], [R. Ahlswede - I. Csiszár]
For interactive communication $\mathbf{F}$ of the Terminals 1 and 2 observing data X_1 and X_2, respectively,

$$I\big(X_1 \wedge X_2 \big| \mathbf{F}\big) \;\leq\; I\big(X_1 \wedge X_2\big).$$

In particular, independent rvs X_1, X_2 remain so upon conditioning on an interactive communication.

Fundamental Property of Interactive Communication

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In particular, independent rvs X_1, X_2 remain so upon conditioning on an interactive communication.

Proof: For interactive communication $\mathbf{F} = \big(F_{11}, F_{12}, \ldots, F_{r1}, F_{r2}\big)$,

$$\begin{aligned} I\big(X_1 \wedge X_2\big) &= I\big(X_1, F_{11} \wedge X_2\big) \\ &\geq I\big(X_1 \wedge X_2 | F_{11}\big) \\ &= I\big(X_1 \wedge X_2, F_{12} | F_{11}\big) \\ &\geq I\big(X_1 \wedge X_2 | F_{11}, F_{12}\big), \end{aligned}$$

followed by iteration. □

An Equivalent Form

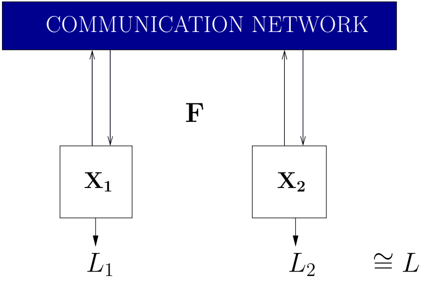
For interactive communication $\mathbf{F}$ of Terminals 1 and 2:

$$I\big(X_1 \wedge X_2 \big| \mathbf{F}\big) \;\leq\; I\big(X_1 \wedge X_2\big)$$

$$\Updownarrow$$

$$H(\mathbf{F}) \;\geq\; H(\mathbf{F}|X_1) + H(\mathbf{F}|X_2).$$

Upper Bound on CR for Two Terminals



Using

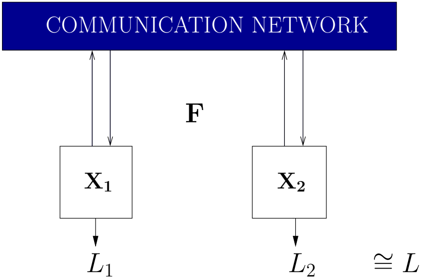
- L is ϵ-CR for Terminals 1 and 2 with interactive communication $\mathbf{F}$; and
- $H(\mathbf{F}) \;\geq\; H(\mathbf{F}|X_1) + H(\mathbf{F}|X_2)$,

we get

$$H\big(L|\mathbf{F}\big) \;\leq\; H\big(X_1, X_2\big) - \Big[H\big(X_1|X_2\big) + H\big(X_2|X_1\big)\Big] + 2\nu(\epsilon),$$

where $\lim_{\epsilon \to 0} \nu(\epsilon) = 0$.

Maximum CR for Two Terminals: Mutual Information



Lemma: [I. Csiszár - P. Narayan] Let L be *any* ϵ-CR for Terminals 1 and 2 observing data X_1 and X_2, respectively, achievable with interactive $\mathbf{F}$. Then

$$H\big(L|\mathbf{F}\big) \lessapprox I\big(X_1 \wedge X_2\big) = D\big(P_{X_1X_2}||P_{X_1} \times P_{X_2}\big).$$

Remark: When $\{(X_{1t}, X_{2t})\}_{t=1}^{\infty}$ is an $\mathcal{X}_1 \times \mathcal{X}_2$-valued i.i.d. process, the upper bound is attained.

Interactive Communication for $m \geq 2$ Terminals

Theorem 1: [I. Csiszár-P. Narayan]
For interactive communication $\mathbf{F}$ of the terminals $i \in \mathcal{M} = \{1, \ldots, m\}$, with Terminal i oberving data X_i,

$$H(\mathbf{F}) \;\geq\; \sum_{B \in \mathcal{B}} \lambda_B H(\mathbf{F}|X_{B^c})$$

for every family $\mathcal{B} = \{B \subsetneq \mathcal{M}, B \neq \emptyset\}$ and set of weights ("*fractional partition*")

$$\lambda \triangleq \left\{ 0 \leq \lambda_B \leq 1,\; B \in \mathcal{B}, \;\; \text{satisfying} \sum_{B \in \mathcal{B}: B \ni i} \lambda_B = 1 \;\forall\; i \in \mathcal{M} \right\}.$$

Equality holds if $X_1, \ldots, X_m$ are mutually independent.

Special case of:
M. Madiman and P. Tetali, "Information inequalities for joint distributions, with interpretations and applications," IEEE Trans. Inform. Theory, June 2010.

CR for $m \geq 2$ Terminals: A Suggestive Analogy

[S. Nitinawarat-P. Narayan]

For interactive communication $\mathbf{F}$ of the terminals $i \in \mathcal{M} = \{1, \ldots, m\}$, with Terminal i observing data X_i,

$$\left(\mathbf{m = 2} : H(\mathbf{F}) \;\geq\; H(\mathbf{F}|X_1) + H(\mathbf{F}|X_2) \;\Leftrightarrow\; I\big(X_1 \wedge X_2 \big| \mathbf{F}\big) \;\leq\; I\big(X_1 \wedge X_2\big) \right)$$

$$H\left(\mathbf{F}\right) \;\geq\; \sum_{B \in \mathcal{B}} \lambda_B H\left(\mathbf{F}|X_{B^c}\right)$$

$$\Updownarrow$$

$$\begin{aligned} H\big(X_1, \ldots, X_m|\mathbf{F}\big) - \sum_{B \in \mathcal{B}} \lambda_B \; H\big(X_B|X_{B^c}, \mathbf{F}\big) & \\ \leq H\big(X_1, \ldots, X_m\big) - \sum_{B \in \mathcal{B}} \lambda_B \; H\big(X_B|X_{B^c}\big). & \end{aligned}$$

An Analogy

[S. Nitinawarat-P. Narayan]

For interactive communication $\mathbf{F}$ of the terminals $i \in \mathcal{M} = \{1, \ldots, m\}$, with Terminal i observing data X_i,

$$H(\mathbf{F}) \geq \sum_{B \in \mathcal{B}} \lambda_B H(\mathbf{F} | X_{B^c})$$

$$\Updownarrow$$

$$H\big(X_1, \ldots, X_m | \mathbf{F}\big) - \sum_{B \in \mathcal{B}} \lambda_B \, H\big(X_B | X_{B^c}, \mathbf{F}\big)$$
$$\leq H\big(X_1, \ldots, X_m\big) - \sum_{B \in \mathcal{B}} \lambda_B \, H\big(X_B | X_{B^c}\big).$$

¿ Does the RHS suggest a measure of mutual dependence

among the rvs $X_1, \ldots, X_m$?

Maximum CR for $m \geq 2$ Terminals: Shared Information

Theorem 2: [I. Csiszár-P. Narayan]

Given $0 \leq \epsilon < 1$, for an ϵ-CR L for $\mathcal{M}$ achieved with interactive communication $\mathbf{F}$,

$$H\big(L|\mathbf{F}\big) \;\leq\; H\big(X_1, \ldots, X_m\big) \;-\; \sum_{B \in \mathcal{B}} \lambda_B H\left(X_B | X_{B^c}\right) + m\nu$$

for every fractional partition λ of $\mathcal{M}$, with $\nu = \nu(\epsilon) = \epsilon \log\big|\mathcal{L}\big| + h(\epsilon)$.

Remarks:

- The proof of Theorem 2 relies on Theorem 1.
- When $\{(X_{1t}, \ldots, X_{mt})\}_{t=1}^{\infty}$ is an i.i.d. process, the upper bound is attained.

Shared Information

Theorem 2: [I. Csiszár-P. Narayan]

$$H\big(L|\mathbf{F}\big) \;\lessapprox\; H\big(X_1,\ldots,X_m\big) \;-\; \max_{\lambda} \sum_{B\in\mathcal{B}} \lambda_B H\left(X_B|X_{B^c}\right)$$

$$\triangleq\; SI\big(X_1,\ldots,X_m\big)$$

Extensions

Theorems 1 and 2 extend to:

- random variables with densities [S. Nitinawarat-P. Narayan]
- a larger class of probability measures [H.Tyagi-P. Narayan].

Shared Information and Kullback-Leibler Divergence

[I. Csiszár-P. Narayan, C. Chan-L. Zheng]

$$SI\big(X_1, \ldots, X_m\big) \;=\; H\big(X_1, \ldots, X_m\big) \;-\; \max_{\lambda} \sum_{B \in \mathcal{B}} \lambda_B H\left(X_B | X_{B^c}\right)$$

$$(m = 2) \;=\; H\big(X_1, X_2\big) - \Big[H\big(X_1|X_2\big) + H\big(X_2|X_1\big)\Big] = I\big(X_1 \wedge X_2\big)$$

$$(m = 2) \;=\; D\big(P_{X_1 X_2} || P_{X_1} \times P_{X_2}\big)$$

Shared Information and Kullback-Leibler Divergence

[I. Csiszár-P. Narayan, C. Chan-L. Zheng]

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$$(m=2) \;=\; H\big(X_1,X_2\big) - \Big[H\big(X_1|X_2\big) + H\big(X_2|X_1\big)\Big] = I\big(X_1 \wedge X_2\big)$$

$$(m=2) \;=\; D\big(P_{X_1X_2}||P_{X_1} \times P_{X_2}\big)$$

$$(m\geq 2) \;=\; \min_{2\leq k\leq m} \;\; \min_{\mathcal{A}_k=(A_1,\ldots,A_k)} \;\; \frac{1}{k-1}\, D\Big(P_{X_1\ldots X_m}||\prod_{i=1}^{k} P_{X_{A_i}}\Big)$$

and equals 0 iff $P_{X_1\ldots X_m} \;=\; P_{X_A}P_{X_{A^c}}$ for some $A \subsetneq \mathcal{M}$.

¿ Does *shared information* have an operational significance as a measure of the mutual dependence among the rvs $X_1,\ldots,X_m$?

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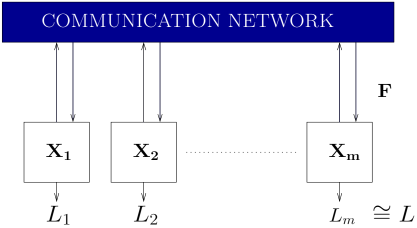
Omniscience



[I. Csiszár-P. Narayan]

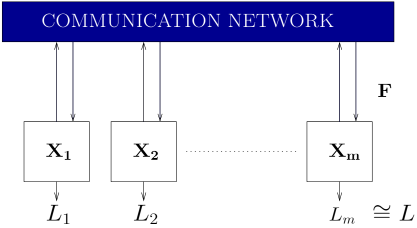
For $L = (X_1, \ldots, X_m)$, Theorem 2 gives

$$H(\mathbf{F}) \gtrsim H(X_1, \ldots, X_m) - SI(X_1, \ldots, X_m),$$

which, for $m = 2$, is

$$H(\mathbf{F}) \gtrsim H(X_1|X_2) + H(X_2|X_1). \quad \text{[Slepian – Wolf]}$$

Signal Recovery: Data Compression



[S. Nitinawarat-P. Narayan]

With $L = X_1$, by Theorem 2

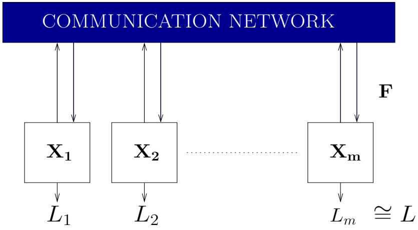
$$H\big(\mathbf{F}\big) \gtrapprox H\left(X_1\right) - SI\left(X_1, \ldots, X_m\right),$$

which, for $m = 2$, gives

$$H\big(\mathbf{F}\big) \gtrapprox H\big(X_1|X_2\big).$$

[Slepian-Wolf]

Secret Common Randomness



Terminals $1, \ldots, m$ generate CR L satisfying the *secrecy condition*

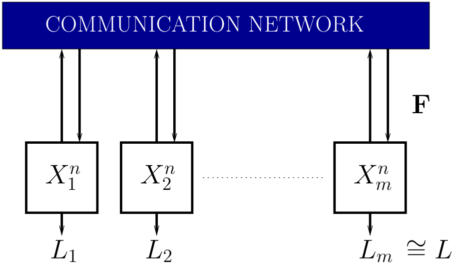
$$I\big(L \wedge \mathbf{F}\big) \;\cong\; 0.$$

By Theorem 2,

$$H\big(L\big) \cong H\big(L|\mathbf{F}\big) \;\lessapprox\; SI\big(X_1, \ldots, X_m\big).$$

- Secret key generation [I. Csiszár-P. Narayan]
- Secure function computation [H. Tyagi-P. Narayan]

Querying Common Randomness

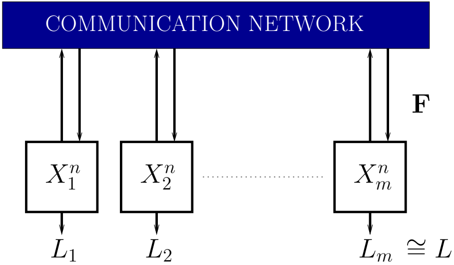


[H. Tyagi-P. Narayan]

- A querier observes communication $\mathbf{F}$ and seeks to resolve the value of CR L by asking questions: "Is $L = l$?" with yes-no answers.
- The terminals in $\mathcal{M}$ seek to generate L using $\mathbf{F}$ so as to make the querier's burden as onerous as possible.

¿ What is the largest query exponent ?

Largest Query Exponent



$$E^* \triangleq \arg\sup_{E} \left[\inf_{q} P\left(q\left(L \mid \mathbf{F}\right) \geq 2^{nE}\right) \to 1 \text{ as } n \to \infty \right]$$

$$E^* = SI\left(X_1, \ldots, X_m\right)$$

Shared information and a Hypothesis Testing Problem

$$SI(X_1, \ldots, X_m) = \min_{2 \le k \le m} \min_{\mathcal{A}_k = (A_1, \ldots, A_k)} \frac{1}{k-1} D\Big(P_{X_1 \ldots X_m} \Big|\Big| \prod_{i=1}^{k} P_{X_{A_i}}\Big)$$

- Related to exponent of "P_e-second kind" for an appropriate binary composite hypothesis testing problem, involving restricted CR L and communication $\mathbf{F}$.

H. Tyagi and S. Watanabe, "Converses for secret key agreement and secure computing," *IEEE Trans. Information Theory*, September 2015.

In Closing ...

¿ How useful is the concept of *shared information* **?**

A: Operational meaning in specific cases of distributed processing ...

In Closing ...

¿ How useful is the concept of *shared information* ?

A: Operational meaning in specific cases of distributed processing ...

For instance

- Consider n i.i.d. repetitions (say, in time) of the rvs $X_1, \ldots, X_m$.
- Data at time instant t is $X_{1t}, \ldots, X_{mt}, \;\; t = 1, \ldots, n$.
- Terminal i observes the i.i.d. data $\left(X_{i1}, \ldots, X_{in}\right), \;\; i \in \mathcal{M}$.
- Shared information-based results are asymptotically tight (in n):
  - *Minimum rate* of communication for omniscience
  - *Maximum rate* of a secret key
  - *Largest* query exponent
  - *Necessary condition* for secure function computation
  - Several problems in information theoretic cryptography.

Shared Information: Many Open Questions ...

- Significance in network source and channel coding ?
- Interactive communication over noisy channels ?
- Data-clustering applications ?
  [C. Chan-A. Al-Bashabsheh-Q. Zhou-T. Kaced-T.Liu, 2016]

$$\vdots$$

$$\vdots$$