Numerical Synthesis of Pontryagin Optimal Control Minimizers Using Sampling-Based Methods

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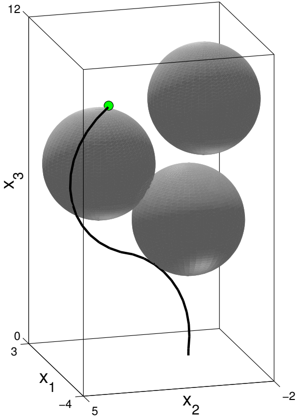
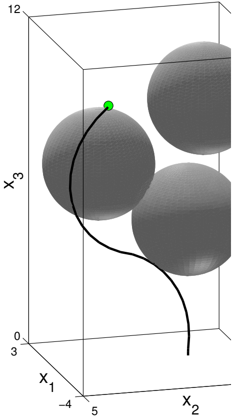
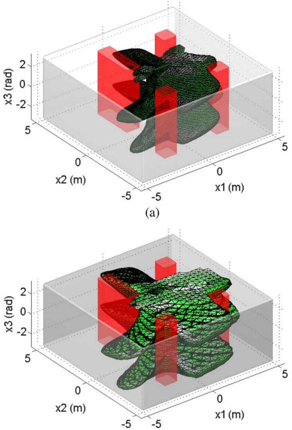
Optimal Control Applications



Path Planning (Vasudevan et al., 2013)

Optimal Control Applications



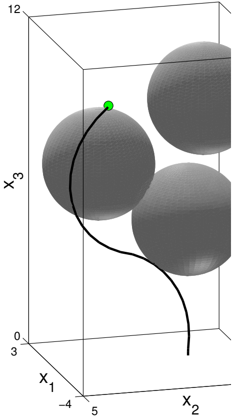


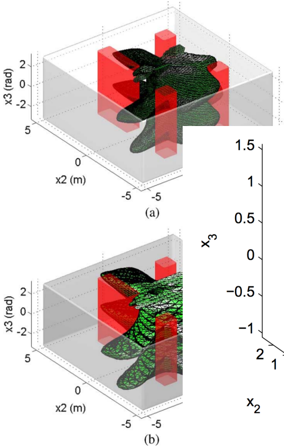
(a)

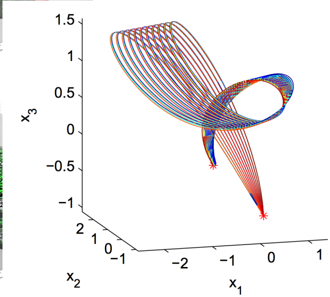
(b)

Dynamic Game Verification
(Margellos & Lygeros, 2011)

Optimal Control Applications







Robust & Ensemble Control
(Zlotnik & Li, 2012)

General Optimal Control Formulation

$$
\begin{aligned}
\min_{u(\cdot),\, x_0,\, T} \quad & \int_0^T L\big(x(s), u(s)\big)\,\mathrm{d}s + \varphi\big(x(T)\big), \\
\text{subject to:} \quad & \dot{x}(t) = f\big(x(t), u(t)\big),\ \forall t \in [0, T], \\
& x(0) = x_0, \\
& x(t) \in \mathcal{X},\ \forall t \in [0, T], \\
& u(t) \in \mathcal{U},\ \forall t \in [0, T], \\
& \big(x_0, x(T)\big) \in \mathcal{S}.
\end{aligned}
$$

From Theory to Practice...

Pros:

- Optimal control framework is *flexible*.
- Decades of accumulated literature.
- Particular cases can be efficiently solved.

Cons:

- General numerical solvers are prone to converge to non-minimizers.
- Computation time can be very long, even for small problems.

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Summary of this Talk

- We developed a new class of numerical optimal control synthesis method.
    - Trade-off between *accuracy* and *efficiency* can be easily configured.
    - Aim to solve large-scale problems.

- Our algorithm avoids *many* undesirable stationary points.
    - Enable real-time applications.

R. He and H. Gonzalez, "Numerical Synthesis of Pontryagin Optimal Control Minimizers Using Sampling-Based Methods," , 2017. arXiv: 1703.10751

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1. Critical Point Characterization

2. Synthesis of Optimal Control Inputs

3. Simulation: Optimal HVAC Operation

Simplified Optimal Control Formulation

Let us define the *original* optimal control problem:

$$
\begin{aligned}
(P_o) \qquad & \min_{u(\cdot)} \varphi\big(x(T)\big), \\
\text{subject to:} \quad & \dot{x}(t) = f\big(x(t), u(t)\big), \ \forall t \in [0, T], \\
& x(0) = x_0, \\
& u(t) \in \mathcal{U}, \ \forall t \in [0, T].
\end{aligned}
$$

where:

- $x_0 \in \mathbb{R}^n$ is given
- $\mathcal{U} \subset \mathbb{R}^m$ is compact and convex.

E. Polak, *Optimization: Algorithms and Consistent Approximations*, ser. Applied Mathematical Sciences. Springer, 1997, Chapter 4.1.2

Relaxed Optimal Control Formulation

Now consider the *relaxed* optimal control problem:

$$(P_r) \qquad \min_{\{\mu_t\}_{t\in[0,T]}} \varphi\big(x(T)\big),$$

$$\text{subject to:} \quad \dot{x}(t) = \int_{\mathbb{R}^m} f\big(x(t),u\big)\,\mathrm{d}\mu_t(u), \ \forall t \in [0,T],$$

$$x(0) = x_0,$$

$$\operatorname{supp}(\mu_t) \subset \mathcal{U}, \ \forall t \in [0,T].$$

where, for each $t \in [0,T]$, μ_t is a unitary Borel measure, i.e.:

$$\int_{\mathbb{R}^m} \mathrm{d}\mu_t(u) = 1.$$

J. Warga, *Optimal Control of Differential and Functional Equations*. Academic Press, 1972

Original-Relaxed Equivalence

Given $\hat{u}(t) \colon [0, T] \to \mathcal{U}$, let $\mu_t(u) = \mathbf{1}[u = \hat{u}(t)]$ for each t.

Then both original and relaxed trajectories generated by $\hat{u}$ and μ are equivalent.

Thus, $\text{Feas}(P_o) \subset \text{Feas}(P_r)$, and $\text{Value}(P_o) > \text{Value}(P_r)$.

Proposition

$\text{Value}(P_o) = \text{Value}(P_r)$.

The proof follows using Berkovitz's Chattering Lemma for switched systems and the fact that L^2 convergence of inputs implies uniform convergence of the trajectories.

L. D. Berkovitz, *Optimal Control Theory*, ser. Applied Mathematical Sciences. Springer, 1974

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Optimality Functions

Definition (Polak, 1997)

Consider the following optimization problem:

$$\min\{\psi(x) \mid x \in \mathcal{X}\}.$$

We say that $\theta\colon \mathcal{X} \to \mathbb{R}$ is an optimality function of this problem if:

- $\theta(x) \leq 0$ for each $x \in \mathcal{X}$; and,
- if x is a minimizer of ψ, then $\theta(x) = 0$.

For example, if $\mathcal{X} \subset \mathbb{R}^n$ then:

$$\theta(x) = \min_{\|h\|\leq 1} \langle \nabla\psi(x), h\rangle, \quad \text{and} \quad \theta(x) = \min_{h} \langle \nabla\psi(x), h\rangle + \|h\|_2^2,$$

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Original Problem Optimality Functions

Let $u_0\colon [0,T] \to \mathcal{U}$ and x_0 be its associated trajectory. Let:

$$\theta_{o,l}(x_0,u_0) = \min_{\delta u(\cdot)} \frac{\partial \varphi}{\partial x}\big(x_0(T)\big)\,\delta x(T),$$

subject to:

$$\delta \dot{x}(t) = \frac{\partial f}{\partial x}\big(x_0(t),u_0(t)\big)\,\delta x(t) + \\ + \frac{\partial f}{\partial u}\big(x_0(t),u_0(t)\big)\,\delta u(t),$$

$$\delta x(0) = 0,$$

$$u_0(t) + \delta u(t) \in \mathcal{U},\ \forall t \in [0,T].$$

We say that $\theta_{o,l}$ is the linearized, or variational, optimality function of P_o.

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Original Problem Optimality Functions

Also, let:

$$\theta_{o,h}(x_0,u_0) = \min_{u(\cdot)} \int_0^T p_0(t)^T \Big(f\big(x_0(t),u(t)\big) - f\big(x_0(t),u_0(t)\big) \Big)\, \mathrm{d}t,$$

subject to:
$$\dot{p}_0(t) = -\frac{\partial f}{\partial x}^T \big(x_0(t),u_0(t)\big)\, p_0(t),$$
$$p_0(T) = \frac{\partial \varphi}{\partial x}\big(x(T)\big),$$
$$u(t) \in \mathcal{U},\ \forall t \in [0,T].$$

We say that $\theta_{o,h}$ is the Pontryagin optimality function of P_o.

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$\theta_{o,h}(x_0,u_0) = 0 \ \Rightarrow\ \theta_{o,l}(x_0,u_0) = 0.$

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$$\delta x(0) = 0,$$

$$\operatorname{supp}(\mu_{0,t} + \delta\mu_t) \subset \mathcal{U},\ \forall t \in [0, T].$$

We say that $\theta_{r,l}$ is the linearized, or variational, optimality function of P_r.

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Proposition

$\theta_{r,h}(x_0, u_0) = \theta_{r,l}(x_0, u_0)$, *and so are their minimizing arguments.*

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Idea...

1. Solve the relaxed problem using (tried and tested) variational-based methods, where at each step we compute $\theta_{r,h}$ to check for optimality.
2. Once we find a minimizer, synthesize an input signal that approximates the optimal relaxed trajectory.

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1. Critical Point Characterization

2. Synthesis of Optimal Control Inputs

3. Simulation: Optimal HVAC Operation

Computing $\theta_{r,h}$

Note that $\theta_{r,h}$ is a convex optimization problem. In fact, it is a linear infinite-dimensional program.

Also note that we can rewrite $\theta_{r,h}$ as:

$$\theta_{r,h}(x_0, \mu_0) = \min_{\{\mu_{1,t}\}_t} \int_0^T p_0(t)^T \Big(\int_{\mathbb{R}^m} f\big(x_0(t), u\big)\, \mathrm{d}\mu_{1,t}(u) + \\ - \int_{\mathbb{R}^m} f\big(x_0(t), u\big)\, \mathrm{d}\mu_{0,t}(u) \Big)\, \mathrm{d}t,$$

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Computing $\theta_{r,h}$

Proposition

Let $x \in \mathbb{R}^n$. Then:

$$\left\{ \int_{\mathbb{R}^m} f(x,u)\,\mathrm{d}\mu(u) \mid \mathrm{supp}(\mu) \subset \mathcal{U} \right\} = \mathrm{co}\{f(x,u) \mid u \in \mathcal{U}\}.$$

Hence:

$$\theta_{r,h}(x_0,\mu_0) = \min_{\{\mu_{1,t}\}_t} \int_0^T p_0(t)^T \left(z(t) - \int_{\mathbb{R}^m} f(x_0(t),u)\,\mathrm{d}\mu_{0,t}(u) \right) \mathrm{d}t,$$

$$\text{subject to:} \quad z(t) \in \mathrm{co}\big\{ f\big(x_0(t),u\big) \mid u \in \mathcal{U} \big\}.$$

Computing Pontryagin-optimal points is as hard as finding a good representation for the convex hull of $f(x,\cdot)$.

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Sampling-Based Synthesis

Let $\{u_i\}_{i=1}^{N} \subset \mathcal{U}$ be a set of samples. Then for each $t \in [0, T]$:

$$\mathrm{co}\{ f(x_0(t), u) \mid u \in \mathcal{U} \} \approx$$

$$\approx \left\{ \sum_{i=1}^{N} \omega_i(t)\, f(x_0(t), u_i) \;\middle|\; \sum_{i=1}^{N} \omega_i(t) = 1,\ \omega_i(t) \geq 0 \right\}.$$

After sampling, our classical dynamical system becomes a switched hybrid system.

Hence, we can use the PWM-based projection method in Vasudevan et al. (SICON, 2013) to synthesize input signals.

R. Vasudevan, H. Gonzalez, R. Bajcsy, and S. S. Sastry, "Consistent Approximations for the Optimal Control of Constrained Switched Systems—Part 1: A Conceptual Algorithm," *Siam journal on control and optimization*, vol. 51, no. 6, pp. 4463–4483, 2013

R. Vasudevan, H. Gonzalez, R. Bajcsy, and S. S. Sastry, "Consistent Approximations for the Optimal Control of Constrained Switched Systems—Part 2: An Implementable Algorithm," *Siam journal on control and optimization*, vol. 51, no. 6, pp. 4484–4503, 2013

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Let $\{u_i\}_{i=1}^N \subset \mathcal{U}$ be a set of samples. Then for each $t \in [0, T]$:

$$\text{co}\{f(x_0(t), u) \mid u \in \mathcal{U}\} \approx$$
$$\approx \left\{ \sum_{i=1}^N \omega_i(t)\, f(x_0(t), u_i) \mid \sum_{i=1}^N \omega_i(t) = 1,\ \omega_i(t) \geq 0 \right\}.$$

After sampling, our classical dynamical system becomes a switched hybrid system.

Hence, we can use the PWM-based projection method in Vasudevan et al. (SICON, 2013) to synthesize input signals.

R. Vasudevan, H. Gonzalez, R. Bajcsy, and S. S. Sastry, "Consistent Approximations for the Optimal Control of Constrained Switched Systems—Part 1: A Conceptual Algorithm," *Siam journal on control and optimization*, vol. 51, no. 6, pp. 4463–4483, 2013

R. Vasudevan, H. Gonzalez, R. Bajcsy, and S. S. Sastry, "Consistent Approximations for the Optimal Control of Constrained Switched Systems—Part 2: An Implementable Algorithm," *Siam journal on control and optimization*, vol. 51, no. 6, pp. 4484–4503, 2013

Sampling-Based Synthesis Algorithm

1. Compute samples of $\mathcal{U}$.
2. Iteratively solve the relaxed optimal control problem using a gradient-based method.
3. Synthesize an approximation of the optimal relaxed input using PWM-based projection.

Numerical Considerations

- Note that if $f(x, u_k)$ lays strictly in the relative interior of $\mathrm{co}\{f(x,u) \mid u \in \{u_i\}_i\}$, then its computation is irrelevant.
  - Use `qhull` to reduce the number of samples.
- In large-scale problems most of the computation time is spent calculating $f\big(x_0(t), u_i\big)$ for each of the samples.
  - Use an ℓ_1 regularizer to force sparsity on the set $\{w_i(t)\}_i$.
- If smoothness is desirable, then use an ℓ_2 regularizer and a large number of samples.

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- If smoothness is desirable, then use an ℓ_2 regularizer and a large number of samples.

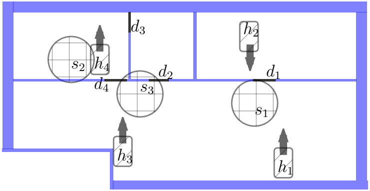
1. Critical Point Characterization

2. Synthesis of Optimal Control Inputs

3. Simulation: Optimal HVAC Operation

Comprehensive Building Operation

Besides controlling the HVAC unit in a building, we can make suggestions to the occupants regarding door configuration to improve efficiency and comfort.



Let $\theta \in \{0, 1\}^4$, where $\theta_i = 1$ if i-th door is open.

CFD Model

Heat convection-diffusion:

$$\frac{\partial T_e}{\partial t}(x,t) - \nabla_x \cdot (\kappa(x,\theta) \nabla_x T_e(x,t)) + \\ + u(x) \cdot \nabla_x T_e(x,t) = g_{T_e}(x,t),$$

Stationary imcompressible Navier-Stokes:

$$-\frac{1}{\mathrm{Re}} \Delta_x u(x) + \big(u(x) \cdot \nabla_x\big) u(x) + \\ + \nabla_x p(x) + \alpha(x,\theta)\, u(x) = g_u(x), \\ \nabla \cdot u = 0$$

Initial condition:

$$T_e(x,0) = \pi_0,$$

Boundary conditions:

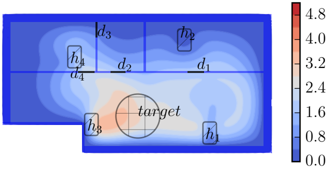
$$T_e(t,x) = 0, \text{ and} \\ u(x) = 0, \ \forall x \in \partial\Omega$$

R. He and H. Gonzalez, "Zoned HVAC Control via PDE-Constrained Optimization," in *Proceedings of the 2016 american control conference*, 2016. arXiv: 1504.04680

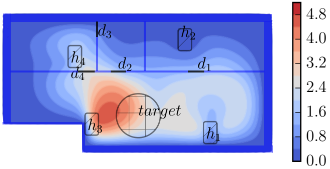
R. He and H. Gonzalez, "Gradient-Based Estimation of Air Flow and Geometry Configurations in a Building Using Fluid Dynamic Adjoint Equations," in *Proceedings of the 4th international high performance buildings conference at purdue*, 2016. arXiv: 1605.05339

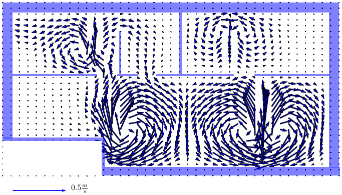
Single Zone Control

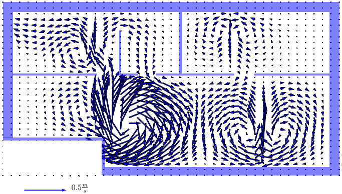
Closed doors:



Open doors:

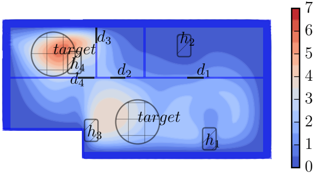




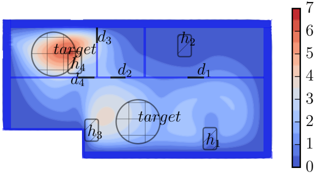


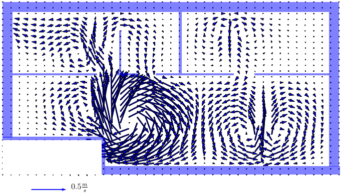
Dual Zone Control

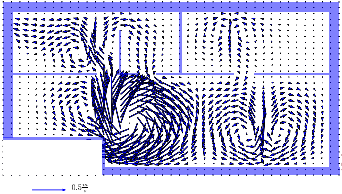
Closed doors:



Open doors:







"There is nothing so practical as a good theory"

— Kurt Lewin