Refinement of Two Fundamental Tools in Information Theory

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Discontinuity of Shannon’s Information Measures

- Shannon’s information measures: $H(X)$, $H(X|Y)$, $I(X;Y)$ and $I(X;Y|Z)$.

- They are described as continuous functions [Shannon 1948] [Csiszár & Körner 1981] [Cover & Thomas 1991] [McEliece 2002] [Yeung 2002].

- All Shannon's information measures are indeed discontinuous everywhere when random variables take values from countably infinite alphabets [Ho & Yeung 2005].

- e.g., $X$ can be any positive integer.
Discontinuity of Entropy

Let $P_X = \{1, 0, 0, ...\}$ and

$$P_{X_n} = \left\{ 1 - \frac{1}{\sqrt{\log n}}, \frac{1}{n \sqrt{\log n}}, \frac{1}{n \sqrt{\log n}}, ..., 0, 0, ... \right\}.$$

As $n \to \infty$, we have

$$\sum_i |P_X(i) - P_{X_n}(i)| = \frac{2}{\sqrt{\log n}} \to 0.$$

However,

$$\lim_{n \to \infty} H(X_n) = \infty.$$
Theorem 1: For any $c \geq 0$ and any $X$ taking values from a countably infinite alphabet with $H(X) < \infty$,

$$\exists P_{X_n} \text{ s.t. } V(P_X, P_{X_n}) = \sum_i |P_X(i) - P_{X_n}(i)| \to 0$$

but

$$H(X_n) \to H(X) + c$$
Theorem 2: For any $c \geq 0$ and any $X$ taking values from countably infinite alphabet with $H(X) < \infty$, 

$$\exists P_{X^n} \text{ s.t. } D(P_X \parallel P_{X^n}) = \sum_i P_X(i) \log \frac{P_X(i)}{P_{X^n}(i)} \to 0$$

but $H(X_n) \to H(X) + c$
Pinsker’s inequality

\[ D(p \parallel q) \geq \frac{1}{2 \ln 2} V^2(p, q) \]

- By Pinsker’s inequality, convergence w.r.t. \( D(\cdot \parallel \cdot) \) implies convergence w.r.t. \( V(\cdot; \cdot) \).
- Therefore, Theorem 2 implies Theorem 1.
Discontinuity of Entropy
Theorem 3: For any $X, Y$ and $Z$ taking values from a countably infinite alphabet with $I(X; Y|Z) < \infty$, there exists $P_{X_nY_nZ_n}$ such that

$$\lim_{n \to \infty} D(P_{XYZ} \parallel P_{X_nY_nZ_n}) = 0$$

but

$$\lim_{n \to \infty} I(X_n; Y_n | Z_n) = \infty.$$
Discontinuity of Shannon’s Information Measures

Applications:
- channel coding theorem
- lossless/lossy source coding theorems, etc.

Typicality

Fano’s Inequality

Shannon’s Information Measures
To Find the Capacity of a Communication Channel

\[ \text{Capacity} \geq C_1 \quad \text{Typicality} \]

\[ \text{Capacity} \leq C_2 \quad \text{Fano’s Inequality} \]
On Countably Infinite Alphabet

Applications:
channel coding theorem
lossless/lossy source coding theorems, etc.

Typicality

Fano’s Inequality

Shannon’s Information Measures

discontinuous!
Typicality

- Weak typicality was first introduced by Shannon [1948] to establish the source coding theorem.
- Strong typicality was first used by Wolfowitz [1964] and then by Berger [1978]. It was further developed into the method of types by Csiszár and Körner [1981].
- Strong typicality possesses stronger properties compared with weak typicality.
- It can be used only for random variables with finite alphabet.
Consider an i.i.d. source \( \{X_k, k \geq 1\} \), where \( X_k \) taking values from a countable alphabet \( \mathcal{X} \).

Let \( P_X = P_{X_k} \) for all \( k \).

Assume \( H(P_X) < \infty \).

Let \( X = (X_1, X_2, ..., X_n) \).

For a sequence \( x = (x_1, x_2, ..., x_n) \in \mathcal{X}^n \),

- \( N(x; x) \) is the number of occurrences of \( x \) in \( x \)
- \( q(x; x) = n^{-1}N(x; x) \) and
- \( Q_X = \{q(x; x)\} \) is the empirical distribution of \( x \)

E.g., \( x = (1, 3, 2, 1, 1) \).

\[ N(1; x) = 3, \quad N(2; x) = N(3; x) = 1 \]

\[ Q_X = \{3/5, 1/5, 1/5\} \].
Definition (Weak typicality): For any $\varepsilon > 0$, the weakly typical set $W_n^{[X],\varepsilon}$ with respect to $P_X$ is the set of sequences $x = (x_1, x_2, ..., x_n) \in X^n$ such that

$$\left| -\frac{1}{n} \log P_X(x) - H(P_X) \right| \leq \varepsilon$$
Weak Typicality

Definition 1 (Weak typicality): For any $\varepsilon > 0$, the weakly typical set $W^n_{[x]_\varepsilon}$ with respect to $P_X$ is the set of sequences $x = (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ such that

$$|D(Q_X \| P_X) + H(Q_X) - H(P_X)| \leq \varepsilon$$

Note that

$$H(Q_X) = -\sum_x Q_X(x) \log Q_X(x)$$

while

Empirical entropy $= -\sum_x Q_X(x) \log P_X(x)$
Theorem 4 (Weak AEP): For any $\varepsilon > 0$:

1) If $x \in W^n_{[X]x} \varepsilon$, then

$$2^{-n(H(X)+\varepsilon)} \leq p(x) \leq 2^{-n(H(X)-\varepsilon)}$$

2) For sufficiently large $n$,

$$\Pr\{X \in W^n_{[X]x} \varepsilon\} > 1 - \varepsilon$$

3) For sufficiently large $n$,

$$(1 - \varepsilon)2^{n(H(X)-\varepsilon)} \leq |W^n_{[X]x} \varepsilon| \leq 2^{n(H(X)+\varepsilon)}$$
Illustration of AEP

$\mathcal{X}^n$ – Set of all $n$-sequences

Typical Set of $n$-sequences: Prob. $\approx 1$
$\approx$ Uniform distribution
Strong Typicality

- Strong typicality has been defined in slightly different forms in the literature.

- **Definition 2 (Strong typicality):** For $|\mathcal{X}| < \infty$ and any $\delta > 0$, the strongly typical set $\mathcal{T}_{[\mathcal{X}]\delta}^n$ with respect to $P_X$ is the set of sequences $x = (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ such that
  \[
  V(P_X, Q_X) = \sum_x |P_X(x) - q(x; x)| \leq \delta
  \]
  the variational distance between the empirical distribution of the sequence $x$ and $P_X$ is small.
Asymptotic Equipartition Property

- **Theorem 5 (Strong AEP):** For a finite alphabet $\mathcal{X}$ and any $\delta > 0$:
  - 1) If $x \in T^n_{\mathcal{X}\delta}$, then
    
    $$2^{-n(H(X)+\delta)} \leq p(x) \leq 2^{-n(H(X)-\delta)}$$
  
  - 2) For sufficiently large $n$,
    
    $$\Pr\{x \in T^n_{\mathcal{X}\delta}\} > 1 - \delta$$
  
  - 3) For sufficiently large $n$,
    
    $$(1 - \delta)2^{n(H(X)-\gamma)} \leq |T^n_{\mathcal{X}\delta}| \leq 2^{n(H(X)+\gamma)}$$
Breakdown of Strong AEP

- If strong typicality is extended (in the natural way) to countably infinite alphabets, strong AEP no longer holds.
- Specifically, Property 2 holds but Properties 1 and 3 do not hold.
Typicality

\( X^n \) finite alphabet

Weak Typicality:

\[
| D(Q_X \parallel P_X) + H(Q_X) - H(P_X) | \leq \varepsilon
\]

Strong Typicality:

\[
V(P_X, Q_X) \leq \delta
\]
Unified Typicality

\( \mathcal{X}^n \) countably infinite alphabet

Weak Typicality:

\[ |D(Q_X \parallel P_X) + H(Q_X) - H(P_X)| \leq \varepsilon \]

Strong Typicality:

\[ V(P_X, Q_X) \leq \delta \]

\( \exists x \) s.t. \( D(Q_X \parallel P_X) \) is small but \( |H(Q_X) - H(P_X)| \) is large
Unified Typicality

$\mathcal{X}^n$ countably infinite alphabet

Weak Typicality:

$\left| D(Q_X \parallel P_X) + H(Q_X) - H(P_X) \right| \leq \varepsilon$

Strong Typicality:

$V(P_X, Q_X) \leq \delta$

Unified Typicality:

$D(Q_X \parallel P_X) + |H(Q_X) - H(P_X)| \leq \eta$. 
Unified Typicality

- Definition 3 (Unified typicality): For any \( \eta > 0 \), the unified typical set \( U^n_{\{x\}\eta} \) with respect to \( P_X \) is the set of sequences \( x = (x_1, x_2, ..., x_n) \in \mathcal{X}^n \) such that

\[
D(Q_X \parallel P_X) + |H(Q_X) - H(P_X)| \leq \eta
\]

- Weak Typicality: \( |D(Q_X \parallel P_X) + H(Q_X) - H(P_X)| \leq \varepsilon \)

- Strong Typicality: \( V(P_X, Q_X) \leq \delta \)

- Each typicality corresponds to a “distance measure”

- Entropy is continuous w.r.t. the distance measure induced by unified typicality
Theorem 6 (Unified AEP): For any $\eta > 0$:

1) If $x \in U^n_{[X]_\eta}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}$$

2) For sufficiently large $n$,

$$\Pr\left\{ X \in U^n_{[X]_\eta} \right\} > 1 - \eta$$

3) For sufficiently large $n$,

$$(1 - \eta)2^{n(H(X)-\mu)} \leq \left| U^n_{[X]_\eta} \right| \leq 2^{n(H(X)+\mu)}$$
Unified Typicality

Theorem 7: For any $x \in X^n$, if $x \in U^n_{[X]\eta}$, then $x \in W^n_{[X]\epsilon}$ and $x \in T^n_{[X]\delta}$, where $\epsilon = \eta$ and $\delta = \sqrt{\eta \cdot 2 \ln 2}$. 
Unified Jointly Typicality

- Consider a bivariate information source \( \{(X_k, Y_k), \ k \geq 1\} \) where \( (X_k, Y_k) \) are i.i.d. with generic distribution \( P_{XY} \).
- We use \( (X, Y) \) to denote the pair of generic random variables.
- Let \( (X, Y) = ((X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)) \).
- For the pair of sequence \( (x, y) \), the empirical distribution is \( Q_{XY} = \{q(x,y; x,y)\} \) where \( q(x,y; x,y) = n^{-1}N(x,y; x,y) \).
Unified Jointly Typicality

- **Definition 4 (Unified jointly typicality):** For any $\eta > 0$, the unified typical set $U_{[XY]n}^{\eta}$ with respect to $P_{XY}$ is the set of sequences $(x, y) \in \mathcal{X}^n \times \mathcal{Y}^n$ such that

$$D(Q_{XY} \parallel P_{XY}) + |H(Q_{XY}) - H(P_{XY})|$$
$$+ |H(Q_X) - H(P_X)| + |H(Q_Y) - H(P_Y)| \leq \eta.$$

- This definition cannot be simplified.
Conditional AEP

- **Definition 5**: For any \( x \in U^n_{[X]\eta} \), the conditional typical set of \( Y \) is defined as

\[
U^n_{[Y|X]\eta}(x) = \{ y \in U^n_{[Y]\eta} : (x, y) \in U^n_{[XY]\eta} \}
\]

- **Theorem 8**: For \( x \in U^n_{[X]\eta} \), if

\[
\left| U^n_{[Y|X]\eta}(x) \right| \geq 1,
\]

then

\[
2^{n(H(Y|X)-\nu)} \leq \left| U^n_{[Y|X]\eta}(x) \right| \leq 2^{n(H(Y|X)+\nu)}
\]

where \( \nu \to 0 \) as \( \eta \to 0 \) and then \( n \to \infty \).
Illustration of Conditional AEP

\[ 2^{nH(Y)} \quad y \in S^n_{[Y]_{\delta}} \]

\[ 2^{nH(X)} \quad x \in S^n_{[X]_{\delta}} \]

\[ (x,y) \in T^n_{[XY]_{\delta}} \]
Applications

- Rate-distortion theory
  - A version of rate-distortion theorem was proved by strong typicality [Cover & Thomas 1991][Yeung 2008]
  - It can be easily generalized to countably infinite alphabet

- Multi-source network coding
  - The achievable information rate region in multisource network coding problem was proved by strong typicality [Yeung 2008]
  - It can be easily generalized to countably infinite alphabet
Fano’s Inequality

- **Fano's inequality**: For discrete random variables $X$ and $Y$ taking values on the same alphabet $\mathcal{X} = \{1, 2, \ldots\}$, let
  \[ \varepsilon = P[X \neq Y] = 1 - \sum_{w \in \mathcal{X}} P_{XY}(w, w) \]

- Then
  \[ H(X \mid Y) \leq \varepsilon \log(\varepsilon) - 1 + h(\varepsilon), \]

where
  \[ h(x) = x \log \frac{1}{x} + (1 - x) \log \frac{1}{1 - x} \]

for $0 < x < 1$ and $h(0) = h(1) = 0$. 
Motivation 1

\[ H(X \mid Y) \leq \varepsilon \log(|X| - 1) + h(\varepsilon) \]

- This upper bound on \( H(X \mid Y) \) is not tight.
- For fixed \( \varepsilon \) and \( |X| \), can always find \( X \) such that
  \[ H(X \mid Y) \leq H(X) < \varepsilon \log(|X| - 1) + h(\varepsilon) \]
- Then we can ask, for fixed \( P_X \) and \( \varepsilon \), what is

\[ \max_{P_{Y \mid X} : P[X \neq Y] = \varepsilon} H(X \mid Y) < \varepsilon \log(|X| - 1) + h(\varepsilon) \]
Motivation 2

- If $\mathcal{X}$ is countably infinite, Fano’s inequality no longer gives an upper bound on $H(X|Y)$.
- It is possible that $H(X|Y) \rightarrow 0$ as $\varepsilon \rightarrow 0$ which can be explained by the discontinuity of entropy.
- \[ P_{X_n} = \left\{ 1 - \frac{1}{\sqrt{\log n}}, \frac{1}{n\sqrt{\log n}}, \ldots, \frac{1}{n\sqrt{\log n}} \right\} \] and $P_{Y_n} = \{1,0,0,...\}$
- Then $H(X_n|Y_n) = H(X_n) \rightarrow \infty$ but $\varepsilon_n = \frac{1}{\sqrt{\log n}} \rightarrow 0$
- Under what conditions $\varepsilon \rightarrow 0 \Rightarrow H(X|Y) \rightarrow 0$ for countably infinite alphabets?
Tight Upper Bound on $H(X|Y)$

Theorem 9: Suppose $\varepsilon = P[X \neq Y] \leq 1 - P_X(1)$, then

$$H(X | Y) \leq \varepsilon \cdot H(Q(P_X, \varepsilon)) + h(\varepsilon)$$

where the right side is the tight bound dependent on $\varepsilon$ and $P_X$. (This is the simplest of the 3 cases.)

Let $\Phi_X(\varepsilon) = \varepsilon \cdot H(Q(P_X, \varepsilon)) + h(\varepsilon)$
Generalizing Fano’s Inequality

- Fano's inequality [Fano 1952] gives an upper bound on the conditional entropy $H(X|Y)$ in terms of the error probability $\varepsilon = \Pr\{X \neq Y\}$.

- e.g. $P_X = [0.4, 0.4, 0.1, 0.1]$
Generalizing Fano’s Inequality

- e.g., $X$ is a Poisson random variable with mean equal to 10.
- Fano's inequality no longer gives an upper bound on $H(X|Y)$. 

$H(X|Y)$

$\varepsilon$
Generalizing Fano’s Inequality

- e.g. $X$ is a Poisson random variable with mean equal to 10.
- Fano's inequality no longer gives an upper bound on $H(X|Y)$.

$H(X|Y)$

[Ho & Verdú 2008]
Joint Source-Channel Coding

\((S_1, S_2, \ldots, S_k) \rightarrow \text{Encoder} \rightarrow (X_1, X_2, \ldots, X_n)\)

\((\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_k) \leftarrow \text{Decoder} \leftarrow (Y_1, Y_2, \ldots, Y_n)\)

\(k\)-to-\(n\) joint source-channel code
Error Probabilities

- The average symbol error probability is defined as
  \[ \lambda_k = \frac{1}{k} \sum_{i=1}^{k} P[S_i \neq \hat{S}_i] \]

- The block error probability is defined as
  \[ \mu_k = P[(S_1, S_2, \ldots, S_k) \neq (\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_k)] \]
Symbol Error Rate

Theorem 10: For any discrete memoryless source and general channel, the rate of a $k$-to-$n$ joint source-channel code with symbol error probability $\lambda_k$ satisfies

$$\frac{k}{n} \leq \frac{\sup_{X^n} n^{-1} I(X^n;Y^n)}{k^{-1} H(S^k) - \Phi_{S^*}(\lambda_k)}$$

where $S^*$ is constructed from $\{S_1, S_2, ..., S_k\}$ according to

$$P_{S^*}(1) = \min_j P_{S_j}(1),$$

$$P_{S^*}(a) = \min_j \sum_{i=1}^{a} P_{S_j}(i) - \sum_{i=1}^{a-1} P_{S^*}(i) \quad a \geq 2.$$
Theorem 11: For any general discrete source and general channel, the block error probability $\mu_k$ of a $k$-to-$n$ joint source-channel code is lower bounded by

$$\Phi_{S^k}^{-1}\left(H(S^k) - \sup_{X^n} I(X^n; Y^n)\right) \leq \mu_k$$
Weak secrecy has been considered in [Csiszár & Körner 78, Broadcast channel] and some seminal papers.

[Wyner 75, Wiretap channel I] only stated that “a large value of the equivocation implies a large value of \( P_{ew} \), where the equivocation refers to \( n^{-1}H(X^n | Y^k) \) and \( P_{ew} \) means \( \mu_n \).

It is important to clarify what exactly weak secrecy implies.
Weak Secrecy

E.g., $P_X = (0.4, 0.4, 0.1, 0.1)$.

$$\varepsilon = P[X \neq Y]$$
Weak Secrecy

- **Theorem 12:** For any discrete stationary memoryless source (i.i.d. source) with distribution \( P_X \), if
  \[
  \lim_{n \to \infty} n^{-1} I(X^n; Y^n) = 0,
  \]

- Then
  \[
  \lim_{n \to \infty} \lambda_n = \lambda_{\text{max}} \quad \text{and} \quad \lim_{n \to \infty} \mu_n = 1.
  \]

- **Remark:**
  - Weak Secrecy together with the stationary source assumption is insufficient to show the maximum error probability.
  - The proof is based on the tight upper bound on \( H(X|Y) \) in terms of error probability.
Summary

Applications:
- channel coding theorem
- lossless/lossy source coding theorems

Typicality
- Weak Typicality
- Strong Typicality

Fano’s Inequality

Shannon’s Information Measures
On Countably Infinite Alphabet

Applications:
- channel coding theorem
- lossless/lossy source coding theorem

Typicality

Weak Typicality

Shannon’s Information Measures

discontinuous!
Unified Typicality

Applications:
- channel coding theorem
- MSNC/lossy SC theorems

Typicality

Unified Typicality

Shannon’s Information Measures
Generalized Fano’s Inequality

Applications:
- results on JSCC, IT security
- MSNC/lossy SC theorems

Typicality

Unified Typicality

Generalized Fano’s Inequality

Shannon’s Information Measures
A lot of fundamental research in information theory are still waiting for us to investigate.
References


Q & A
Why Countably Infinite Alphabet?

- An important mathematical theory can provide some insights which cannot be obtained from other means.
- Problems involve random variables taking values from countably infinite alphabets.
- Finite alphabet is the special case.
- Benefits: tighter bounds, faster convergent rates, etc.
- In source coding, the alphabet size can be very large, infinite or unknown.
Discontinuity of Entropy

- Entropy is a measure of uncertainty.

- *We can be more and more sure that a particular event will happen as time goes, but at the same time, the uncertainty of the whole picture keeps on increasing.*

- If one found the above statement counter-intuitive, he/she may have the concept that entropy is continuous rooted in his/her mind.

- The limiting probability distribution may not fully characterize the asymptotic behavior of a Markov chain.
Discontinuity of Entropy

Suppose a child hides in a shopping mall where the floor plan is shown in the next slide.

In each case, the chance for him to hide in a room is directly proportional to the size of the room.

We are only interested in which room the child locates in but not his exact position inside a room.

Which case do you expect is the easiest to locate the child?
### Case A
1 blue room + 2 green rooms

### Case B
1 blue room + 16 green rooms

### Case C
1 blue room + 256 green rooms

### Case D
1 blue room + 4096 green rooms

<table>
<thead>
<tr>
<th></th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
<th>Case D</th>
</tr>
</thead>
<tbody>
<tr>
<td>The chance in the blue room</td>
<td>0.5</td>
<td>0.622</td>
<td>0.698</td>
<td>0.742</td>
</tr>
<tr>
<td>The chance in a green room</td>
<td>0.25</td>
<td>0.0326</td>
<td>0.00118</td>
<td>0.000063</td>
</tr>
</tbody>
</table>
Discontinuity of Entropy

From Case A to Case D, the difficulty is increasing. By the Shannon entropy, the uncertainty is increasing although the probability of the child being in the blue room is also increasing.

We can continue to construct this example and make the chance in the blue room approaching to 1!

The critical assumption is that the number of rooms can be unbounded. So we have seen that “There is a very sure event” and “large uncertainty of the whole picture” can exist at the same time.

Imagine there is a city where everyone has a normal life everyday with probability 0.99. With probability 0.01, however, any kind of accident that beyond our imagination can happen. Would you feel a big uncertainty about your life if you were living in that city?
Weak secrecy is insufficient to show the maximum error probability.

Example 1: Let $W$, $V$ and $X_i$ be binary random variables.

Suppose $W$ and $V$ are independent and uniform.

Let

$$X_i = \begin{cases} W & V = 0 \\ \text{independent and uniform} & V = 1 \end{cases}$$

$$\lambda_{\text{max}} = 1 - \max_x P_X (x) = 0.5$$

$$\mu_{\text{max}} = \lim_{n \to \infty} \left(1 - \max_{x^n} P_{X^n} (x^n)\right) = \lim_{n \to \infty} \left(1 - \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{4}$$
Example 1

- Let

\[
\begin{array}{c|ccccc}
Y_1 & Y_2 & Y_3 & Y_4 & \ldots \\
\hline
X_1 & X_4 & X_9 & X_{16} \\
X_2 & X_3 & X_8 & X_{15} \\
X_5 & X_6 & X_7 & X_{14} \\
X_{10} & X_{11} & X_{12} & X_{13} \\
\end{array}
\]

\[
0 \leq \lim_{n \to \infty} n^{-1} I(X^n; Y^k) \leq \lim_{n \to \infty} n^{-1} \sqrt{n} = 0
\]

Choose \( \hat{x}_n = \begin{cases} 
(0,0,\ldots,0) & \text{if } Y_i = 0 & \forall i \\
(1,1,\ldots,1) & \text{if } Y_i = 1 & \forall i.
\end{cases} \)

- Then

\[
\lim_{n \to \infty} \mu_n = P[V = 1] = \frac{1}{2} < \tilde{\mu}_{\max} = \frac{3}{4}
\]

\[
\lim_{n \to \infty} \lambda_n = P[V = 1] \cdot \frac{1}{2} = \frac{1}{4} < \tilde{\lambda}_{\max} = \frac{1}{2}
\]
Joint Unified Typicality

Can be changed to

\[ D(Q_{XY} \parallel P_{XY}) + |H(Q_{XY}) - H(P_{XY})| + |H(Q_X) - H(P_X)| + |H(Q_Y) - H(P_Y)| \leq \eta. \]

Ans:

\[ Q = \{q(xy)\} \]

\[ P = \{p(xy)\} \]

\[ D(Q \parallel P) << 1 \]
Joint Unified Typicality

Can be changed to

$$D(Q_{XY} \parallel P_{XY}) + |H(Q_{XY}) - H(P_{XY})| + |H(Q_X) - H(P_X)| + |H(Q_Y) - H(P_Y)| \leq \eta.$$ 

Ans:

$$Q = \{q(xy)\}$$

$$P = \{p(xy)\}$$

$$D(Q \parallel P) << 1$$
Asymptotic Equipartition Property

- **Theorem 5 (Consistency):** For any \((x, y) \in \mathcal{X}^n \times \mathcal{Y}^n\),
  \[
  \text{if } (x, y) \in U_{[XY]}^{n,\eta} \text{, then } x \in U_{[X]}^{n,\eta} \text{ and } y \in U_{[Y]}^{n,\eta}.
  \]

- **Theorem 6 (Unified JAEP):** For any \(\eta > 0\):
  1) If \((x, y) \in U_{[XY]}^{n,\eta}\), then
  \[
  2^{-n(H(XY)+\eta)} \leq p(x, y) \leq 2^{-n(H(XY)-\eta)}
  \]
  2) For sufficiently large \(n\),
  \[
  \Pr \left\{ (X, Y) \in U_{[XY]}^{n,\eta} \right\} > 1 - \eta
  \]
  3) For sufficiently large \(n\),
  \[
  (1-\eta)2^{n(H(XY)-\eta)} \leq \left| U_{[XY]}^{n,\eta} \right| \leq 2^{n(H(XY)+\eta)}
  \]