Solution of a Class of Second Order Functional Difference Equations in Electromagnetic Diffraction Theory

by

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Nous devons l'unique science
Que l'on puisse conquérir
Aux chercheurs dont la patience
En a laissé les fruits mûrir.

Adapted from Sully-Prudhomme, Le Bonheur.

Les chercheurs qui cherchent, on en trouve.
Les chercheurs qui trouvent, on en cherche.

Unknown.

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CHAPTER 1

INTRODUCTION

DIFFRACTION theory originated over 100 years ago from the consideration of wave scattering by edges [POINCARÉ, 1892; SOMMERFELD, 1896], a topic which remains to this day at the heart of this challenging discipline. This stems not only from interest in the intrinsic scattering properties of edges but also, more pragmatically, from their role in determining the scattering characteristics of complex bodies. Indeed, under the localization principle, scattering from intricate geometries can be expressed in terms of coherent contributions from distinct scattering centers, such as edges and corners, together with appropriate specular components. Applications abound in a wide range of fields concerned with electromagnetic, acoustic and elastic wave scattering. In electromagnetics these are epitomized by radar cross section analysis and synthesis as well as the study of wave propagation in the presence of man-made obstacles. For the latter, the problem of propagation in urban environments leaps to mind, and this has been the subject of recent interest with the spread of wireless technology. Whereas these applications are mainly concerned with (exterior) edges having reentrant angles less than π, there are also cases where one is predominantly concerned with interior edges (or corners) — those whose reentrant angle is greater than π — and a good example is provided by rectangular guiding structures such as waveguides and aperture antennas with non-perfectly conducting or corrugated walls. There is therefore a strong incentive for a thorough characterization of edge scattering and the canonical geometry of choice is the wedge which includes the half-plane as a special case. The wedge geometry provides much flexibility: a wide range of configurations are possible by varying its opening angle as well as the boundary conditions characterizing each of its faces. The boundary conditions may be, in the simplest cases, of the Dirichlet or Neumann type or, in more complicated cases, mixed conditions approximating non-penetrable surfaces or transition conditions for penetrable ones. Despite the disarmingly simple nature of the geometry, the solution of Helmholtz's equation in wedge-shaped regions remains notoriously difficult, if at all possible, especially when mixed or penetrable boundary conditions are used.

It was not until the later part of the the XIXth century that the mathematical theory of diffraction was put on a rigorous footing. The first publication on the subject is by the French mathematician POINCARÉ [1892] who, using separation of variables, studied polarization by diffraction of cylindrical waves impinging on half-planes and wedges. His work dealt mainly with perfectly conducting surfaces though he did provide some discussion of metallic (non-perfectly conducting) structures. It is however SOMMERFELD [1896], with his rigorous solution for the perfectly conducting half-plane, that is today acknowledged as the founding father of mathematical diffraction theory. In that celebrated paper Sommerfeld solved the problem by introducing an angular spectral representation for the fields, an approach which would some 50 years later form the basis of the powerful Sommerfeld-Maliuzhinets technique. An English version of the solution is presented in his *Lectures on Theoretical Physics* [SOMMERFELD, 1964] and a rigorous treatment of the angular spectrum representation within the framework of the Laplace transform is provided by BUDAEV [1995]. Sommerfeld's paper catalyzed a number of research efforts. MACDONALD [1902] provided an extensive analysis of the perfectly conducting wedge six years later in which he considered the two dimensional cases of scattering due to plane waves and line sources as well as the three dimensional case of point source illumination. The case of arbitrary plane wave incidence on a wedge was also examined by CARSLAW [1919] using an extension of Sommerfeld's approach. See JONES [1964], BOWMAN *et al.* [1969] and RUCK [1970] for more thorough surveys of works published in that period.

The brunt of the geometries examined in those early days had perfectly conducting surfaces corresponding to the cases of greatest mathematical simplicity since only Dirichlet and Neumann boundary conditions are required. However, the growing interest in electromagnetic waves and their applications would spur a desire to model more accurately actual geometries and attention would soon turn to problems having non-perfectly conducting surfaces. There are unfortunately no exact solutions in such cases and it is only with the advent of adequate approximate boundary conditions [see for reference SENIOR AND VOLAKIS, 1995] in the early 1940s that they would be dealt with — albeit approximately — rigorously and accurately. The two most widely-used boundary conditions approximating non-perfectly conducting surfaces are the impedance and resistive boundary conditions. The former approximate interfaces (possibly corrugated or rough) with non-penetrable (lossy) media and yield surprisingly good accuracy when properly employed [SENIOR AND VOLAKIS, 1995]. The operating principle behind impedance boundary conditions is actually quite intuitive. If we consider an interface with a lossy medium, electromagnetic waves are increasingly confined to the interface with increasing loss; it is therefore not surprising that, if the loss is sufficiently high, a properly defined boundary condition will succeed in capturing the salient properties of the interface. The most widely used impedance boundary condition is the first order standard impedance boundary condition which, since it has historically been attributed to LEONTOVICH [1948] — but see [SENIOR AND VOLAKIS, 1995, pp. 10–11] for an interesting discussion of its origins — it is also known as the Leontovich boundary condition. It was conceived while studying radio wave propagation over lossy media and can be considered as a special (lowest order) case of the generalized impedance boundary condition which arose from the work of RYTOV [1940]. The resistive boundary condition is used to approximate thin lossy dielectric sheets and its origins can be traced to the early part of the XX$^{\text{th}}$ century. The reader is referred again to [SENIOR AND VOLAKIS, 1995] for a thorough discussion. It is also referred to, more accurately, as a resistive transition condition since it imposes a relationship on field components on either side of thin penetrable sheets. Wedges characterized with one or two resistive conditions present significant challenges as the fields must be solved in both the interior and exterior of the structure.

The appearance of these boundary conditions would enable the modeling of the material characteristics of real-life geometries with much more fidelity. Furthermore, the advent of more sophisticated analysis techniques at roughly the same period in time would make it possible to obtain solutions even when the faces of the wedge are characterized with mixed boundary conditions, leading to the first major advances in electromagnetic diffraction theory in this area since the early part of the XX$^{\text{th}}$ century. The problem of scattering from a half-plane with common (isotropic) impedance faces illuminated at normal incidence was solved by SENIOR [1952] and he subsequently extended it to the case of oblique incidence [SENIOR, 1959*a*]. Senior used the Wiener-Hopf function-theoretic technique — see for reference [NOBLE, 1958] — which provides the means of finding solutions for a large number of canonical problems but can be brought to bear only on rectangular geometries such as half-planes and right-angled junctions. A technique suitable to more general wedge configurations was proposed by MALIUZHINETS [1950, 1958*a*,*b*]. Maliuzhinets[1], a student of Fock and Leontovich, used as a starting point the angular spectral representation proposed over 50 years earlier by Sommerfeld and successfully reduced the problem to equivalent functional difference equations for the spectra. This function-theoretic technique is now referred to as the Sommerfeld-Maliuzhinets method and has been recently reviewed by OSIPOV AND NORRIS [1999] and NORRIS AND OSIPOV [1999]. Following this approach, Maliuzhinets obtained a solution to the problem of the wedge of arbitrary angle and differing

[1]See the short biography by MANARA *et al.* [1998] for interesting notes and a list of publications.

face impedances [Maliuzhinets, 1958*a*]. It is also noted that Senior [1959*b*] and Williams [1959], using the Wiener-Hopf technique, provided solutions for the wedge with common face impedances under condition of non-oblique illumination. Maliuzhinets's approach and the (Maliuzhinets) functions he derived to solve the difference equations have been extended in a number of investigations: a few examples are provided by the first in a series of papers by Tuzhilin [1970] and the recent effort of Avdeev [1994].

The majority of the problems involving impedance wedges for which exact solutions are available correspond to the case of normal incidence, that is, when the incident wave impinges normally on the edge of the structure. It turns out that diffraction from impedance structures under oblique (non-normal) incidence is much more difficult due to coupling between the field components. The handful of geometries that have been solved, using mainly either the Wiener-Hopf or the Sommerfeld-Maliuzhinets technique, are the planar impedance junction [Vaccaro, 1980; Senior, 1986; Rojas, 1988*b*], the impedance half-plane [Bucci and Franceschetti, 1976; Rojas, 1988*b*], the right-angled wedge with one perfect electrically conducting face [Vaccaro, 1981; Senior and Volakis, 1986; Rojas, 1988*a*; Manara and Nepa, 2000], the polarization independent anisotropic impedance wedge [Bernard, 1998], and the right-angled wedge with specific anisotropic impedance faces examined by Lyalinov and Zhu [1999]. The problem of the impedance half-plane with distinct face impedances has also been recently reexamined by Lüneberg and Serbest [2000] using the Wiener-Hopf technique. Hurd and Lüneberg [1985] considered the difficult problem of the anisotropic half-plane under skew illumination (see also Senior and Legault [1998] for an approximate solution) but the results are complicated and not amenable to a solution. The fundamental problem of the right-angled wedge with common isotropic impedance faces at skew incidence remains to this day unsolved, a testament to the complicated nature of the analysis involved.

In the context of the Sommerfeld-Maliuzhinets approach, with which we concern ourselves in this work, the fields tangential to the edge are first expressed in terms of integrals of unknown angular spectra. These must be meromorphic and free of poles and zeros in a prescribed strip of analyticity (save for an optics pole required to reproduce the specular components of the incident and reflected fields) and fulfill a specific growth condition dictated by the edge condition. The latter is an important consideration to fully specify the solution of the problem and until recently [Idemen, 2000], particularly in the case of impedance and resistive surfaces, it had not yet received its due share of scrutiny.[2] The imposition of the boundary conditions together with a theorem put forward by Maliuzhinets [1958*b*] then leads to a pair of first order difference equations for the spectra having pe-

[2]See also the texts by Jones [1964] and Van Bladel [1995] for further discussion and references.

riods related to the open angle of the wedge. In the special case of normal incidence the technique leads to uncoupled difference equations whose coefficients are rational trigonometric functions and solutions subject to the required constraints are readily obtained in terms of Maliuzhinets functions [Maliuzhinets, 1958*a*]. At skew (non-normal) incidence, the equation pair is generally coupled and solutions, listed in the previous paragraph, are obtainable only for a few particular wedge/angle combinations for which uncoupled first order equations for linear combinations of the spectra can be found. In general the equation pair cannot be decoupled and we are faced with, equivalently, solving a second order functional difference equation whose coefficients are rational functions of trigonometric polynomials. A pair of associated first order difference equations can be obtained from the second order one but these, as we shall see in Chapter 2, typically involve branched functions and Maliuzhinets technique does not apply.

The solution of functional second order difference equations is not a well understood process, and only a few specialized attempts have been published. An example is the second order difference equation obtained by Demetrescu *et al.* [1998] in their consideration of the penetrable composite right-angled wedge using the Sommerfeld-Maliuzhinets technique. Unfortunately, the solution given is flawed since it fails to fulfill the requirement for meromorphism in the entire plane. Another example is the work of Gaudin [1978] who considers the second order difference equation that arises in the study of the quantum mechanical problem of two electrons interacting with a localized magnetic moment. The particular equation studied is of a high order of complexity and the ensuing analysis prohibitively complicated. The objective of the present work is to develop a technique for solving functional second order difference equations of the class that arises in the application of the Sommerfeld-Maliuzhinets technique.

To first shed light on the origin of the difference equations, Chapter 2 provides an overview of the formulation required for the anisotropic impedance wedge of arbitrary open angle illuminated by an obliquely incident plane wave. Using the Sommerfeld-Maliuzhinets technique, the second order difference equation and its associated first order equation pair are derived. The results are then specialized to the cases of the half-plane with common anisotropic impedance faces as well as to the cases of both the exterior and interior right-angled wedges with common isotropic impedance faces. The final section presents the difference equations that arise in the study of a certain penetrable composite wedge [Demetrescu *et al.*, 1998]. These are simpler in nature than the ones obtained for the impedances structures and are used to develop the solution technique. We bear in mind however that the ultimate objective is to solve the previously derived equations for the impedance half-plane and the right-angled wedges.

The foundations of the approach are presented in Chapter 3. Exploiting the inherent periodicity involved, the second order difference equation can be factored as a product of two first order difference operators as demonstrated in Chapter 2. General solutions of the second order equation can therefore be expressed as linear combinations of solutions to the first order equations together with arbitrary periodic coefficients. A problem that immediately arises is that the first order equations generally have branched coefficients, a consequence of the factorization process. Fortunately, there exist independent meromorphic linear combinations of branched solutions and these are discussed in detail at the beginning of the chapter. With this information in hand, the focus is on the difficult problem of solving branched first order difference equations. Proceeding by construction, it is shown how inhomogeneous solutions to the first order equations can be immediately found by taking their logarithmic derivative and exploiting the periodicity. The resulting solutions are however ill-defined: they are expressed in terms of a path integral in the complex angular plane and the strip of analyticity which is of concern is populated by both poles and branch point singularities. Appropriate homogeneous solutions, which take the form of multiplicative factors, are introduced in order to eliminate the undesired singularities while preserving the desired analytical properties of the solution. This is not a trivial task as very specific parity and order requirements must be fulfilled. The poles are cancelled by introducing homogeneous solutions such that the residues of their poles exactly cancel those of the inhomogeneous solution. The branch point singularities now remain and it is shown that the requirement for a continuous meromorphic spectral function is equivalent to requiring that all branch point to branch point integrals vanish within the strip. In the language of elliptic integral theory, this is the same as requiring that all modules of periodicity of the integrals involved vanish. This is accomplished once again by adding appropriate homogeneous solutions, this time such that the sum of the periodicity modules appearing is null. This process may involve, depending on the degree of complexity, trigonometric elliptic or hyper-elliptic forms and the resulting system of equations is solved using a bilinear relation of Riemann. The latter is derived by applying the residue theorem on a two-sheeted Riemann surface. Meromorphic solutions of the second order difference equation follow immediately from the results given at the beginning of the chapter. The approach is conceptually simple but a number of subtleties are encountered and discussed. The bilinear relations of Riemann, the key to carrying out the solution fully analytically, are discussed at the end of the chapter.

In Chapter 4, the aforementioned equation associated with the penetrable composite wedge is solved using the proposed technique. The solution is carried out step by step and the appropriate bilinear relation of Riemann corresponding to this specific case is also derived. A pair of meromorphic solutions are constructed and it is shown that they both recover known limiting functions. The chapter closes with the discussion of a pair of mathematical generalizations of the equation just solved. In terms of complexity, they correspond to intermediates between the penetrable wedge and the anisotropic half-plane. The second of these generalized equations distinguishes itself by requiring a much more complicated analysis for the elimination of branch point contributions and is examined in detail in Chapter 5. The construction of meromorphic solutions fulfilling the analyticity requirements is also more difficult due to a shortfall in the number of degrees of freedom. Both analytical and partially numerical solutions are provided.

Experience from Chapters 4 and 5 suggests that the problem of the anisotropic half-plane, which in the context of this approach has a large number of singularities in the strip of analyticity, will be difficult using the direct method followed in those two chapters. A variant of the technique which circumvents the problems due to a large number of singularities is developed in Chapter 6. This consists in the insertion of an intermediate step where the first order equations are manipulated to obtain another pair of first order equations where the period is now reduced by half. This halves the strip of analyticity and greatly simplifies the elimination of singularities. However, it is achieved at the price of compromising the order and this, together with a large number of poles, significantly complicates the construction of solutions to the second order equation. Since these difficulties cannot be overcome analytically at this time, various approaches requiring the numerical identification of zeros are presented.

The problem of the anisotropic half-plane is examined in Chapter 7 and, despite the advances made in the preceding chapters, it remains a formidable problem; an approximate solution is first presented. Two analytical procedures for constructing exact solutions are next examined. The first is based on the direct approach taken in Chapters 4 and 5. The preliminary analysis follows through quite easily but an impasse is reached at the stage where contributions from branch points must be eliminated. The cause of this is a shortage of degrees of freedom and indications are that some form of beneficial symmetry is being overlooked. The second approach relies on the method of period reduction discussed in Chapter 6 and the elimination of branch point contributions can be now carried through. However, the number of poles introduced in the strip of analyticity is such that the construction of meromorphic solutions, though possible in principle, is not now feasible in practice.

Chapter 8 closes this thesis by summarizing the results obtained and making a number of recommendations for future work.

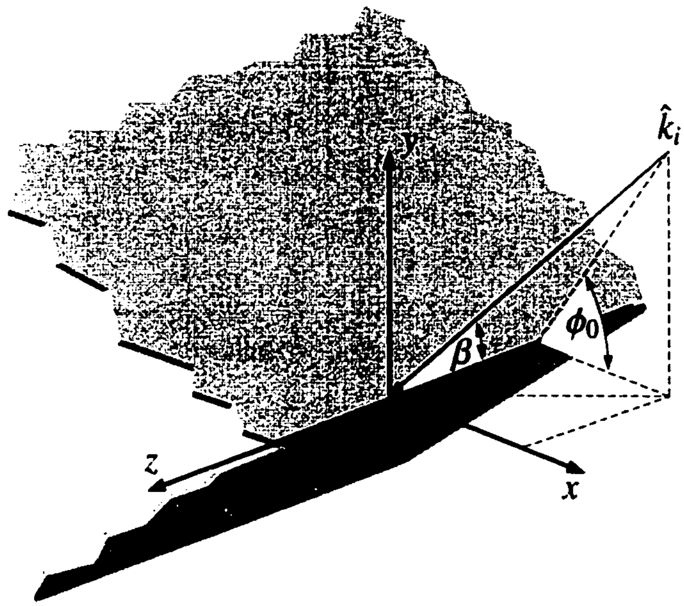
CHAPTER 2

Genesis of the Difference Equations

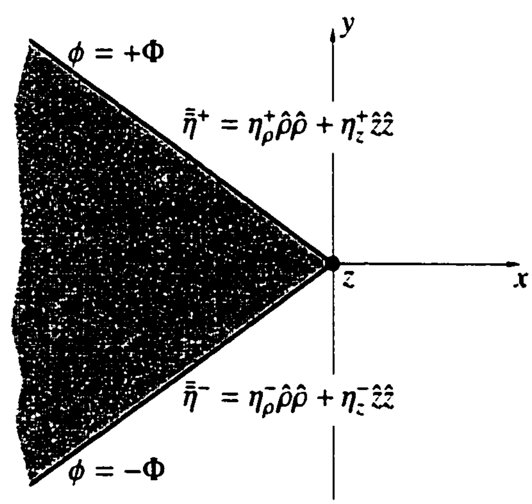
THE Sommerfeld-Maliuzhinets technique is applied to the unsolved problem of the anisotropic impedance wedge illuminated by an obliquely incident plane wave and the difference equations, the second order one as well as the subordinate first order pair, are derived. This work focuses on finding solutions to a few particular subcases of the problem for which the difference equations are of the same nature. Indeed, it will be shown below that the equations for the anisotropic impedance half-plane as well as the right-angled isotropic impedance wedge, a canonical problem whose solution is one of the most sought after, are of the same form. These equations are however of a relatively high degree of complexity and expressions of lower complexity are first considered in order to develop a suitable solution technique as well as to build insight. It turns out that a generalization of the difference equation that arises in the analysis of a penetrable composite right-angled wedge fulfills this requirement.

2.1 The anisotropic impedance wedge

The formulation provided is, for the time being, general and applies to a wedge with an angular opening of 2Φ whose faces are characterized by impedance (mixed) boundary conditions as shown on Figure 2.1. Such boundary conditions are used to approximate interfaces with non-penetrable (lossy) media and provide good accuracy under those conditions, making it possible to analyze a two media problem as a one medium problem with approximate boundary conditions. In terms of the cylindrical polar coordinates (ρ, ϕ, z) with the z axis coincident with the edge of the wedge, the upper and lower faces are $\phi = \Phi$ and $\phi = -\Phi$ respectively (Figure 2.1(b)), and subject to the first order impedance boundary



(a) Three dimensional view of the impedance wedge.



(b) Two dimensional cross-section of the wedge.

Figure 2.1: A three-dimensional view (a) and two dimensional cross section (b) of the anisotropic impedance wedge of opening angle 2Φ with upper and lower faces respectively characterized by the impedance tensors $\bar{\bar{\eta}}^+$ and $\bar{\bar{\eta}}^-$. The edge of the structure coincides with the z axis and is illuminated by a plane wave propagating along the unit vector $\hat{k}_i$.

condition

$$\hat{n} \times \mathbf{E} = Z\,\bar{\bar{\eta}}^{\pm} \cdot \hat{n} \times (\hat{n} \times \mathbf{H}), \qquad \phi = \pm\Phi, \tag{2.1}$$

where $\hat{n} = \mp\hat{\phi}$ is the outward unit vector normal and $\bar{\bar{\eta}}^{\pm}$ are the tensor impedances normalized to the free space impedance Z. The surface impedances are

$$\bar{\bar{\eta}}^{\pm} = \eta_{\rho}^{\pm}\hat{\rho}\hat{\rho} + \eta_{z}^{\pm}\hat{z}\hat{z} \tag{2.2}$$

and since the tensors are diagonal in the ρ, z coordinates, the special impedance combinations recently considered by LYALINOV AND ZHU [1999] are excluded. For isotropic impedances

$$\eta_{\rho}^{+} = \eta_{z}^{+} = \eta^{+}\,, \qquad \eta_{\rho}^{-} = \eta_{z}^{-} = \eta^{-}\,; \tag{2.3}$$

for anisotropic impedances the same on both faces

$$\eta_{\rho}^{+} = \eta_{\rho}^{-} = \eta_{\rho}\,, \qquad \eta_{z}^{+} = \eta_{z}^{-} = \eta_{z}\,; \tag{2.4}$$

and for polarization independent impedances

$$\eta_{\rho}^{+}\eta_{z}^{+} = 1\,, \qquad \eta_{\rho}^{-}\eta_{z}^{-} = 1\,. \tag{2.5}$$

If the lower face, for example, is a perfect electric conductor (pec)

$$\eta_{\rho}^{-} = 0\,, \qquad \eta_{z}^{-} = 0\,, \tag{2.6}$$

and if it is a perfect magnetic conductor (pmc)

$$\eta_{\rho}^{-} = \infty\,, \qquad \eta_{z}^{-} = \infty\,. \tag{2.7}$$

The wedge is illuminated by a plane electromagnetic wave whose z components are

$$E_{z}^{i} = e_{z}\mathrm{e}^{-jk\hat{k}_i\cdot\boldsymbol{r}}\,, \qquad ZH_{z}^{i} = h_{z}\mathrm{e}^{-jk\hat{k}_i\cdot\boldsymbol{r}} \tag{2.8}$$

with (see Figure 2.1(a))

$$\hat{k}_i = -\hat{x}\cos\phi_0\sin\beta - \hat{y}\sin\phi_0\sin\beta + \hat{z}\cos\beta \tag{2.9}$$

where $-\Phi \le \phi_0 \le \Phi$ is the direction of incidence and β is the angle measuring the skewness of the incident wave. A time factor $e^{j\omega t}$ has been suppressed. Owing to the symmetry of the structure, the scattered (and therefore the total) field must have the same z dependence as the incident field. Taking advantage of this and using Maxwell's equations to express the field in terms of E_z and H_z, the ρ components become

$$E_\rho = \frac{1}{jk\sin^2\beta}\left\{\cos\beta\frac{\partial}{\partial\rho}E_z + \frac{1}{\rho}\frac{\partial}{\partial\phi}(ZH_z)\right\}, \tag{2.10a}$$

$$ZH_\rho = \frac{1}{jk\sin^2\beta}\left\{\cos\beta\frac{\partial}{\partial\rho}(ZH_z) - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_z\right\}, \tag{2.10b}$$

enabling us to write the boundary conditions (2.1) as

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \cos\beta\frac{\partial}{\partial\rho}(ZH_z) \pm \frac{jk}{\eta_\rho^\pm}\sin^2\beta E_z = 0, \tag{2.11a}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}(ZH_z) + \cos\beta\frac{\partial}{\partial\rho}E_z \pm jk\eta_z^\pm\sin^2\beta ZH_z = 0, \tag{2.11b}$$

on $\phi = \pm\Phi$. Because the surface impedances are anisotropic, it is not possible to express the boundary conditions in terms E_ϕ and H_ϕ, and this rules out the simplifying procedure used, for example, by Senior and Volakis [1995].

Following Maliuzhinets [1958*a*] the z components of the total fields are written as

$$E_z(\rho,\phi,z) = \frac{e^{-jkz\cos\beta}}{2\pi j}\int_\gamma e^{jk\rho\sin\beta\cos\alpha}s_e(\alpha+\phi)d\alpha, \tag{2.12a}$$

$$ZH_z(\rho,\phi,z) = \frac{e^{-jkz\cos\beta}}{2\pi j}\int_\gamma e^{jk\rho\sin\beta\cos\alpha}s_h(\alpha+\phi)d\alpha, \tag{2.12b}$$

where γ is the Sommerfeld double loop contour shown in Figure 2.2 and $s_{e,h}(\alpha)$ are unknown angular spectra. It is required that $s_{e,h}(\alpha)$ be free of branch points in the entire α plane, free of poles in the strip $|\mathrm{Re}\,\alpha| \le \Phi$ apart from an optics pole at $\alpha = \phi_0$ necessary to reproduce the incident field (and since this can be inserted in the final stages of the analysis it will be ignored), and of an order as $|\mathrm{Im}\,\alpha| \to \infty$ consistent with the edge condition. More precisely, this stipulates that if $|E_z, H_z| < M\rho^{-1+\varepsilon}e^{b\rho}$ where M, ε and b are positive and bounded, then $s_{e,h}(\alpha) \sim O\{\exp[(1-\varepsilon)|\mathrm{Im}\,\alpha|]\}$ as $|\mathrm{Im}\,\alpha| \to \infty$ [Maliuzhinets, 1958*b*; Osipov and Norris, 1999]. The quantity ε depends on the angle of the wedge as well as the type of boundary conditions used. When (2.12) are inserted into (2.11), $\partial/\partial\rho$ can be

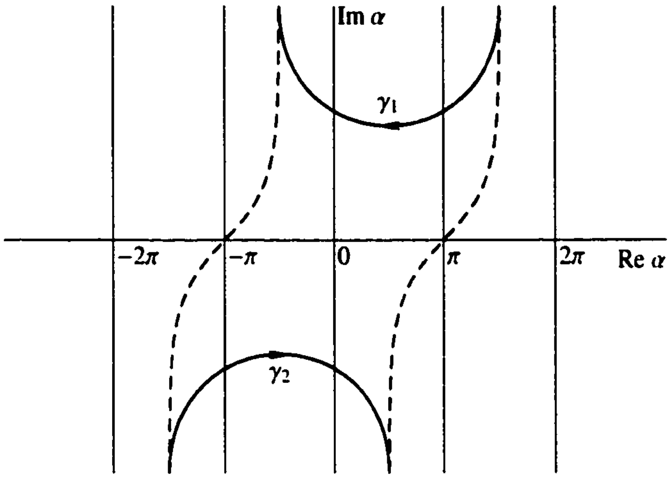


Figure 2.2: The Sommerfeld contour $\gamma = \gamma_1 + \gamma_2$. The associated steepest descent paths are indicated by the dashed lines.

replaced by $jk \sin\beta \cos\alpha$ and, using integration by parts, $\partial/\partial\phi$ by $jk\rho \sin\beta \sin\alpha$. If we write

$$s(\alpha) = \begin{bmatrix} s_e(\alpha) \\ s_h(\alpha) \end{bmatrix}, \qquad \mathcal{M}^{\pm}(\alpha) = \begin{bmatrix} m_{ee}^{\pm}(\alpha) & m_{eh}(\alpha) \\ m_{he}(\alpha) & m_{hh}^{\pm}(\alpha) \end{bmatrix} \tag{2.13}$$

where

$$m_{ee}^{\pm}(\alpha) = \sin\alpha \pm \frac{1}{\eta_\rho^{\pm}} \sin\beta, \tag{2.14a}$$

$$m_{hh}^{\pm}(\alpha) = \sin\alpha \pm \eta_z^{\pm} \sin\beta, \tag{2.14b}$$

$$m_{eh}(\alpha) = -\cos\alpha\cos\beta, \tag{2.14c}$$

$$m_{he}(\alpha) = -m_{eh}(\alpha), \tag{2.14d}$$

the boundary conditions then demand

$$\mathcal{M}^{\pm}(\alpha)s(\alpha \pm \Phi) = \mathcal{M}^{\pm}(-\alpha)s(-\alpha \pm \Phi). \tag{2.15}$$

It is clear from the symmetry of the contour $\gamma = \gamma_1 \cup \gamma_2$ in Figure 2.2 that (2.15) is sufficient to satisfy (2.11), and that it is also necessary was shown by Maliuzhinets [1958*b*].

If $\beta = \pi/2$, corresponding to incidence in a plane perpendicular to the edge, $m_{eh}(\alpha) = 0$

and the matrices $\mathcal{M}^{\pm}(\alpha)$ diagonalize. It is then a simple matter to show that

$$\frac{s_e(\alpha + 2\Phi)}{s_e(\alpha - 2\Phi)} = \frac{m^+_{ee}(-\alpha - \Phi)}{m^+_{ee}(\alpha + \Phi)} \frac{m^-_{ee}(\alpha - \Phi)}{m^-_{ee}(-\alpha + \Phi)} \tag{2.16}$$

with a similar equation for $s_h(\alpha)$. This is a first order difference equation of period 4Φ whose right-hand side is a rational function of $\sin\alpha$ and $\cos\alpha$, and the solution satisfying the required conditions is easily expressed in terms of Maliuzhinets functions [Maliuzhinets, 1958*a*]. While solutions are readily obtained for the cases of normal incidence, the case of oblique incidence is more complicated. When $\beta \neq \pi/2$ the matrices are no longer diagonal, but if some linear combinations of $s_e(\alpha)$ and $s_h(\alpha)$ can be found to make them so, the problem can still be solved. There are a few particular wedge angles/impedance combinations for which this is possible, *e.g.* one face pec or pmc, the other impedance isotropic, and $\Phi = n\pi/4$ ($n = 1,2,3$ or 4), and most of these special cases have been explored [Vaccaro, 1981; Senior, 1986] with the most recent contributions being provided by Bernard [1998] and Manara and Nepa [2000]. In general, however, there is no linear combination that diagonalizes $\mathcal{M}^{\pm}(\alpha)$, and we are then faced with second order difference equations.

2.1.1 The second order difference equations

To see how these equations arise, suppose we diagonalize the matrix for the upper face by writing

$$\boldsymbol{t}(\alpha) = \begin{bmatrix} t_e(\alpha) \\ t_h(\alpha) \end{bmatrix} = \mathcal{M}^+(\alpha)\boldsymbol{s}(\alpha + \Phi) \tag{2.17}$$

so that

$$\boldsymbol{s}(\alpha + \Phi) = \{\mathcal{M}^+(\alpha)\}^{-1}\,\boldsymbol{t}(\alpha) \tag{2.18}$$

and (2.15) then implies

$$\boldsymbol{t}(\alpha) = \boldsymbol{t}(-\alpha), \qquad \boldsymbol{t}(\alpha + 2\Phi) = \boldsymbol{t}(-\alpha - 2\Phi). \tag{2.19}$$

Note that the requirement on $s_{e,h}(\alpha)$ to be free of poles in the strip $|\mathrm{Re}\,\alpha| \leq \Phi$ together with (2.17) and (2.19) implies a similar one on $t_{e,h}(\alpha)$ in the strip $|\mathrm{Re}\,\alpha| \leq 2\Phi$. Furthermore, in the limits where first order equations are obtained the solutions are reciprocals of one another and must therefore also be free of zeros in the same strip. It is assumed that $\boldsymbol{t}(\alpha)$

must fulfill the same requirement. We now proceed to obtain another matrix equations involving $\boldsymbol{t}(\alpha)$ using the boundary condition for the lower face in (2.15)

$$\mathcal{M}^{-}(\alpha)\boldsymbol{s}(\alpha-\Phi)=\mathcal{M}^{-}(-\alpha)\boldsymbol{s}(-\alpha-\Phi). \tag{2.20}$$

In terms of $\boldsymbol{t}(\alpha)$, this becomes

$$\mathcal{M}^{-}(\alpha)\left\{\mathcal{M}^{-}(\alpha-2\Phi)\right\}^{-1}\boldsymbol{t}(\alpha-2\Phi)=\mathcal{M}^{-}(-\alpha)\left\{\mathcal{M}^{-}(-\alpha-2\Phi)\right\}^{-1}\boldsymbol{t}(\alpha+2\Phi). \tag{2.21}$$

This can be rewritten more compactly as

$$\mathcal{N}(\alpha)\boldsymbol{t}(\alpha-2\Phi)=\mathcal{N}(-\alpha)\boldsymbol{t}(\alpha+2\Phi) \tag{2.22}$$

with

$$\mathcal{N}(\alpha)=\mathcal{M}^{-}(\alpha)\left\{\mathcal{M}^{-}(\alpha-2\Phi)\right\}^{-1}=\begin{bmatrix} n_{ee}(\alpha) & n_{eh}(\alpha) \\ n_{he}(\alpha) & n_{hh}(\alpha)\end{bmatrix}, \tag{2.23}$$

where

$$n_{ee}(\alpha)=\frac{1}{|\mathcal{M}^{-}(\alpha-2\Phi)|}\left\{m_{ee}^{-}(\alpha)m_{hh}^{+}(\alpha-2\Phi)+m_{eh}(\alpha)m_{eh}(\alpha-2\Phi)\right\}, \tag{2.24a}$$

$$n_{eh}(\alpha)=\frac{1}{|\mathcal{M}^{-}(\alpha-2\Phi)|}\left\{m_{ee}^{-}(\alpha)m_{eh}(\alpha-2\Phi)-m_{eh}(\alpha)m_{ee}^{+}(\alpha-2\Phi)\right\}, \tag{2.24b}$$

$$n_{he}(\alpha)=\frac{1}{|\mathcal{M}^{-}(\alpha-2\Phi)|}\left\{m_{eh}(\alpha)m_{hh}^{+}(\alpha-2\Phi)-m_{hh}^{-}(\alpha)m_{eh}(\alpha-2\Phi)\right\}, \tag{2.24c}$$

$$n_{hh}(\alpha)=\frac{1}{|\mathcal{M}^{-}(\alpha-2\Phi)|}\left\{m_{hh}^{-}(\alpha)m_{ee}^{+}(\alpha-2\Phi)+m_{eh}(\alpha)m_{eh}(\alpha-2\Phi)\right\}. \tag{2.24d}$$

The boundary condition at the lower face then gives

$$n_{ee}(\alpha)t_e(\alpha-2\Phi)+n_{eh}(\alpha)t_h(\alpha-2\Phi)=n_{ee}(-\alpha)t_e(\alpha+2\Phi)+n_{eh}(-\alpha)t_h(\alpha+2\Phi), \tag{2.25a}$$

$$n_{he}(\alpha)t_e(\alpha-2\Phi)+n_{hh}(\alpha)t_h(\alpha-2\Phi)=n_{he}(-\alpha)t_e(\alpha+2\Phi)+n_{hh}(-\alpha)t_h(\alpha+2\Phi), \tag{2.25b}$$

on making use of (2.19). By eliminating $t_h(\alpha+2\Phi)$ we obtain

$$t_h(\alpha-2\Phi)=X_e(\alpha)t_e(\alpha+2\Phi)-Y_e(\alpha)t_e(\alpha-2\Phi) \tag{2.26}$$

where

$$X_e(\alpha) = \frac{n_{ee}(-\alpha)n_{hh}(-\alpha) - n_{eh}(-\alpha)n_{he}(-\alpha)}{n_{eh}(\alpha)n_{hh}(-\alpha) - n_{hh}(\alpha)n_{eh}(-\alpha)}, \tag{2.27}$$

$$Y_e(\alpha) = \frac{n_{ee}(\alpha)n_{hh}(-\alpha) - n_{he}(\alpha)n_{eh}(-\alpha)}{n_{eh}(\alpha)n_{hh}(-\alpha) - n_{hh}(\alpha)n_{eh}(-\alpha)} \, . \tag{2.28}$$

Hence

$$t_h(\alpha + 2\Phi) = X_e(\alpha + 4\Phi)t_e(\alpha + 6\Phi) - Y_e(\alpha + 4\Phi)t_e(\alpha + 2\Phi) \, , \tag{2.29}$$

and when (2.26) and (2.29) are substituted into (say) the first of the equations (2.25) the result is

$$t_e(\alpha + 6\Phi) - a_e(\alpha)t_e(\alpha + 2\Phi) + b_e(\alpha)t_e(\alpha - 2\Phi) = 0 \tag{2.30}$$

with

$$a_e(\alpha) = \frac{Y_e(\alpha + 4\Phi) - Y_e(-\alpha)}{X_e(\alpha + 4\Phi)}, \tag{2.31}$$

$$b_e(\alpha) = -\frac{X_e(-\alpha)}{X_e(\alpha + 4\Phi)} \, . \tag{2.32}$$

Equation (2.30) is a second order difference equation of period 4Φ for $t_e(\alpha)$. The analogous equations for $t_h(\alpha)$ can be deduced by making the duality transformation $n_{ee}(\alpha) \leftrightarrow n_{hh}(\alpha)$, $n_{eh}(\alpha) \leftrightarrow -n_{he}(\alpha)$, and is

$$t_h(\alpha + 6\Phi) - a_h(\alpha)t_h(\alpha + 2\Phi) + b_h(\alpha)t_h(\alpha - 2\Phi) = 0 \tag{2.33}$$

where

$$a_h(\alpha) = \frac{Y_h(\alpha + 4\Phi) - Y_h(-\alpha)}{X_h(\alpha + 4\Phi)}, \tag{2.34}$$

$$b_h(\alpha) = -\frac{X_h(-\alpha)}{X_h(\alpha + 4\Phi)}, \tag{2.35}$$

with

$$X_h(\alpha) = \frac{n_{ee}(-\alpha)n_{hh}(-\alpha) - n_{eh}(-\alpha)n_{he}(-\alpha)}{n_{ee}(\alpha)n_{he}(-\alpha) - n_{he}(\alpha)n_{ee}(-\alpha)}, \tag{2.36}$$

$$Y_h(\alpha) = \frac{n_{he}(\alpha)n_{ee}(-\alpha) - n_{eh}(\alpha)n_{he}(-\alpha)}{n_{ee}(\alpha)n_{he}(-\alpha) - n_{he}(\alpha)n_{ee}(-\alpha)}. \tag{2.37}$$

Equations (2.33) has the same form as (2.30), and in both cases the coefficients are rational functions of $\sin\alpha$ and $\cos\alpha$. No methods to directly solve such second order difference equations are currently known. We proceed by obtaining a pair of related first order equations which are more amenable to a solution.

2.1.2 The first order difference equations

The key to the methods presented here for reducing the second order difference equation to a pair of associated first order ones lies in the periodicity of the functional coefficients. If the coefficients have the same period as the difference equation (examples are the half-plane for which $\Phi = \pi$ and the junction of two half-planes for which $\Phi = \pi/2$), it is then a simple matter to reduce the equation to a pair of first order difference equations. Writing (2.30) as

$$t_e(\alpha + 6\Phi) - p(\alpha)t_e(\alpha + 2\Phi) - q(\alpha)\left\{t_e(\alpha + 2\Phi) - p(\alpha)t_e(\alpha - 2\Phi)\right\} = 0 \tag{2.38}$$

with

$$p(\alpha) + q(\alpha) = a_e(\alpha), \tag{2.39a}$$

$$p(\alpha)q(\alpha) = b_e(\alpha), \tag{2.39b}$$

shows

$$r(\alpha) + \frac{1}{r(\alpha)} = \frac{a_e(\alpha)}{\sqrt{b_e(\alpha)}} \tag{2.40}$$

where $\sqrt{b_e(\alpha)}r(\alpha) = p(\alpha)$, and both $r(\alpha)$ and its reciprocal are thus admissible solutions. Alternatively, one could proceed by rewriting (2.30) as

$$\mathcal{L}t_e(\alpha) = 0 \tag{2.41}$$

where $\mathcal{L} = D_{8\Phi} - a_e(\alpha)D_{4\Phi} + b_e(\alpha)$ and $D_{4\Phi}$ is the 4Φ difference operator. Since both $a_e(\alpha)$ and $b_e(\alpha)$ are 4Φ periodic, the difference operator can be factored and written as

$$\begin{aligned}\mathcal{L} = D_{8\Phi} - a_e(\alpha)D_{4\Phi} + b_e(\alpha) &= \{D_{4\Phi} + p(\alpha)\}\{D_{4\Phi} + q(\alpha)\} \\ &= D_{8\Phi} - \{p(\alpha) + q(\alpha)\}D_{4\Phi} + p(\alpha)q(\alpha)\end{aligned} \tag{2.42}$$

which recovers the equations for $p(\alpha)$ and $q(\alpha)$ given above. This approach is equivalent to the preceding one save that it arguably puts more emphasis on the equivalence between the second order difference equation and the pair of associated first order difference equations. It is expedient to proceed by postulating solutions of the form

$$r(\alpha) = \frac{r_1(\alpha) - r_2(\alpha)}{r_1(\alpha) + r_2(\alpha)} \tag{2.43}$$

and doing so yields

$$r(\alpha) = \frac{\sqrt{\dfrac{a_e(\alpha)}{2\sqrt{b_e(\alpha)}} + 1} - \sqrt{\dfrac{a_e(\alpha)}{2\sqrt{b_e(\alpha)}} - 1}}{\sqrt{\dfrac{a_e(\alpha)}{2\sqrt{b_e(\alpha)}} + 1} + \sqrt{\dfrac{a_e(\alpha)}{2\sqrt{b_e(\alpha)}} - 1}} = \frac{\sqrt{\dfrac{4\sqrt{b_e(\alpha)}}{a_e(\alpha) - 2\sqrt{b_e(\alpha)}} + 1} - 1}{\sqrt{\dfrac{4\sqrt{b_e(\alpha)}}{a_e(\alpha) - 2\sqrt{b_e(\alpha)}} + 1} + 1}. \tag{2.44}$$

The beauty of the above is that $r(\alpha)$ goes explicitly to its reciprocal, the other solution of (2.40), by changing the branch of the square root. The associated first order difference equations, one for each branch of the square root, are then

$$\frac{t_e(\alpha + 2\Phi)}{t_e(\alpha - 2\Phi)} = \sqrt{b_e(\alpha)}\,\frac{\pm\sqrt{z(\alpha) + 1} - 1}{\pm\sqrt{z(\alpha) + 1} + 1} \tag{2.45}$$

where

$$z(\alpha) = \frac{4\sqrt{b_e(\alpha)}}{a_e(\alpha) - 2\sqrt{b_e(\alpha)}}. \tag{2.46}$$

However, the right-hand side of (2.45) is not, in general, a rational function of $\sin\alpha$ and $\cos\alpha$, and the occurrence of branched functions is a major complication. Indeed, in such instances Maliuzhinets's technique does not apply and a more general approach, which constitutes the core of this work, must be sought.

If the period of the coefficients is an integer multiple of the period 4Φ of the equation, the method proposed in [Demetrescu *et al.*, 1998] is to increase the period of the equation to at least equal the period of the coefficients. Consider, for example, a $\pi/2$ or $3\pi/2$ wedge for which $\Phi = \pi/4$ or $3\pi/4$. Writing

$$t_e(\alpha) = t'_e(\alpha + 4\Phi) + d_1(\alpha)t'_e(\alpha) + d_2(\alpha)t'_e(\alpha - 4\Phi), \tag{2.47}$$

substituting into (2.30), and then choosing $d_1(\alpha)$ and $d_2(\alpha)$ to eliminate the terms involving $t'_e(\alpha + 6\Phi)$ and $t'_e(\alpha - 2\Phi)$, we obtain

$$d_1(\alpha) = a_e(\alpha + 2\Phi)\,, \qquad d_2(\alpha) = b_e(\alpha - 2\Phi). \tag{2.48}$$

Use has been made above of the fact that, for the values of Φ being considered, both $a_e(\alpha)$ and $b_e(\alpha)$ are 8Φ periodic. The resulting equation for $t'_e(\alpha)$ is

$$t'_e(\alpha + 10\Phi) - a'_e(\alpha)t'_e(\alpha + 2\Phi) + b'_e(\alpha)t'_e(\alpha - 6\Phi) = 0 \tag{2.49}$$

whose period is 8Φ, and the coefficients are

$$\begin{aligned} a'_e(\alpha) &= a_e(\alpha + 4\Phi)a_e(\alpha) - b_e(\alpha + 4\Phi) - b_e(\alpha), \\ b'_e(\alpha) &= b_e(\alpha + 4\Phi)b_e(\alpha), \end{aligned} \tag{2.50}$$

whose period is 4Φ, a consequence of the 8Φ periodicity of $a_e(\alpha)$ and $b_e(\alpha)$. Since this is a submultiple of the period of the equation, the previous method can now be used to reduce (2.49) to a pair of first order equations, and in terms of the solution

$$t_e(\alpha) = t'_e(\alpha + 4\Phi) + a_e(\alpha + 2\Phi)t'_e(\alpha) + b_e(\alpha - 2\Phi)t'_e(\alpha - 4\Phi). \tag{2.51}$$

2.2 The anisotropic half-plane

We specialize the result of the previous section to the case of the isolated anisotropic half-plane where $\Phi = \pi$. In the general case where the upper and lower faces have different impedances, the coefficients of the second order difference equation have period 2π and the second order difference equation has period 4π, (2.30) can be reduced to a pair of first order equations using the technique described in Section 3, but no solutions of these equations have not yet been obtained. Even if one of the faces is a pec or pmc, the problem remains intractable.[1] If the impedances are different on the two faces but isotropic, so that $\eta^+_{\rho z} = \eta^+$, $\eta^-_{\rho z} = \eta^-$, the second order difference equation satisfied by both $t_e(\alpha)$ and $t_h(\alpha)$ now factors by inspection into first order equations and their solutions are easily obtained in terms of

[1]The recent work of Manara and Nepa [2000] may however provide clues on how to diagonalize the matrix in (2.15).

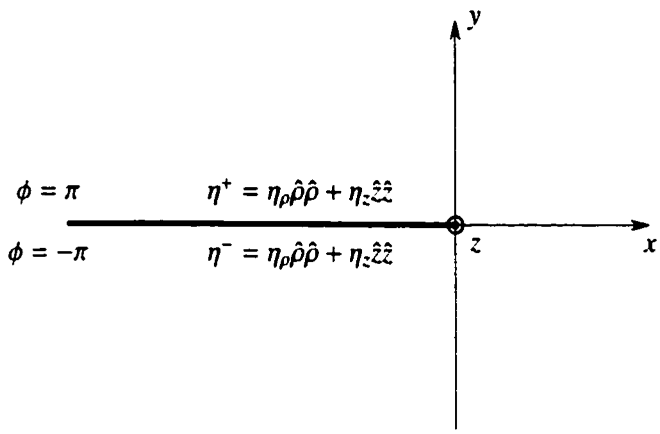


Figure 2.3: The anisotropic half-plane with $\Phi = \pi$.

Maliuzhinets functions [MALIUZHINETS, 1958*a*].

If the impedances are anisotropic but the same on both faces, see Figure 2.3, then $\eta_\rho^- = \eta_\rho^+ = \eta_\rho$, $\eta_z^- = \eta_z^+ = \eta_z$ and $m_{ee}^-(\pm\alpha) = -m_{ee}^+(\mp\alpha)$, $m_{hh}^-(\pm\alpha) = -m_{hh}^+(\mp\alpha)$, giving

$$a_e(\alpha) = a_h(\alpha) = \frac{m_{eh}^2(\alpha) + m_{ee}^+(-\alpha)m_{hh}^+(-\alpha)}{m_{eh}^2(\alpha) + m_{ee}^+(\alpha)m_{hh}^+(\alpha)} \cdot \left[2 + \frac{\left\{m_{ee}^+(-\alpha)m_{hh}^+(\alpha) - m_{ee}^+(\alpha)m_{hh}^+(-\alpha)\right\}^2}{\left\{m_{eh}^2(\alpha) + m_{ee}^+(\alpha)m_{hh}^+(\alpha)\right\}\left\{m_{eh}^2(\alpha) + m_{ee}^+(-\alpha)m_{hh}^+(-\alpha)\right\}}\right]$$

$$b_e(\alpha) = b_h(\alpha) = \left\{\frac{m_{eh}^2(\alpha) + m_{ee}^+(-\alpha)m_{hh}^+(-\alpha)}{m_{eh}^2(\alpha) + m_{ee}^+(\alpha)m_{hh}^+(\alpha)}\right\}^2 . \tag{2.52}$$

From (2.45), the corresponding first order equations satisfied by both $t_e(\alpha)$ and $t_h(\alpha)$ are

$$\frac{t(\alpha + 2\pi)}{t(\alpha - 2\pi)} = \frac{m_{eh}^2(\alpha) + m_{ee}^+(-\alpha)m_{hh}^+(-\alpha)}{m_{eh}^2(\alpha) + m_{ee}^+(\alpha)n_{hh}^+(\alpha)} \; \frac{\pm\sqrt{z(\alpha) + 1} - 1}{\pm\sqrt{z(\alpha) + 1} + 1} \tag{2.53}$$

with

$$z(\alpha) = \frac{\left\{m_{eh}^2(\alpha) + m_{ee}^+(\alpha)m_{hh}^+(\alpha)\right\}\left\{m_{eh}^2(\alpha) + m_{ee}^+(-\alpha)m_{hh}^+(-\alpha)\right\}}{\left\{\left(\eta_z - \frac{1}{\eta_\rho}\right)\sin\alpha\sin\beta\right\}^2} . \tag{2.54}$$

When the expressions (2.14) for $m_{he}(\alpha)$, $m_{ee}^{+}(\pm\alpha)$ and $m_{hh}^{+}(\pm\alpha)$ are inserted, we obtain

$$t(\alpha) = t_1(\alpha)t_2(\alpha) \tag{2.55}$$

where

$$\frac{t_1(\alpha+2\pi)}{t_1(\alpha-2\pi)} = \frac{(\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)} \tag{2.56}$$

and

$$\frac{t_2(\alpha+2\pi)}{t_2(\alpha-2\pi)} = \frac{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} - \gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} + \gamma\sin\alpha}, \tag{2.57}$$

with

$$\sin\theta_{1,2} = \frac{1}{2\sin\beta}\left\{\eta_z + \frac{1}{\eta_\rho} \mp \sqrt{\left(\eta_z - \frac{1}{\eta_\rho}\right)^2 + 4\left(\frac{\eta_\rho}{\eta_z} - 1\right)\cos^2\beta}\right\}, \tag{2.58}$$

$$\sin\delta_{1,2} = \frac{1}{\sin\beta}\left\{\sqrt{\frac{\eta_z}{\eta_\rho}} \pm \cos\beta\sqrt{\frac{\eta_z}{\eta_\rho} - 1}\right\}, \tag{2.59}$$

and

$$\gamma = \left(\eta_z - \frac{1}{\eta_\rho}\right)\frac{1}{\sin\beta}. \tag{2.60}$$

The required solution of (2.56) is

$$t_1(\alpha) = \psi_\pi\left(\alpha + \frac{\pi}{2} - \theta_1\right)\psi_\pi\left(\alpha - \frac{\pi}{2} + \theta_1\right)\psi_\pi\left(\alpha + \frac{\pi}{2} - \theta_2\right)\psi_\pi\left(\alpha - \frac{\pi}{2} + \theta_2\right) \tag{2.61}$$

where $\psi_\pi(\alpha)$ is the Maliuzhinets half-plane function [Maliuzhinets, 1958*a*]. The nature of (2.57) is complicated by the presence of the square root and both approximate and exact solutions are discussed in Chapter 7. Note that the square root has no less than sixteen branch points in the 4π wide strip of analyticity, as illustrated in Figure 2.4.

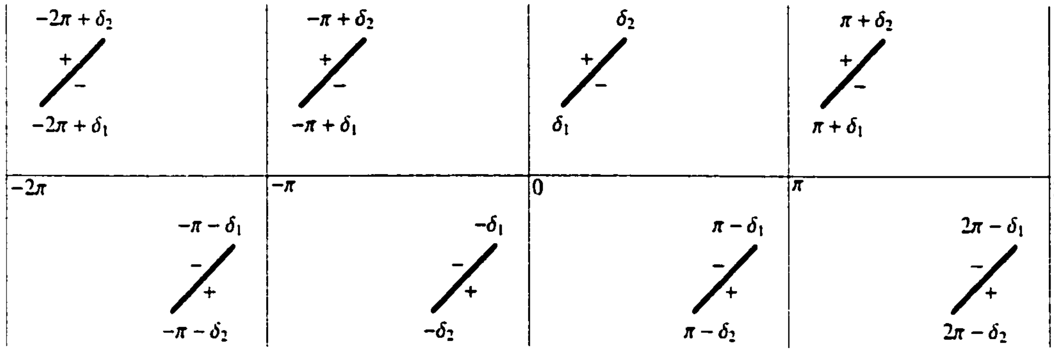
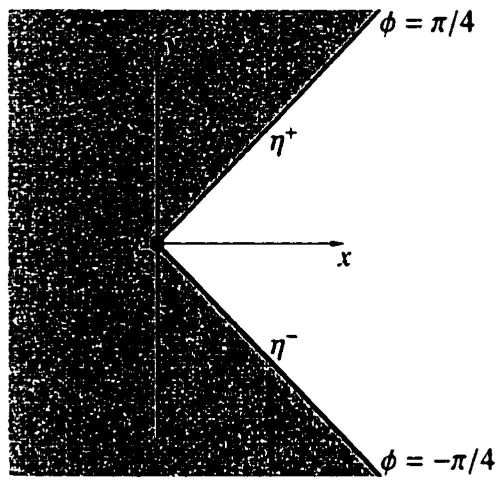


Figure 2.4: The 4π strip of analyticity with the branch points and associated cuts for the square root appearing in (2.57).

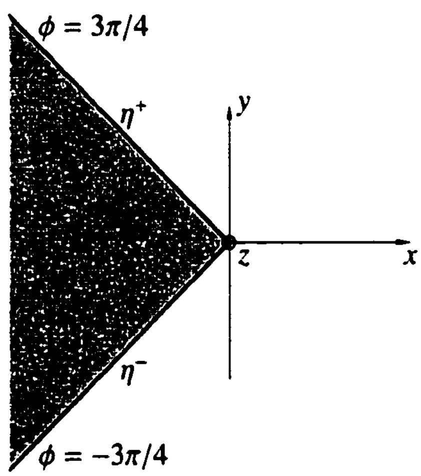
2.3 The isotropic right-angled wedge

The cases where $\Phi = \pi/4$ (see Figure 2.5(a)) and $\Phi = 3\pi/4$ (see Figure 2.5(b)) corresponding to right-angled wedges having open angles $\pi/2$ (the interior right-angled wedge) and $3\pi/2$ (the exterior right-angled wedge) are now examined. Because of the complicated nature of the resulting expressions, attention is confined to isotropic impedances for which $\eta_\rho^+ = \eta_z^+ = \eta^+$ and $\eta_\rho^- = \eta_z^- = \eta^-$. In the equations for the coefficients that immediately follow, the upper (lower) sign applies to $\Phi = \pi/4$ ($\Phi = 3\pi/4$). Making the appropriate substitution for Φ and simplifying, the coefficients of the second order difference equation (2.30) for $t_e(\alpha)$ are

$$\begin{aligned} a_e(\alpha) = {} & \pm 2\cos\alpha\sin\beta\left\{\left(1 \pm \eta^+\cos\alpha\sin\beta\right)\left(1 \pm \frac{1}{\eta^+}\cos\alpha\sin\beta\right)\left(1 - \eta^-\sin\alpha\sin\beta\right)\right. \\ & \left.\cdot\left(1 - \frac{1}{\eta^-}\sin\alpha\sin\beta\right)\left(1 - \sin^2\alpha\sin^2\beta \mp \frac{1}{\eta^+}\cos\alpha\sin\beta\right)\right\}^{-1} \\ & \cdot\left\{\left(1 + \eta^-\sin\alpha\sin\beta\right)\left(1 - \frac{1}{\eta^-}\sin\alpha\sin\beta\right)\left[\eta^+(1 - \sin^2\alpha\sin^2\beta) - \frac{1}{\eta^+}\sin^2\beta\right]\right. \\ & \left. + \frac{2}{\eta^+}\sin^2\alpha\sin^2\beta\cos^2\beta\right\} \end{aligned} \tag{2.62a}$$



(a)



(b)

Figure 2.5: The isotropic right-angled (a) interior wedge with $\Phi = \pi/4$ and (b) exterior wedge with $\Phi = 3\pi/4$.

and

$$b_e(\alpha) = -\frac{\left(1 \mp \eta^+ \cos\alpha \sin\beta\right)\left(1 \mp \frac{1}{\eta^+}\cos\alpha\sin\beta\right)\left(1 - \sin^2\alpha \sin^2\beta \pm \frac{1}{\eta^+}\cos\alpha\sin\beta\right)}{\left(1 \pm \eta^+ \cos\alpha \sin\beta\right)\left(1 \pm \frac{1}{\eta^+}\cos\alpha\sin\beta\right)\left(1 - \sin^2\alpha \sin^2\beta \mp \frac{1}{\eta^+}\cos\alpha\sin\beta\right)}. \tag{2.62b}$$

The analogous results for $a_h(\alpha)$ and $b_h(\alpha)$ can be obtained by making the transformation $\eta^\pm \to 1/\eta^\pm$.

When $\beta = \pi/2$ (incidence in a plane perpendicular to the edge), the associated first order difference equation is

$$\frac{t_e(\alpha + 2\Phi)}{t_e(\alpha - 2\Phi)} = 1 \tag{2.63}$$

and $t_e(\alpha)$ is simply a 4Φ periodic function, as is $t_h(\alpha)$. Alternatively, if the lower face of the wedge is perfectly conducting ($\eta^- = 0$), then

$$\frac{t_e(\alpha + 2\Phi)}{t_e(\alpha - 2\Phi)} = -1, \tag{2.64}$$

showing that $t_e(\alpha)$ is an 8Φ periodic function. For both wedge angles, the solutions of the diffraction problems are available [Senior, 1986; Senior and Volakis, 1986].

Since the coefficients (2.62) have period 2π, whereas the difference equation (2.30) has period π (or 3π), it is convenient to double the period of the equation so that its period becomes an integer multiple of the period of the coefficients. The resulting equation is (2.51) and its coefficients are as given in (2.50). These are the same for both wedge angles, and by the method described in Section 2.1.2, the second order difference equation can be reduced to the same two first order equations for both $t_e'(\alpha)$ and $t_h'(\alpha)$.

If $\sin\delta_e^\pm = 1/(\eta^\pm \sin\beta)$ and $\sin\delta_h^\pm = \eta^\pm/\sin\beta$, and if $\tilde{t}(\alpha) = t(\alpha/2)$, the first order difference equation can be written as

$$\frac{\tilde{t}(\alpha' + 8\Phi)}{\tilde{t}(\alpha' - 8\Phi)} = \frac{\pm\sqrt{y(\alpha')} - \gamma\sin\alpha'}{\pm\sqrt{y(\alpha')} + \gamma\sin\alpha'} \tag{2.65}$$

where $\alpha' = 2\alpha$ and

$$y(\alpha') = \left(\cos\alpha' + \cos 2\delta_e^+\right)\left(\cos\alpha' - \cos 2\delta_e^-\right) \cdot \left(\cos\alpha' + \cos 2\delta_h^+\right)\left(\cos\alpha' - \cos 2\delta_h^-\right) + \gamma^2\sin^2\alpha' \tag{2.66}$$

with

$$\gamma = 4\mathrm{j}\frac{\cos\beta}{\sin^2\beta}\,. \tag{2.67}$$

In the particular case when the impedances are the same on both faces of the wedge ($\eta^+ = \eta^- = \eta$ implying $\delta_e^+ = \delta_e^- = \delta_e$, $\delta_h^+ = \delta_h^- = \delta_h$), we have

$$y(\alpha') = \left(\sin^2\alpha' - \sin^2\delta_1\right)\left(\sin^2\alpha' - \sin^2\delta_2\right) \tag{2.68}$$

where

$$\begin{aligned} \sin^2\delta_1 + \sin^2\delta_2 &= \sin^2 2\delta_e + \sin^2 2\delta_h - \gamma^2, \\ \sin^2\delta_1 \sin^2\delta_2 &= \sin^2 2\delta_e \sin^2 2\delta_h\,, \end{aligned} \tag{2.69}$$

and the first order difference equations (2.65) are then

$$\frac{\bar{t}(\alpha' + 8\Phi)}{\bar{t}(\alpha' - 8\Phi)} = \frac{\pm\sqrt{\left(\sin^2\alpha' - \sin^2\delta_1\right)\left(\sin^2\alpha' - \sin^2\delta_2\right)} - \gamma\sin\alpha'}{\pm\sqrt{\left(\sin^2\alpha' - \sin^2\delta_1\right)\left(\sin^2\alpha' - \sin^2\delta_2\right)} + \gamma\sin\alpha'}\,. \tag{2.70}$$

For a $\pi/2$ wedge these are identical to the equations (2.57) for an isotropic impedance half-plane, and for a $3\pi/2$ wedge the equations differ only in having period 12π rather than 4π. It is thus recognized that obtaining solutions to the class of first order difference equations given in (2.70) will unlock both the problems of the interior isotropic impedance wedge as well as the anisotropic impedance half-plane. It would also provide insight into the problem of the exterior isotropic wedge though the latter, whose difference equations are of period 12π, is expected to be significantly more challenging.

2.4 The penetrable composite right-angled wedge

The first order difference equations obtained in the previous sections for the anisotropic half-plane and the right-angled wedge have a relatively high degree of complexity as will become apparent later in this work. It is advantageous to consider an equation of lower complexity to simplify the development of a technique to solve second order difference equations. The penetrable composite right-angled wedge examined by Demetrescu *et al.* [1998] and illustrated in Figure 2.6 leads to a considerably simpler equation. The structure consists of abutted semi-infinite perfectly conducting and resistive half-planes illuminated at normal incidence. The added complication here arises from the fact that we are dealing

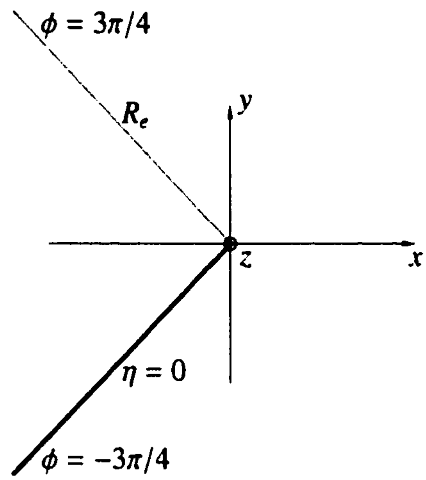


Figure 2.6: The penetrable composite right-angled wedge with $\Phi = 3\pi/4$. The upper face is a resistive sheet with resistivity R_e and the lower one is a perfectly conducting half-plane.

with a penetrable structure, leading to a different class of problems than the previously discussed impedance wedge. However, it yields the somewhat simpler difference equation[2]

$$t(\alpha + 2\pi) - 2\frac{\cos^2\alpha - \frac{1}{2}\sin^2\bar{\theta}}{\cos^2\alpha - \sin^2\bar{\theta}}t(\alpha) + t(\alpha - 2\pi) = 0 \tag{2.71}$$

which has period 2π, where $\sin\bar{\theta} = Z_0/2R_e$, R_e is the surface resistivity of the resistive sheet and Z_0 is the free-space impedance. The related first order difference equations are

$$\frac{t(\alpha+\pi)}{t(\alpha-\pi)} = \frac{\pm\sqrt{\cos^2\alpha - \frac{3}{4}\sin^2\bar{\theta}} - \frac{1}{2}\sin\bar{\theta}}{\pm\sqrt{\cos^2\alpha - \frac{3}{4}\sin^2\bar{\theta}} + \frac{1}{2}\sin\bar{\theta}}. \tag{2.72}$$

A more general second order difference equation which has the above as a special case is

$$t(\alpha + 3\pi) - 2\left\{1 - 2\frac{\cos^2\delta - \cos^2\theta}{\cos^2\alpha - \cos^2\theta}\right\}t(\alpha+\pi) + t(\alpha-\pi) = 0 \tag{2.73}$$

which has associated first order equations

$$\frac{t(\alpha+\pi)}{t(\alpha-\pi)} = g(\alpha) = \frac{u(\alpha) - u(\theta)}{u(\alpha) + u(\theta)}, \tag{2.74a}$$

$$\frac{t(\alpha+\pi)}{t(\alpha-\pi)} = \frac{1}{g(\alpha)} = \frac{u(\alpha) + u(\theta)}{u(\alpha) - u(\theta)}, \tag{2.74b}$$

[2]For convenience and without loss of generality, a negative sign is associated with the functional coefficient as opposed to the positive one in [Demetrescu *et al.*, 1998].

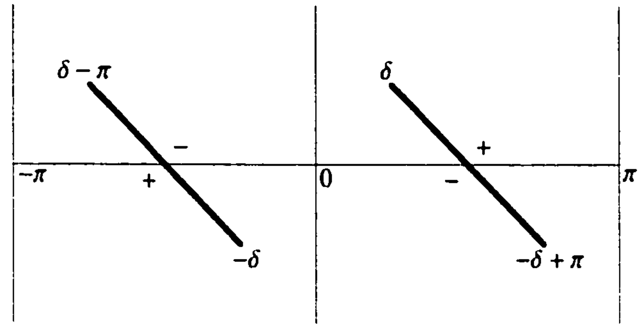


Figure 2.7: The 2π strip of analyticity with the branch points and associated cuts for the square root (2.75).

with

$$u(\alpha) = \sqrt{\cos^2 \alpha - \cos^2 \delta}. \tag{2.75}$$

The associated cuts in the 2π wide strip of analyticity are illustrated in Figure 2.7 and, relative to the case of the anisotropic half-plane shown in Figure 2.4, it is seen to have a much simpler nature with only four branch points in the strip of analyticity. The parameters θ and δ are allowed to vary independently in what follows. We note however that equation (2.71) is recovered from (2.73) by letting

$$\cos^2 \delta = \frac{3}{4} \sin^2 \bar{\theta}, \qquad \cos^2 \theta = \sin^2 \bar{\theta}, \tag{2.76}$$

and that, within the context of the problem of the penetrable wedge, the parameters θ and δ are not independent.

2.5 Summary

An abridged derivation — the reader is referred to [Senior and Legault, 2001] for more details — of the difference equations for an anisotropic impedance wedge of open angle 2Φ under oblique plane wave illumination was provided. The second order functional difference equation, itself obtained from a pair of coupled meromorphic first order difference equations, was recast as a pair of uncoupled, but branched, first order difference equations. The expressions were specialized to the cases of the anisotropic half-plane with common face impedances as well as the isotropic impedance exterior and interior right-angled wedges, also with common face impedances. Interestingly, the equations obtained all share the same form, the only difference being in the period of the equations for the exterior right-angled wedge. Their solution is discussed Chapter 7. Lastly, a generalized equation (solved in Chapter 4) based on the one obtained for a penetrable composite right-angled wedge at normal incidence was presented; its inherently simpler nature makes it ideal to develop a solution technique. The proposed approach, together with a number of important notions, is presented in the chapter that follows.

CHAPTER 3

FUNDAMENTAL NOTIONS

A NUMBER of important fundamental notions which constitute the core of the proposed method are now developed. It was shown in the previous chapter that the second order difference equation can be reduced to a pair of associated first order difference equations at the cost of introducing branch points. Since the solution to the second order equation must be expressed in terms of the branched solutions of the first order equation, meromorphic combinations of branched functions are first examined in detail. The construction of the branched solutions, a process fraught with subtleties, is next discussed. The initial step is the application of a logarithmic derivative to the first order equation and, to obtain a well-defined solution, a procedure for the successive elimination of contributions from singularities is presented. The completion of this task requires bilinear relations of Riemann which are also discussed. The relevant elements are then gathered to provide an overview of the proposed method. The chapter closes with a brief discussion of elliptic integrals of the first kind and two useful mappings are presented.

3.1 Meromorphic solutions

The solutions of the second order difference equation (2.30) are required to be meromorphic in the complex α plane as well as free of poles and zeros in the 4Φ wide strip which we denote by $S_{4\Phi} = \{|\text{Re}\,\alpha| \leq 2\Phi\}$. The solution must also satisfy a certain order condition when $|\text{Im}\,\alpha| \to \infty$ as governed by the edge condition. In the majority of the cases examined in this work the desired function will be $O(1)$ as $|\text{Im}\,\alpha| \to \infty$. We obtained a pair of related first order difference equations (2.45) and they must now be used to construct solutions to the parent second order equation. This procedure is, at first glance at least, far from obvious since the solutions to the first order equations, a general representation of which is

$$\frac{w(\alpha+2\Phi,+u)}{w(\alpha-2\Phi,+u)} = g(\alpha,+u) = \frac{u(\alpha)-f(\alpha)}{u(\alpha)+f(\alpha)}, \tag{3.1a}$$

$$\frac{w(\alpha+2\Phi,-u)}{w(\alpha-2\Phi,-u)} = g(\alpha,-u) = \frac{u(\alpha)+f(\alpha)}{u(\alpha)-f(\alpha)}, \tag{3.1b}$$

where $u(\alpha)$ denotes a square root term, are necessarily branched. These are representative of the equations that arise for the cases of interest, see for example the homologous equations for the anisotropic half-plane (2.57), those for the isotropic right-angled wedge (2.70) or those for the composite right-angled wedge (2.74). A quick note on notation is in order. Throughout this work $u(\alpha)$ is used to denote square root terms, $t(\alpha)$ denotes solutions to the second order difference equations and solutions to first order difference equations are identified by $w(\alpha, u)$. Note also that the inclusion of $u(\alpha)$ — or just u to lighten the notation — as an argument indicates a branched function. Before resuming our discussion, we also note that, of course, $w(\alpha, u)$ is also a solution to the second order difference equation but the converse is not true for $t(\alpha)$. The two equations in (3.1) only differ in the branch of $u(\alpha)$. Hence, if $w(\alpha, u)$ is a solution of (3.1a), then $w(\alpha, -u)$ is a solution of (3.1b), and this follows from the reciprocal symmetry of the right-hand sides of (3.1a) and (3.1b) with respect to the branch of $u(\alpha)$. Both solutions to the first order difference equation pair are therefore embodied in $w(\alpha, u)$ and it is sufficient to restrict out attention to (3.1a). In terms of the solutions of the first order difference equations, solutions to the second order equation (2.30) are the linear combinations

$$t(\alpha) = C_1(\alpha)w(\alpha, u) + C_2(\alpha)w(\alpha, -u) \tag{3.2}$$

where $C_{1,2}(\alpha)$ are 4Φ periodic functions. We recall however that meromorphic solutions $t(\alpha)$ are sought, a requirement which such combinations generally fail to fulfill. Before attempting to solve for $w(\alpha, u)$, it is imperative to find out whether or not combinations of the form (3.2) can be used to obtain branch-free solutions $t(\alpha)$.

3.1.1 The meromorphic constructs

Fortunately, it is indeed possible to linearly combine branched expressions such that the result is branch-free. The answer lies in constructing solutions which are symmetric with respect to the choice of the branch of $u(\alpha)$, a long-known procedure mentioned by Sommerfeld [1964] as well as Appell [1976]. The two simplest such expressions that are linearly independent are

$$t_\Sigma(\alpha) = w(\alpha, u(\alpha)) + w(\alpha, -u(\alpha)), \tag{3.3a}$$

$$t_\Delta(\alpha) = \frac{w(\alpha, u(\alpha)) - w(\alpha, -u(\alpha))}{u(\alpha)}, \tag{3.3b}$$

and both are specialized versions of the solution (3.2). Heuristically, they are seen to be invariant if we let $u(\alpha) \to -u(\alpha)$ and therefore certainly seem to provide a basis for branch-free solutions. The property is verified more rigorously by examining, using Taylor expansions, the behavior of the combinations in the neighborhood of branch points of $u(\alpha)$. As made obvious in Section 3.2.1, it is appropriate to consider expressions of the form

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \frac{p(\xi)}{u(\xi)} d\xi \tag{3.4}$$

where $p(\alpha)$ is a rational meromorphic function free of singularities in the neighborhood of branch points of $u(\alpha)$. In the neighborhood of a branch point[1] δ, we therefore have

$$\frac{p(\alpha)}{u(\alpha)} \simeq \sqrt{\alpha - \delta} \sum_{n \geq -1} a_n (\alpha - \delta)^n \tag{3.5}$$

where the a_n are constants, and, integrating term by term,

$$\int_{\delta}^{\alpha} \frac{p(\xi)}{u(\xi)} d\xi \simeq \sqrt{\alpha - \delta} \sum_{n \geq 0} A_n (\alpha - \delta)^n = \sqrt{\alpha - \delta} P(\alpha) \tag{3.6}$$

where $P(\alpha)$ is a polynomial and A_n are constants. Therefore, in the neighborhood of a branch-point δ,

$$w(\alpha, \pm u) = \exp \int_{\delta}^{\alpha} \frac{p(\xi)}{\pm u(\xi)} d\xi \simeq \sum_{n \geq 0} (\pm 1)^n (\alpha - \delta)^{n/2} P^n(\alpha). \tag{3.7}$$

Upon substitution into (3.3a) the fractional powers mutually cancel and

$$t_\Sigma(\alpha) \simeq 2 \sum_{n \geq 0} (\alpha - \delta)^n P^{2n}(\alpha), \tag{3.8}$$

[1] The letter δ is used to denote a branch point of $u(\alpha)$ throughout this work.

an expression which is independent of the branch of $u(\alpha)$. Since $1/u(\alpha) \simeq Q(\alpha)/\sqrt{\alpha-\delta}$ for $\alpha \simeq \delta$ with $Q(\alpha)$ a polynomial, a similar procedure for (3.3b) gives

$$t_\Delta(\alpha) \simeq 2\frac{Q(\alpha)}{\sqrt{\alpha-\delta}}\sum_{n\geq 0}(\alpha-\delta)^{(2n+1)/2}P^{2n+1}(\alpha) = 2Q(\alpha)\sum_{n\geq 0}(\alpha-\delta)^n P(\alpha)^{2n+1}. \tag{3.9}$$

Both (3.3a) and (3.3b) are therefore meromorphic in the neighborhood of the branch points of $u(\alpha)$; we conclude that it is indeed possible to construct such expressions from linear combinations of branched expressions which, in this instance, are the solutions to the first order equations. Their examination is postponed for a short while, it is first verified that the branch-free forms (3.3) are meromorphic in the entire complex α plane.

3.1.2 Analytic continuation

It is worthwhile to examine at this juncture the continuation of $t(\alpha)$ outside the strip of analyticity via the first order difference equations for $w(\alpha, u)$. This will explicitly show that the solution $t(\alpha)$ is free of branch points outside the strip of analyticity $S_{4\Phi}$ and has both poles and zeros there, a characteristic shared with the Maliuzhinets functions. Suppose that $\operatorname{Re}\alpha' \in [2\Phi, 4\Phi]$ such that α' lies in a strip adjacent to the strip of analyticity. Then $\alpha = \alpha' - 4\Phi$ so that $\operatorname{Re}\alpha \in [-2\Phi, 2\Phi]$, and from the first order difference equation (3.1a)

$$w(\alpha', u) = w(\alpha + 4\Phi, u) = g(\alpha + 2\Phi, u)w(\alpha, u) = g(\alpha, u)^{\pm 1}w(\alpha, u), \tag{3.10}$$

where it is assumed, consistent with the cases we are concerned with in this work, that $g(\alpha + 2\Phi, u) = g(\alpha, u)$ or $g(\alpha + 2\Phi, u) = 1/g(\alpha, u)$. The branch-free expression $t_\Sigma(\alpha')$ is then

$$\begin{aligned} t_\Sigma(\alpha') &= w(\alpha', u) + w(\alpha', -u) \\ &= g(\alpha, u)^{\pm 1}w(\alpha, u) + g(\alpha, u)^{\mp 1}w(\alpha, -u) \\ &= \frac{u(\alpha)^2 + f(\alpha)^2}{u(\alpha)^2 - f(\alpha)^2}\left\{w(\alpha, u) + w(\alpha, -u)\right\} \mp \frac{2u(\alpha)f(\alpha)}{u(\alpha)^2 - f(\alpha)^2}\left\{w(\alpha, u) - w(\alpha, -u)\right\} \\ &= \frac{u(\alpha)^2 + f(\alpha)^2}{u(\alpha)^2 - f(\alpha)^2}t_\Sigma(\alpha) \mp \frac{2u(\alpha)^2 f(\alpha)}{u(\alpha)^2 - f(\alpha)^2}t_\Delta(\alpha), \end{aligned} \tag{3.11}$$

a meromorphic linear combination of $t_\Sigma(\alpha)$ and $t_\Delta(\alpha)$ which are themselves meromorphic. Similarly, assuming[2] $u(\alpha') = u(\alpha + 4\Phi) = u(\alpha)$, then

$$\begin{aligned} t_\Delta(\alpha') &= \frac{1}{u(\alpha')}\left\{w(\alpha', u) - w(\alpha', -u)\right\} \\ &= \frac{u(\alpha)^2 + f(\alpha)^2}{u(\alpha)^2 - f(\alpha)^2}\,\frac{w(\alpha, u) - w(\alpha, -u)}{u(\alpha)} \mp \frac{2f(\alpha)}{u(\alpha)^2 - f(\alpha)^2}\left\{w(\alpha, u) + w(\alpha, -u)\right\} \\ &= \frac{u(\alpha)^2 + f(\alpha)^2}{u(\alpha)^2 - f(\alpha)^2}\,t_\Delta(\alpha) \mp \frac{2f(\alpha)}{u(\alpha)^2 - f(\alpha)^2}\,t_\Sigma(\alpha), \end{aligned} \tag{3.12}$$

and a meromorphic linear combination of meromorphic functions is again recovered. Hence, the continuation of both of the branch-free forms $t_\Sigma(\alpha)$ and $t_\Delta(\alpha)$ outside the strip of analyticity can be expressed as meromorphic linear combinations of the branch-free functions evaluated within the strip of analyticity. The expressions are thus branch-free but may now have both poles and zeros. In fact, by iterating this procedure, the function $t_\Delta(\alpha)$ and $t_\Sigma(\alpha)$ can be continued to any arbitrary point in the plane, and it can be appreciated from (3.11) and (3.12) that the order of the poles will increase as we move further away from the strip of analyticity. The solution therefore has an infinite number of denumerable poles in the α plane and, consequently, if it is $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$ say, it must also have an infinite number of denumerable zeros. More generally, this also applies if the function is $O\{\exp(1 - \varepsilon)|\mathrm{Im}\,\alpha|\}$ (with ε positive real). This property is shared with the Maliuzhinets functions, see for example [Maliuzhinets, 1958*a*].

Provided we can construct branched solutions $w(\alpha, \pm u)$ which are well-defined, then the expressions (3.3) will allow us to construct solutions to the second order difference equation that are meromorphic in the entire complex α plane. Of course, this addresses only the singularities associated with branch points, any poles of $w(\alpha, u)$ must be eliminated subsequently if pole-free (and zero-free) solutions $t(\alpha)$ are sought in some strip of the complex α plane. The impetus is now to find solutions to the first order difference equations.

3.2 Solution to first order equations

The second order equation was reduced to a pair of branched first order equations and it was shown in the previous section that their solutions could be linearly combined to construct meromorphic solutions to the parent second order equation. Unfortunately, the resulting first order difference equations generally involve branched expressions and the

[2]The other possibility of interest $u(\alpha + 4\Phi) = -u(\alpha)$ is also acceptable. It results in an overall change of sign.

methodology put forward by MALIUZHINETS [1958*a*] does not apply. In fact, it turns out that the cases which can be solved by Maliuzhinets's approach are special cases of the more general branched first order equations obtained. The intricacies of obtaining well-defined branched solutions of these first order equations are now examined. The approach followed, while conceptually simple, has its share of subtleties. We proceed by applying a logarithmic derivative to the first order difference equation which results in a solution expressed in terms of a path integral. However, the resulting integrand has both poles and branch points meaning that the path integral is initially ill-defined. In order to mitigate this, appropriate 4Φ periodic expressions offsetting the singularities of the integrand are added.

3.2.1 The logarithmic derivative

Throughout much of this work, solutions are sought to first order difference equations of the form

$$\frac{w(\alpha+2\Phi,u)}{w(\alpha-2\Phi,u)} = g(\alpha,u) = \frac{u(\alpha)-f(\alpha)}{u(\alpha)+f(\alpha)} \tag{3.13}$$

where $u(\alpha)$ is the square root of a trigonometric polynomial such as the ones found in the equations of the anisotropic half-plane (2.57), the isotropic right-angled wedge (2.70), or the composite right-angled wedge(2.74a). Consistent with the cases of interest, the function $f(\alpha)$ is a trigonometric polynomial, possibly a constant, which is such that $f(\alpha)/u(\alpha) \to 0$ as $|\mathrm{Im}\,\alpha| \to \infty$. The initial step in solving for $w(\alpha,u)$ is to apply a logarithmic derivative to the first order equation (3.13). Doing so results in

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha+2\Phi,u) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha-2\Phi,u) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u) \tag{3.14}$$

and, consistent with the form assumed for $g(\alpha,u)$ in (3.13) as well as that of $u(\alpha)$ mentioned above, if we write $\mathrm{d}u(\alpha)/\mathrm{d}\alpha = x(\alpha)/u(\alpha)$ where $x(\alpha)$ is a trigonometric polynomial, then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\frac{u(\alpha)-f(\alpha)}{u(\alpha)+f(\alpha)} = \frac{1}{u(\alpha)}\frac{x(\alpha)f(\alpha)-f'(\alpha)u(\alpha)^2}{u(\alpha)^2-f^2(\alpha)} = \frac{h(\alpha)}{u(\alpha)} \tag{3.15}$$

where $h(\alpha)$ is a meromorphic function. It can also be shown that, with the assumptions made above, $h(\alpha)/u(\alpha) \to 0$ (exponentially) as $|\mathrm{Im}\,\alpha| \to \infty$. For the cases considered herein, it is sufficient to restrict ourselves to cases where $h(\alpha\pm2\Phi)/u(\alpha\pm2\Phi) = +h(\alpha)/u(\alpha)$ or $h(\alpha\pm2\Phi)/u(\alpha\pm2\Phi) = -h(\alpha)/u(\alpha)$. This periodicity is exploited by writing

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha,u) = v_0(\alpha,u) = C\alpha\frac{h(\alpha)}{u(\alpha)} \tag{3.16}$$

where C is some constant to be determined. Substituting in the left-hand side of (3.14),

$$\begin{aligned} v_0(\alpha+2\Phi)-v_0(\alpha-2\Phi) &= C(\alpha+2\Phi)\frac{h(\alpha+2\Phi)}{u(\alpha+2\Phi)} - C(\alpha+2\Phi)\frac{h(\alpha-2\Phi)}{u(\alpha-2\Phi)} \\ &= \pm C(\alpha+2\Phi)\frac{h(\alpha)}{u(\alpha)} \mp C(\alpha+2\Phi)\frac{h(\alpha)}{u(\alpha)} \\ &= \pm C4\Phi\frac{h(\alpha)}{u(\alpha)} \end{aligned} \tag{3.17}$$

and (3.14) is satisfied if $C = \pm 1/4\Phi$. A tentative solution to the first order equation (3.13) is then[3]

$$w(\alpha,u) = \exp\int_{\alpha_0}^{\alpha} v_0(\xi,u)\mathrm{d}\xi \tag{3.18}$$

with

$$v_0(\alpha,u) = \pm\frac{\alpha}{4\Phi}\frac{h(\alpha)}{u(\alpha)}, \qquad \frac{h(\alpha+2\Phi)}{u(\alpha+2\Phi)} = \frac{h(\alpha-2\Phi)}{u(\alpha-2\Phi)} = \pm\frac{h(\alpha)}{u(\alpha)}. \tag{3.19}$$

Consistent with equation (3.13), it is seen that $v_0(\alpha,u) = -v_0(\alpha,-u)$ and hence $w(\alpha,u) = 1/w(\alpha,-u)$, in concordance with $g(\alpha,u) = 1/g(\alpha,-u)$. Furthermore, since $h(\alpha)/u(\alpha)$ vanishes exponentially fast as $|\mathrm{Im}\,\alpha| \to \infty$, it follows that $v_0(\alpha,u)$ also vanishes in that limit. The form proposed in (3.18) is however ill-defined owing to the presence of the polar and the cyclic periods[4] due to, respectively, the poles of $h(\alpha)$ and the branch points of $1/u(\alpha)$. In order to obtain a single-valued integral expression, we must consider instead

$$w(\alpha,u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi,u) + v_{4\Phi}(\xi,u)\right\}\mathrm{d}\xi, \tag{3.20}$$

where the added term $v_{4\Phi}(\alpha,u)$ represents a sum of 4Φ periodic terms — also referred to as unit periodics — having the same symmetries as $v_0(\alpha,u)$ and specifically selected to remove the offending periods.

It is appropriate to discuss more precisely at this juncture what is meant by 4Φ periodic functions as, not surprisingly, not all 4Φ periodics are acceptable. For example, it can be shown with little effort that arbitrary constants will fail to provide proper "homogeneous solutions". To clarify this, consider the case where $v_{4\Phi}(\alpha)$ is temporarily assumed to be meromorphic and has poles with integer residues. The 4Φ periodic functions added must

[3]The integral is taken on a Riemann sheet dissected with branch cuts. As shown later, the branched function $u(\alpha)$ is added to the limits to denote integrals taken on the Riemann surface.

[4]We borrow here the terminology used by Springer [1981] when characterizing differentials of the third kind.

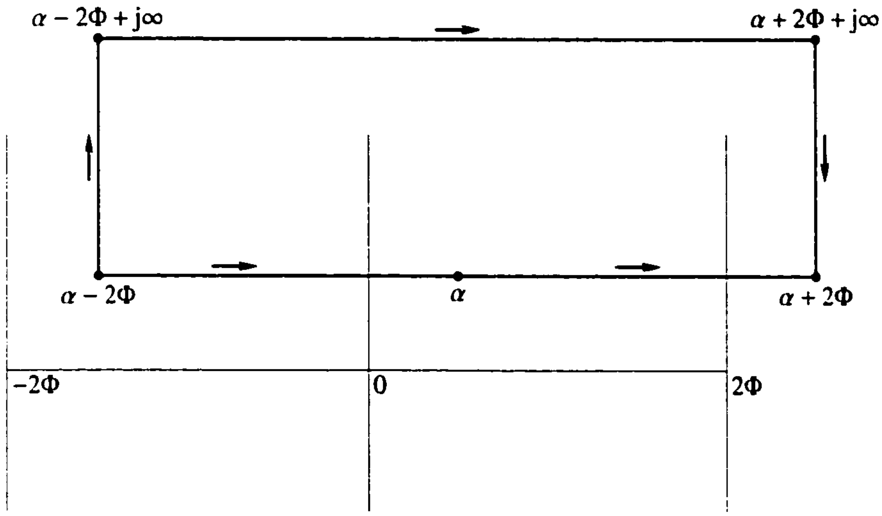


Figure 3.1: The strip of analyticity $S_{4\Phi} = \{\alpha : |\text{Re}\ \alpha| \leq 2\Phi\}$ with the original path of integration from $\alpha - 2\Phi$ to $\alpha + 2\Phi$ as well as the resulting path after deformation out to $j\infty$.

be such that, if

$$w_{4\Phi}(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} v_{4\Phi}(\xi)\mathrm{d}\xi \tag{3.21}$$

then

$$\frac{w_{4\Phi}(\alpha + 2\Phi, u)}{w_{4\Phi}(\alpha - 2\Phi, u)} = 1, \tag{3.22}$$

so that its addition to the integrand of $w(\alpha, u)$ in (3.20) does not jeopardize the equality in (3.13). It is insightful to examine (3.22) when $|\text{Im}\ \alpha| \to \infty$. By direct substitution,

$$\frac{w_{4\Phi}(\alpha + 2\Phi, u)}{w_{4\Phi}(\alpha - 2\Phi, u)} = \frac{\exp \int_{\alpha_0}^{\alpha+2\Phi} v_{4\Phi}(\xi)\mathrm{d}\xi}{\exp \int_{\alpha_0}^{\alpha-2\Phi} v_{4\Phi}(\xi)\mathrm{d}\xi} = \exp \int_{\alpha-2\Phi}^{\alpha+2\Phi} v_{4\Phi}(\xi)\mathrm{d}\xi \tag{3.23}$$

and closure of the contour at infinity, as shown in Figure 3.1, yields

$$\exp\int_{\alpha-2\Phi}^{\alpha+2\Phi} v_{4\Phi}(\xi)\mathrm{d}\xi = \exp\left(2\pi\mathrm{j}\sum \mathrm{Res}\right)$$

$$\cdot\exp\left(\int_{\alpha-2\Phi}^{\alpha-2\Phi+\mathrm{j}\infty} + \int_{\alpha-2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi+\mathrm{j}\infty} + \int_{\alpha+2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi}\right) v_{4\Phi}(\xi)\mathrm{d}\xi$$

$$= \exp\int_{\alpha-2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi+\mathrm{j}\infty} v_{4\Phi}(\xi)\mathrm{d}\xi \tag{3.24}$$

where the contributions from the vertical paths along $\alpha \pm 2\Phi$ vanish by virtue of the periodicity of the integrand, and the contributions from any pole vanish due to the assumption of integer residues. This highlights, besides the needed periodicity, specific requirements on the behavior of $v_{4\Phi}(\alpha)$ as $|\mathrm{Im}\,\alpha| \to \infty$ which, in this instance, must be such that the above expression equals unity in order for (3.22) to hold. The two simplest cases where this holds are (i) when $v_{4\Phi}(\alpha) \to 0$ as $|\mathrm{Im}\,\alpha| \to \infty$ and (ii) when $v_{4\Phi}(\alpha) \to C_\infty$ as $|\mathrm{Im}\,\alpha| \to \infty$, where C_∞ is a specific constant. Indeed, the above expression then requires

$$\exp\int_{\alpha-2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi+\mathrm{j}\infty} v_{4\Phi}(\xi)\mathrm{d}\xi = \exp(4\Phi C_\infty) \tag{3.25}$$

and since we must have $4\Phi C_\infty = 2\pi\mathrm{j}\mathbf{Z}$, then[5] $C_\infty = \mathrm{j}\pi\mathbf{Z}/2\Phi$. An example of such a function is $v_{4\Phi}(\alpha) = (\pi\mathbf{Z}/2\Phi)\tan\pi\alpha/4\Phi$. In the majority of the cases examined in this work, it will be sufficient to restrict our attention to the first case given above, *i.e.*, $v_{4\Phi}(\alpha) \to 0$ as $|\mathrm{Im}\,\alpha| \to \infty$. The above also implies that if the right-hand side of (3.22) was some arbitrary constant C, then equality would only hold for specific 4Φ periodic integrands that do not vanish as $|\mathrm{Im}\,\alpha| \to \infty$ and it would then be required that $\exp(4\Phi C_\infty) = C$. There will be instances in Chapters 6 and 7 where $C = -1$; appropriately chosen unit periodics that do not vanish as $|\mathrm{Im}\,\alpha| \to \infty$ will be introduced. This analysis also applies when the integrand is branched provided the cyclic periods have all been eliminated.

The above technique of path closure at infinity also makes it possible to verify explicitly that $w(\alpha, u)$ as given in (3.20) is indeed a solution of the difference equation (3.13). From (3.24),

$$\frac{w(\alpha+2\Phi,u)}{w(\alpha-2\Phi,u)} = \exp\left(2\pi\mathrm{j}\sum \mathrm{Res}\right)$$

$$\cdot\exp\left(\int_{\alpha-2\Phi}^{\alpha+2\Phi-\mathrm{j}\infty} + \int_{\alpha-2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi+\mathrm{j}\infty} + \int_{\alpha+2\Phi+\mathrm{j}\infty}^{\alpha+2\Phi}\right)\left\{v_0(\xi,u) + v_{4\Phi}(\xi,u)\right\}\mathrm{d}\xi \tag{3.26}$$

and the restriction to meromorphic functions has now been lifted since both $v_0(\alpha, u)$ and $v_{4\Phi}(\alpha, u)$ are functions of $u(\alpha)$. Without loss of generality, the restriction to integer residues

[5] Throughout, $\mathbf{Z}$ denotes the set of integers, $\mathbf{C}$ the set of complex numbers, and $\mathbf{R}$ the set of real numbers.

is maintained; the reason for this is discussed in Section 3.2.3. We also recall that $v_{4\Phi}(\alpha, u)$, now assumed to vanish as $|\text{Im}\,\alpha| \to \infty$, is chosen such that the sum $v_0(\alpha, u) + v_{4\Phi}(\alpha, u)$ is free of cyclic periods in the strip and this implies that the Cauchy-Goursat theorem and, by extension, the residue theorem apply. In other words, we need only concern ourselves with contributions from eventual poles with integer residues. The contribution from the portion of the path in (3.26) that lies entirely at infinity vanishes as both members of the integrand vanish in that limit, the remaining portion of the path integral becomes

$$\begin{aligned}
\left(\int_{\alpha-2\Phi+j\infty}^{\alpha-2\Phi} + \int_{\alpha+2\Phi}^{\alpha+2\Phi+j\infty}\right)&\left\{v_0(\xi, u) + v_1(\xi, u)\right\}\mathrm{d}\xi \\
&= \int_{\alpha+j\infty}^{\alpha} \left\{v_0(\xi + 2\Phi, u) - v_0(\xi - 2\Phi, u)\right\}\mathrm{d}\xi \\
&= \int_{\alpha+j\infty}^{\alpha} \frac{\mathrm{d}}{\mathrm{d}\xi} \ln g(\xi, u)\mathrm{d}\xi \\
&= \ln g(\alpha, u)
\end{aligned} \tag{3.27}$$

where use has been made of (3.14), (3.16) and the fact that $g(\alpha, u)$ goes to unity as $|\text{Im}\,\alpha| \to \infty$. This allows for the explicit recovery of $g(\alpha, u)$ by direct substitution of $w(\alpha, u)$ as given in (3.20) into the first order difference equation (3.1). There is however one problem with the above to be addressed: the logarithmic derivative of a functional quantity multiplied by an arbitrary constant is independent of that constant.

This ambiguity can be resolved by examining the solution and the first order equation in the neighborhood of $\pm j\infty$. Indeed, suppose we had instead of (3.1) the equation

$$\frac{w(\alpha + 2\Phi, u)}{w(\alpha - 2\Phi, u)} = C\frac{u(\alpha) - f(\alpha)}{u(\alpha) + f(\alpha)} \tag{3.28}$$

for some arbitrary constant C. It is disquieting to realize that the logarithmic derivative of the right-hand side is *independent* of the constant C, casting a veil of suspicion on the entire procedure given above. The development shown in (3.27) is unsatisfactory in this respect and a variation of this procedure is now considered where, instead of closing the contour of integration out to $j\infty$, the behavior of the solution is directly examined in the neighborhood of $\pm j\infty$. Letting $\alpha \to \alpha + j\infty$ in the first order equation (3.13) results in, recalling that $f(\alpha)/u(\alpha) \to 0$ as $|\text{Im}\,\alpha| \to \infty$,

$$\frac{w(\alpha + 2\Phi + j\infty, u)}{w(\alpha - 2\Phi + j\infty, u)} = \frac{u(\alpha + j\infty) - f(\alpha + j\infty)}{u(\alpha + j\infty) + f(\alpha + j\infty)} = 1 \tag{3.29}$$

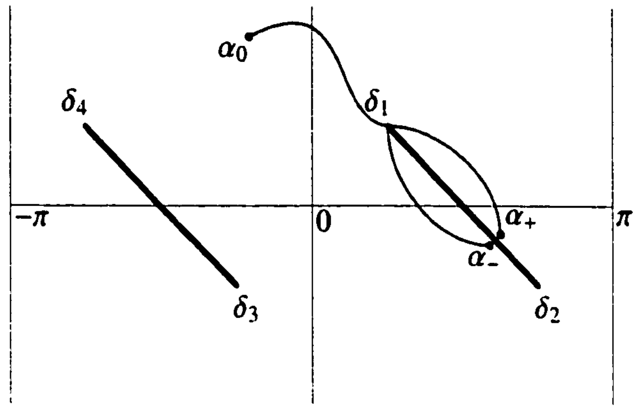


Figure 3.2: The strip of analyticity $S_{2\pi} = \{\alpha : |\text{Re}\,\alpha| \le 2\pi\}$ with the branch cuts of $u(\alpha)$ indicated by the thick solid lines. The thinner line indicates a path starting from an arbitrary point α_0, going to the branch point δ and then connecting to points $\alpha_\pm$ located on either side of a branch cut.

implying

$$\exp \int_{\alpha-2\Phi+j\infty}^{\alpha+2\Phi+j\infty} \Big\{ v_0(\xi, u) + v_{4\Phi}(\xi, u) \Big\} d\xi = 1. \tag{3.30}$$

Since $v_0(\alpha, u) \to 0$ as $|\text{Im}\,\alpha| \to \infty$ the above reduces to

$$\exp \int_{\alpha-2\Phi+j\infty}^{\alpha+2\Phi+j\infty} v_{4\Phi}(\xi, u) d\xi = 1 \tag{3.31}$$

from which it is concluded that $v_0(\alpha, u)$ corresponds to the case where $C = 1$ in (3.28). Any other value of C would have to be accommodated with the appropriate choice of $v_{4\Phi}(\alpha, u)$. Demonstration (3.27) is on its own ambiguous to a multiplicative constant C but, since the analysis just carried out shows that this is unity, the proposed $v_0(\alpha, u)$, once its periods are eliminated, does indeed satisfy the desired first order equation.

3.2.2 The lower limit of integration

An important point which has yet to be addressed is the nature of the lower limit α_0 in (3.20). It is clear that as far as the solution $w(\alpha, u)$ given in (3.20) is concerned, an arbitrary lower limit will result in a scaling of the solution. This may not seem of consequence on its own but it does have serious ramifications for the branch-free forms (3.3). To see this, consider the points α_+ and α_- which are adjacent and on opposite sides of a branch cut as shown in Figure 3.2. For the sake of illustration, the branch point configuration for $u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta}$ is used and this has branch points at $\pm\delta$ and $\pm(\pi - \delta)$. With an arbitrary α_0, we have from (3.3),

$$t_\Sigma(\alpha_+) = \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_+}\right) v(\alpha, u)\mathrm{d}\alpha + \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_+}\right) v(\alpha, -u)\mathrm{d}\alpha, \tag{3.32}$$

$$t_\Sigma(\alpha_-) = \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_-}\right) v(\alpha, u)\mathrm{d}\alpha + \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_-}\right) v(\alpha, -u)\mathrm{d}\alpha, \tag{3.33}$$

where $v(\alpha, u) = v_0(\alpha, u) + v_{4\Phi}(\alpha, u)$. In order for the branch-free function $t_\Sigma(\alpha)$ to be continuous, we require $t_\Sigma(\alpha_+) = t_\Sigma(\alpha_-)$. This can be verified by rewriting, say, the expression for $t_\Sigma(\alpha_-)$ in terms of α_+ using

$$\int_{\delta}^{\alpha_-} v(\alpha, u)\mathrm{d}\alpha = -\int_{\delta}^{\alpha_+} v(\alpha, u)\mathrm{d}\alpha = \int_{\delta}^{\alpha_+} v(\alpha, -u)\mathrm{d}\alpha \tag{3.34}$$

since $v(\alpha, u) = -v(\alpha, -u)$. We therefore have

$$\begin{aligned} t_\Sigma(\alpha_-) &= \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_-}\right) v(\alpha, u)\mathrm{d}\alpha + \exp\left(\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_-}\right) v(\alpha, -u)\mathrm{d}\alpha \\ &= \exp\left(\int_{\alpha_0}^{\delta} - \int_{\delta}^{\alpha_+}\right) v(\alpha, u)\mathrm{d}\alpha + \exp\left(\int_{\alpha_0}^{\delta} - \int_{\delta}^{\alpha_+}\right) v(\alpha, -u)\mathrm{d}\alpha \\ &= \exp\left(-\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_+}\right) v(\alpha, -u)\mathrm{d}\alpha + \exp\left(-\int_{\alpha_0}^{\delta} + \int_{\delta}^{\alpha_+}\right) v(\alpha, u)\mathrm{d}\alpha \end{aligned} \tag{3.35}$$

and equality between (3.32) and (3.35) requires

$$\exp\int_{\alpha_0}^{\delta} v(\alpha, u)\mathrm{d}\alpha = \exp - \int_{\alpha_0}^{\delta} v(\alpha, u)\mathrm{d}\alpha \tag{3.36}$$

which, since it must hold for all values of δ, implies $\alpha_0 = \delta$. Assuming that α_0 is a branch point distinct from δ, a similar analysis now shows

$$\exp\int_{\delta_m}^{\delta_n} v(\alpha, u)\mathrm{d}\alpha = \exp - \int_{\delta_m}^{\delta_n} v(\alpha, u)\mathrm{d}\alpha, \tag{3.37}$$

where δ_m and δ_n are distinct branch points, and the equality is generally satisfied if

$$\int_{\delta_m}^{\delta_n} v(\alpha, u)\mathrm{d}\alpha = \mathrm{j}\pi\mathbf{Z}. \tag{3.38}$$

In fact, the branch point to branch point integrals must all identically vanish. Such an assertion is unfounded within the scope of the analysis just presented and will be made clear in the upcoming section which deals with the elimination of the singularities. It is shown there that in order to avoid poles that arise in the limits where $\delta \to \pi/2$ or 0, the branch point to branch point integrals must all vanish. It can therefore be concluded that we require

$$\int_{\delta_m}^{\delta_n} v(\alpha, u)\mathrm{d}\alpha = 0, \tag{3.39}$$

where δ_m and δ_n are any two branch points in the strip of analyticity. This also eliminates any ambiguity with respect to the choice of the lower limit. Indeed, in this light, (3.39) implies that the lower limit α_0 must simply be a branch point — any branch point — in the strip of analyticity. Apart from its simple nature, this requirement also has computational benefits since the numerical evaluation of integrals can be made less computationally demanding by choosing the branch point that lies closest to the upper limit α, a welcome advantage when carrying out integrations on a dissected plane.

3.2.3 Elimination of singularities

As mentioned above, the 4Φ periodic function $v_{4\Phi}$ are required in the integrand of $w(\alpha, u)$ in (3.20) to render the path integral well-defined despite the presence of poles, leading to polar contributions, as well as branch points, leading to cyclic contributions, in $v_0(\alpha, u)$. The presence of poles will generally make a path integral such as the one in (3.20) multi-valued. A pole of residue Res will give rise to a polar period equal to $2\pi\mathrm{j}\mathrm{Res}$ and, depending on the orientation of the integration path and its winding number around the pole, the contribution to the integral will be $2\pi\mathrm{j}\mathbf{Z}\mathrm{Res}$. In the case of (3.20), polar periods arise at the zeros of $u(\alpha)^2 - f^2(\alpha)$ — see (3.15). Fortunately, the quantities involves are all trigonometric polynomials and the poles will therefore arise in a periodic fashion, simplifying their elimination. This serves two purposes as it not only eliminates their associated polar periods but goes towards fulfilling the requirement for a solution that is pole-free in the strip of analyticity. They can be eliminated by introducing the 4Φ periodic

$$v_1(\alpha, u) = \frac{1}{u(\alpha)} \frac{\varphi_0 + \sum_{n\geq 1} \varphi_n \cos\left(\frac{\pi\alpha}{2\Phi}\right) + \vartheta_n \sin\left(\frac{\pi\alpha}{2\Phi}\right)}{u^2(\alpha) - f^2(\alpha)}, \tag{3.40}$$

and the number of degrees of freedoms φ_n and ϑ_n is considerably reduced by the need to match the parity of $v_0(\alpha, u)$ as well as the order requirement. The remaining constants φ_n or ϑ_n are chosen to eliminate the residues and obtained by straightforward algebra. Interestingly, it will also be shown in future chapters when specific examples are considered that

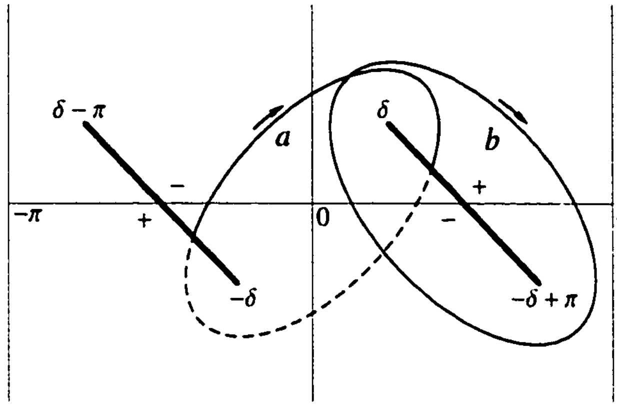


Figure 3.3: The strip of analyticity $S_{2\pi} = \{\alpha : |\text{Re}\,\alpha| \leq 2\pi\}$. The thick lines indicate the branch cuts of $u(\alpha)$, the positive and negative signs indicate relative changes in sign of $u(\alpha)$ across the different cuts. The clockwise cycles a and b used to define the cyclic periods are as indicated. Note that cycle a crosses from the upper Riemann sheet (solid line) to the lower Riemann sheet (dashed line) whereas b is confined to the upper sheet.

the sum of $v_0(\alpha, u)$ and $v_1(\alpha, u)$ recovers the logarithmic derivative of the known solution in the particular limit when the branch points vanish. In other words, in the instance where the right-hand side in (3.13) is branch-free, the above procedure of pole elimination is sufficient for the construction of the known solutions expressed in terms of Maliuzhinets functions. We also note in passing that poles with integer residue $\mathbf{Z}$ do not compromise path independence. Indeed, their capture leads to an additive $2\pi j\mathbf{Z}$ contribution in the exponent of (3.20) which has no effect on the final value of $w(\alpha, u)$. *This is the reason why integer residues are assumed in most of the developments which require applying the residue theorem.*

In a fashion similar to polar periods, a cyclic period arises from the non-zero contribution incurred when integrating along a loop encircling a branch cut in $S_{4\Phi}$, thereby making the path integral multi-valued. To better illustrate this, consider the case where $u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta}$ (see (2.75)) whose branch points are shown in Figure 3.3 along with a particular configuration of branch cuts. For future reference, the figure also provides the definition of the cycles a and b required when considering the matter of the cyclic periods. A cyclic period is then, for example, obtained when integrating $v_0 + v_1$ along the cycle b which encircles the branch cut joining the branch points δ and $-\delta + \pi$. As in the case of the polar periods, it is strictly speaking not required for the cyclic periods to vanish identically to avoid jeopardizing single-valuedness since periods equal to $2\pi j\mathbf{Z}$ do not change the value of (3.20). This ambiguity is resolved by considering values of δ for which the branch points vanish. As $\delta \to \pi/2$ the branch points of $1/u(\alpha)$ in $v_{0,1}(\alpha, u)$ coalesce into poles at $\pm\pi/2$ and their associated cyclic periods then become polar periods. Consequently, the cyclic periods associated with the cycle b in Figure 3.3 must be annulled, as opposed to setting them equal to some non-zero integer multiple of $2\pi j$, to eliminate poles that would otherwise arise as $\delta \to \pi/2$. We observe that this requirement is equivalent to annulling the integral of $v_0 + v_{4\Phi}$ along the cuts between δ and $\pi - \delta$. There is also a similar requirement, which is not obvious when solely considering integration on either of the Riemann sheets, on the cyclic periods associated with the cycle a which loops from one Riemann sheet to the other. Its necessity is revealed by examining either of the symmetric forms in (3.3) together with the requirement for continuity as carried out in the previous section. More crucially perhaps, it can be appreciated that in the limit where $\delta \to 0$, $1/u(\alpha)$ gives rise to a pole at $\alpha = 0$ whose residue will be null if the cyclic period on cycle a is set to zero. In short, the above implies, taking advantage of the even parity, the need to annul the cyclic periods of $v_0(\alpha, u) + v_{4\Phi}(\alpha, u)$ on the cycles a and b shown in Figure 3.3. Alternatively, this can be thought of as requiring that all branch point to branch point integrals vanish in $S_{4\Phi}$ and under this condition all branch points within the strip are equivalent integration-wise. This is stated mathematically in (3.39) and it ensures that the resulting expressions will remain free of poles despite coalescing branch points in certain limits. Interestingly, the requirement for vanishing branch point to branch point integrals is independent of the choice of branch cuts: the same result would be obtained if we were carrying out the analysis on the Riemann surface (as opposed to a dissected Riemann sheet).

The degrees of freedom required to eliminate the cyclic periods will be provided by 4Φ periodic functions which give rise to elliptic integrals of the first and third kind. Since integrals of the first kind have cyclic periods which are not parameter dependent, meaning that they are constant for a given configuration of branch points, their contributions on the various cycles will be adjusted by means of an associated multiplicative constant. On the other hand, the cyclic periods of integrals of the third kind are functions of the location of its logarithmic singularities, their contributions along the cycles can be adjusted by changing the location of the singularities. This will be made clearer when elliptic integrals are discussed in the next section. Adding such expressions as required to the function $v_{4\Phi}(\alpha, u)$ and enforcing vanishing cyclic periods will produce an equation system consisting of N equations in N unknowns, N being the number of cyclic periods to eliminate. The resulting system is however highly non-linear but it will be shown that it can be solved analytically by resorting to relationships known as the bilinear relations of Riemann. We defer any further discussion of the cancellation process until the next chapter where a specific example is considered.

3.3 The bilinear relations of Riemann

We now examine, albeit briefly, some rather remarkable relationships involving the cyclic periods (or periodicity modules) of elliptic and hyper-elliptic integrals. The motivation for this is that expressions obtained when enforcing vanishing cyclic periods have the same nature as relationships known as the bilinear relations of Riemann. This allows for rewriting quantities involving cyclic periods, some of which are functions of an unknown pole, in terms of an integral of the first kind where the pole now appears explicitly as its argument. It is unclear how, within the framework of this technique, one could otherwise proceed to obtain an analytical solution to the problem. The treatment presented here is more classical in nature and is tailored after the one given by Appell [1976]. A more contemporary approach is provided by Springer [1981].

We consider the rational function $f(z,u)$ which, for simplicity, we assume to have null residues. In accordance with the notation used above, $u(z)$ denotes a branched function where

$$u(z) = \sqrt{(z-\delta_1)(z-\delta_2)(z-\delta_3)(z-\delta_4)} \tag{3.41}$$

and the branch points and associated cuts are illustrated in Figure 3.4. The function exists on a two-sheeted Riemann surface whose members are connected by the pair of cuts illustrated in the figure. This is topologically equivalent to a torus (a handlebody of genus one) as illustrated in Figure 3.5, and can be visualized by first deforming each of the Riemann sheets into a sphere with two dissections (the branch cuts) and then joining the spheres at the cuts. The behavior of the integral of a function $f(z,u)$ defined on such a Riemann surface is now considered. Define[6]

$$F(z,u) = \int_{(z_0,u_0)}^{(z,u)} f(\xi,u)\mathrm{d}\xi \tag{3.42}$$

which is assumed to represent an elliptic integral. It is ill-defined on the Riemann surface since the latter is not simply connected. In order to correct this, the dissection pair $\mathfrak{a}, \mathfrak{b}$ are introduced with the understanding that they cannot be crossed when carrying out the path integral in (3.42). Dissection $\mathfrak{b}$ forms a loop (or a cycle) on the (say) top Riemann sheet whereas dissection $\mathfrak{a}$ loops from one Riemann sheet to the other by crossing the branch cuts as shown in Figure 3.4. The portion of the path on the bottom sheet is indicated by the dashed line. The cycles associated with the dissections, also identified as $\mathfrak{a}$ and $\mathfrak{b}$, are

[6]The inclusion of $u(\alpha)$ in the limits of the integral indicates that it is carried on the (usually dissected) Riemann surface.

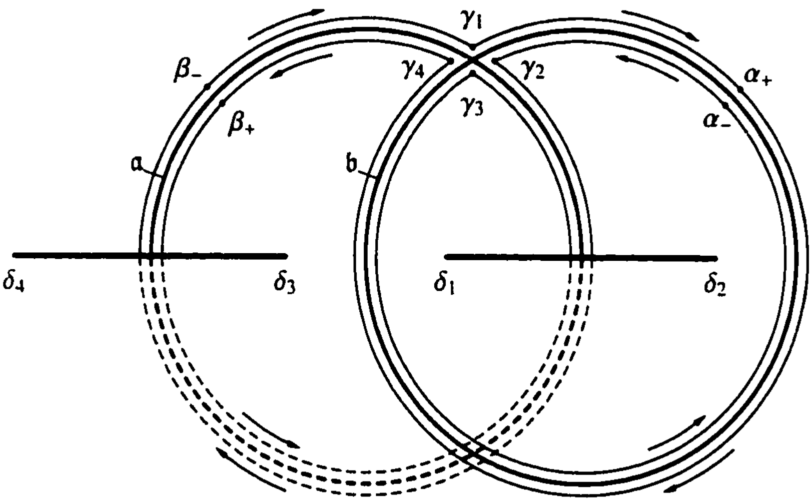


Figure 3.4: A generic Riemann surface of genus one with branch points δ_k and associated cuts. The dissections $\mathfrak{a}$ and $\mathfrak{b}$ have been introduced to make the surface simply connected. Dissection $\mathfrak{b}$ is confined to the upper sheet while dissection $\mathfrak{a}$ loops from the upper sheet (solid line) to the lower sheet (dashed line).

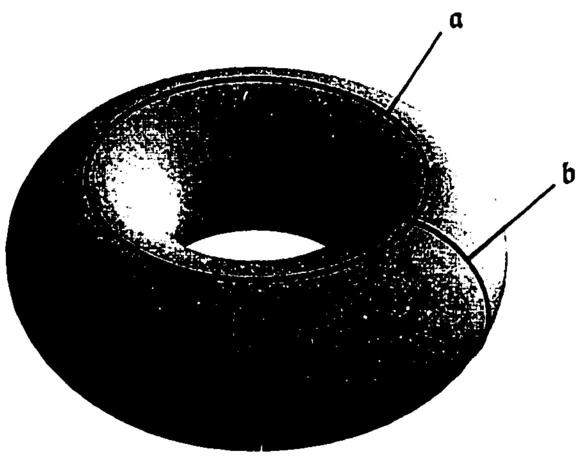


Figure 3.5: The torus, handlebody of genus one, which is topologically equivalent to the Riemann surface shown in Figure 3.4. It has been made simply connected by the dissections $\mathfrak{a}$ and $\mathfrak{b}$.

termed the *canonical cycles*. Provided the dissections are not crossed, the resulting surface is now simply connected. In other words, all paths between a pair of given points on the surface are homotopic (or equivalent.) This is perhaps better appreciated on the torus in Figure 3.5 where the dissections are indicated. If one proceeds to cut the torus along the dissections, the resulting surface can be flattened into a manifestly simply connected rectangular sheet. This rectangle is known as the Poincaré fundamental polygon — contemporary treatments of this material such as the one in SPRINGER [1981] are often carried out on the fundamental polygon of the Riemann surface of interest. It also follows that Riemann surfaces associated with higher genera require the introduction of larger numbers of dissections [APPELL, 1976; SPRINGER, 1981]. The process of dissecting the surface, besides making $F(\alpha, u)$ well-defined, also implies that the Cauchy-Goursat theorem as well as the residue theorem are now applicable on the surface. This is an important consideration since the residue theorem is required to derive the bilinear relations of Riemann.

3.3.1 The cyclic periods

The branch point contributions to the path integral, the cyclic periods, are now defined. We proceed by evaluating the elliptic integral $F(z, u)$ at the two points β_+ and β_- which are adjacent but on opposite sides of the $\mathfrak{a}$ cycle as shown in Figure 3.4. It follows that

$$F(\beta_+, u) = F(\gamma_1, u) + \int_{(\gamma_1,u)}^{(\beta_+,u)} f(\xi, u)\mathrm{d}\xi, \tag{3.43}$$

$$F(\beta_-, u) = F(\gamma_4, u) + \int_{(\gamma_4,u)}^{(\beta_+,u)} f(\xi, u)\mathrm{d}\xi, \tag{3.44}$$

and $F(\beta_+, u) - F(\beta_-, u) = F(\gamma_1, u) - F(\gamma_4, u)$ since the infinitesimally distant path integrals are equal. This holds for all the β_+, β_- pairs along the cut $\mathfrak{a}$ and implies that the discontinuity across it is constant and given by

$$\mathfrak{B} = F(\beta_+, u) - F(\beta_-, u) = F(\gamma_1, u) - F(\gamma_4, u) = \int_{(\gamma_1,u)}^{(\gamma_4,u)} f(\xi, u)\mathrm{d}\xi = \int_{\mathfrak{b}} f(\xi, u)\mathrm{d}\xi. \tag{3.45}$$

Similarly, by considering the adjacent points α_+ and α_- on each side of the dissection $\mathfrak{b}$, one obtains

$$F(\alpha_+, u) = F(\gamma_1, u) + \int_{(\gamma_1,u)}^{(\alpha_+,u)} f(\xi, u)\mathrm{d}\xi, \tag{3.46}$$

$$F(\alpha_-, u) = F(\gamma_2, u) + \int_{(\gamma_2,u)}^{(\alpha_+,u)} f(\xi, u)\mathrm{d}\xi, \tag{3.47}$$

so that $F(\alpha_+, u) - F(\alpha_-, u) = F(\gamma_1, u) - F(\gamma_2, u)$ and

$$\mathfrak{A} = F(\alpha_+, u) - F(\alpha_-, u) = F(\gamma_1, u) - F(\gamma_2, u) = \int_{(\gamma_2,u)}^{(\gamma_1,u)} f(\xi, u)\mathrm{d}\xi = \int_{\mathfrak{a}} f(\xi, u)\mathrm{d}\xi. \quad (3.48)$$

The quantities $\mathfrak{A}$ and $\mathfrak{B}$ are defined as the *cyclic periods* or the *periodicity modules* of $f(z, u)$. It is worth noting that if the dissections $\mathfrak{a}$ and $\mathfrak{b}$ are removed from the surface, the path integral (3.42) becomes multi-valued and given by

$$F(z, u) = \int_{z_0}^{z} f(\xi, u)\mathrm{d}\xi + \mathbf{Z}_1\mathfrak{A} + \mathbf{Z}_2\mathfrak{B}, \quad (3.49)$$

where the integers $\mathbf{Z}_{1,2}$ indicate the capture of an arbitrary number of cyclic periods. It can be appreciated that functions defined on Riemann surfaces with higher genus have a larger number of cyclic periods, see for example the analysis carried out on a surface of genus three in Chapter 5. Despite the added complications, the concepts introduced herein are sufficient.

3.3.2 The residue theorem on a Riemann surface

An extension of the Cauchy-Goursat theorem, the residue theorem is one of the fundamental results of complex analysis and it simply states that, over a simply connected region, the integral along a closed path equals $2\pi\mathrm{j}$ times the sum of the residues of the integrand that are captured [Churchill and Brown, 1990; Ahlfors, 1979]. For the Riemann surface of genus one considered in the previous section, it was seen that a pair of dissections was required to make it simply connected. Any contour integral enclosing the dissection pair must therefore take into account the contribution from the path, denoted $\mathfrak{a}\cup\mathfrak{b}$, enclosing the dissections. Applying the residue theorem to the path $\mathfrak{a}\cup\mathfrak{b}$ that encloses the two dissections on Figure 3.4 therefore yields

$$\int_{\mathfrak{a}\cup\mathfrak{b}} f(\alpha, u)\mathrm{d}\alpha = 2\pi\mathrm{j}\sum \mathrm{Res} \quad (3.50)$$

where the residues are those of $f(\alpha, u)$ on the *entire* Riemann surface. With this information in hand, now define the elliptic integral of the first kind

$$F_1(z, u) = \int_{(z_0,u_0)}^{(z,u)} \frac{\mathrm{d}\xi}{u(\xi)}, \quad (3.51)$$

which is finite and free of poles on the Riemann surface. An example of an elliptic integral of the third kind is

$$F_3(z,u) = \int_{(z_0,u_0)}^{(z,u)} \frac{1}{2u(\xi)} \left(\frac{u(\xi)+u(a)}{\xi - a} - \frac{u(\xi)+u(b)}{\xi - b} \right) \mathrm{d}\xi \tag{3.52}$$

which has a logarithmic residues $+1$ at $z = (a, u(a))$ and -1 at $z = (b, u(b))$. Note that the poles at $(a, u(a))$ and $(b, u(b))$ are arbitrarily located on the Riemann surface. Looking ahead to the integrals of the third kind encountered throughout this work, it is illustrative to consider the case where we have a pole of residue $+1$ at $z = (a, u(a))$ and a pole of residue -1 at $z = (a, -u(a))$ on the other Riemann sheet. This produces a co-located pole-zero pair where a change of Riemann sheet leads to a change in sign of the residue. The above expression then becomes

$$F_3(z,u) = \int_{(z_0,u_0)}^{(z,u)} \frac{u(a)}{u(\xi)} \frac{1}{\xi - a} \mathrm{d}\xi. \tag{3.53}$$

We denote the cyclic periods associated with the integral of the first kind $F_1(z,u)$ by $\mathfrak{A}_1$ and $\mathfrak{B}_1$ and those with the integral of the third kind $F_3(z,u)$ by $\mathfrak{A}_3$ and $\mathfrak{B}_3$. Note that while $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are constant, the cyclic periods of the integral of the third kind $\mathfrak{A}_3$ and $\mathfrak{B}_3$ are functions of the location a of the pole of the integrand.

An enchanting relationship between the cyclic periods of $F_1(z,u)$ and $F_3(z,u)$ can be obtained by applying the residue theorem on the Riemann surface in Figure 3.4 in the following fashion

$$\int_{\mathfrak{a}\cup\mathfrak{b}} F_1(z,u)\mathrm{d}F_3(z,u) = \int_{\mathfrak{a}\cup\mathfrak{b}} F_1(z,u) \frac{u(a)}{u(z)} \frac{1}{z-a} \mathrm{d}z = 2\pi \mathrm{j} \sum \mathrm{Res}. \tag{3.54}$$

We first show by explicitly evaluating the path integral that, remarkably enough,

$$\int_{\mathfrak{a}\cup\mathfrak{b}} F_1(z,u)\mathrm{d}F_3(z,u) = \mathfrak{A}_1\mathfrak{B}_3 - \mathfrak{B}_1\mathfrak{A}_3. \tag{3.55}$$

Consider first the dissection $\mathfrak{b}$ in Figure 3.4. The contribution $I_{\mathfrak{b}}$ from the path integral enclosing it is given by the sum of the integrals from γ_1 to γ_4 (outer path or + side) and from γ_3 to γ_2 (inner path or – side), *viz.*

$$I_{\mathfrak{b}} = \int_{(\gamma_1,u)}^{(\gamma_4,u)} F_1(\alpha_+)\mathrm{d}F_3(\alpha_+) + \int_{(\gamma_3,u)}^{(\gamma_2,u)} F_1(\alpha_-)\mathrm{d}F_3(\alpha_-)$$

$$= \int_{(\gamma_1,u)}^{(\gamma_4,u)} F_1(\alpha_+)\mathrm{d}F_3(\alpha_+) - \int_{(\gamma_2,u)}^{(\gamma_3,u)} F_1(\alpha_-)\mathrm{d}F_3(\alpha_-) \quad (3.56)$$

and, for the sake of clarity, α_+ is used to indicate the path on the positive (exterior) edge of $\mathfrak{b}$ and α_- the path on the negative (interior) edge of the dissection. The last member of the right-hand side of the above can be rewritten in terms of an integral on the outer path (α_+) since, from (3.48), $\mathfrak{A}_k = F_k(\alpha_+, u) - F_k(\alpha_-, u)$. Moreover, since $F_3(\alpha_+, u)$ and $F_3(\alpha_-, u)$ differ by a constant (see (3.45)) along the dissection, $\mathrm{d}F_3(\alpha_+, u) = \mathrm{d}F_3(\alpha_-, u)$. Making use of both of these properties

$$\int_{(\gamma_2,u)}^{(\gamma_3,u)} F_1(\alpha_-)\mathrm{d}F_3(\alpha_-) = \int_{(\gamma_1,u)}^{(\gamma_4,u)} \left\{F_1(\alpha_+, u) - \mathfrak{A}_1\right\}\mathrm{d}F_3(\alpha_+) \quad (3.57)$$

and the expression for $I_{\mathfrak{b}}$ is then

$$\begin{aligned} I_{\mathfrak{b}} &= \int_{(\gamma_1,u)}^{(\gamma_4,u)} F_1(\alpha_+)\mathrm{d}F_3(\alpha_+) - \int_{(\gamma_1,u)}^{(\gamma_4,u)} \left\{F_1(\alpha_+, u) - \mathfrak{A}_1\right\}\mathrm{d}F_3(\alpha_+) \\ &= \mathfrak{A}_1 \int_{(\gamma_1,u)}^{(\gamma_4,u)} \mathrm{d}F_3(\alpha_+) \\ &= \mathfrak{A}_1\left\{F_3(\gamma_4, u) - F_3(\gamma_1, u)\right\} \\ &= \mathfrak{A}_1\mathfrak{B}_3 \end{aligned} \quad (3.58)$$

where use has been made of (3.45). Similarly, the contribution from the portion of the path enclosing dissection $\mathfrak{a}$ is given by

$$\begin{aligned} I_{\mathfrak{a}} &= \int_{(\gamma_2,u)}^{(\gamma_1,u)} F_1(\beta_-)\mathrm{d}F_3(\beta_-) + \int_{(\gamma_4,u)}^{(\gamma_3,u)} F_1(\beta_+, u)\mathrm{d}F_3(\beta_+, u) \\ &= -\int_{(\gamma_4,u)}^{(\gamma_3,u)} \left\{F_1(\beta_+) - \mathfrak{B}_1\right\}\mathrm{d}F_3(\beta_-) + \int_{(\gamma_4,u)}^{(\gamma_3,u)} F_1(\beta_+, u)\mathrm{d}F_3(\beta_+, u) \\ &= \mathfrak{B}_1 \int_{(\gamma_4,u)}^{(\gamma_3,u)} \mathrm{d}F_3(\beta_+, u) \\ &= -\mathfrak{B}_1\mathfrak{A}_3. \end{aligned} \quad (3.59)$$

Finally, summing the contributions $I_{\mathfrak{a}}$ from (3.59) and $I_{\mathfrak{b}}$ from (3.58),

$$\int_{\mathfrak{a}\cup\mathfrak{b}} F_1(z,u)\mathrm{d}F_3(z,u) = \mathfrak{A}_1\mathfrak{B}_3 - \mathfrak{B}_1\mathfrak{A}_3. \tag{3.60}$$

Repeated use will be made of this fundamental result throughout the remainder of this work. By the residue theorem (3.50) this quantity must equal the sum of the residues of the integrand. In this instance there are two such contributions, namely,

$$\operatorname{Res}\left\{F_1(z,u)\frac{u(a)}{u(z)}\frac{1}{z-a}\right\}\bigg|_{z=(a,u(a))} = F_1(a,u(a)), \tag{3.61}$$

and

$$\operatorname{Res}\left\{F_1(z,u)\frac{u(a)}{u(z)}\frac{1}{z-a}\right\}\bigg|_{z=(a,-u(a))} = -F_1(a,-u(a)). \tag{3.62}$$

The application of the residue theorem (3.54) on the Riemann surface depicted in Figure 3.4 therefore yields

$$\mathfrak{A}_1\mathfrak{B}_3 - \mathfrak{B}_1\mathfrak{A}_3 = 4\pi\mathrm{j}\left\{F_1(a,u(a)) - F_1(a,-u(a))\right\} \tag{3.63}$$

and the difference of products of cyclic periods is expressed in terms of an elliptic integral of the first kind. This is often derived by considering differentials on Riemann surfaces and is thus known as a bilinear relation for differentials of the first and the third kind. Such relationships can be derived for elliptic and hyper-elliptic integrals and, together with a related set involving integrals of the second kind, are commonly referred to as *Riemann's bilinear relations* [Springer, 1981]. Note that if two integrals of the first kind were used in (3.54), the sum of residues would be zero. The salient feature of equation (3.63) is that whereas the dependence on the location of the pole a is implied on the left-hand side, it shows up explicitly on the right-hand side and its extraction is then, in principle at least, a relatively simple matter. It is this essential feature that makes possible an analytical treatment of the problem in the context of the current approach. Indeed, it will be seen that the process of elimination of the cyclic periods leads to equalities between known quantities and expressions equivalent to the left-hand side of (3.63), the only unknown being the location a of the pole. Invoking (3.63), an equality can then be obtained between known quantities and expressions equivalent to the right-hand side of (3.63). The inversion for the unknown pole is then possible using the Jacobian elliptic sine function (see Section 3.5). We recall that since periodic expressions — elements of $v_{4\Phi}(\alpha,u)$ — are required

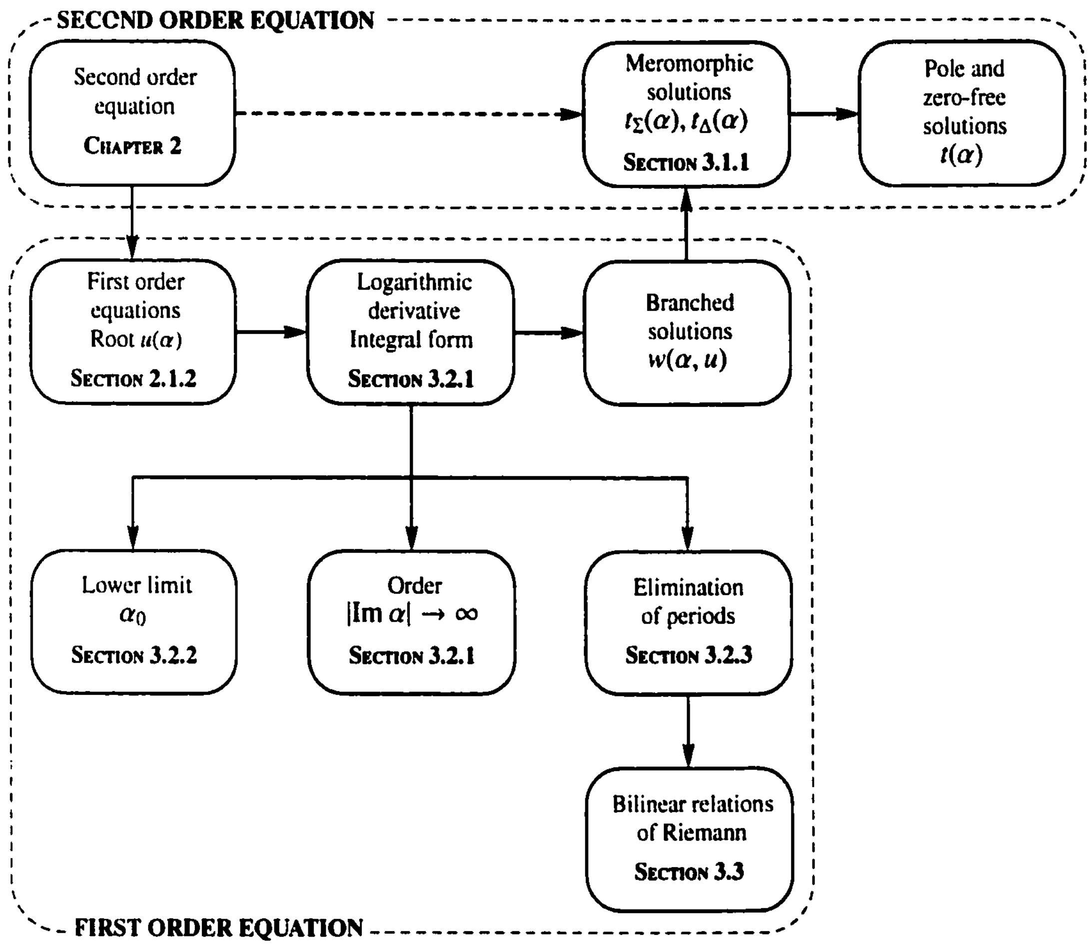


Figure 3.6: Schematic overview of the proposed technique.

when eliminating the cyclic periods, the derivation of the appropriate bilinear relation in the forthcoming chapters is carried out with some modifications. Likewise, the inversion for the unknown pole will require a transformation to recast trigonometric integrals of the first kind to the standard form, a topic addressed in the last section of this chapter.

3.4 Overview of proposed technique

It is useful at this juncture to briefly pause to paint a coherent picture of the technique proposed by summarizing the material covered so far. This short overview, a schematic representation of which is provided in Figure 3.6, follows more closely the order in which the various steps of the analysis must be carried out. The starting point of the analysis is a second order difference equation with period 4Φ to which a pair of meromorphic solutions are sought that are free of poles and zeros in a strip of width 4Φ. The solutions, which are related to unknown spectral functions, must also satisfy a certain order requirement dictated by the edge condition as $|\text{Im } \alpha| \to \infty$. This is geometry specific and, in this work, solutions that are $O(1)$ as $|\text{Im } \alpha| \to \infty$ are sought. We proceed to derive, as shown in Section 2.1.2, a pair of associated first order difference equations which are more easily solved. Their solution is however a subtle affair since the obtained expressions are generally branched and the standard solution technique put forward by Maliuzhinets does not apply. A general way to proceed is to take the logarithmic derivative of the first order equations, as discussed in Section 3.2.1, to obtain a tentative solution in path integral form. The latter is unfortunately ill-defined due to the singularities, both poles and branch points, of the integrand that arise in the strip $S_{4\Phi}$. Appropriate 4Φ periodic functions — they must satisfy certain growth conditions as discussed in Section 3.2.1 — must be introduced to negate the effect of poles (polar periods) and branch points (cyclic periods) as discussed in Section 3.2.3. While the elimination of the polar periods is algebraic in nature, the elimination of the cyclic periods is much more challenging and requires the use of bilinear relations of Riemann, an example of which is derived in Section 3.3. We also note that by combining the requirement for continuous meromorphic solutions and the elimination of the cyclic periods, it was shown in Section 3.2.2 that the lower limit of integration of the path integral can be any branch point in $S_{4\Phi}$. Once the effect of the singularities has been mitigated, the solution $w(\alpha, u)$ is well-defined and, using the meromorphic constructs of Section 3.1.1, meromorphic solutions of the second order equation are obtained. The final step is the elimination of unwanted poles and zeros in the strip of analyticity. This is surprisingly challenging and not discussed in this chapter. Instead, it is dealt with on an individual basis in the chapters that follow. In Chapter 4, the technique is applied to solve the generalized equation (2.73) of period 2π related to the composite wedge. Chapter 5 provides another application of technique where the same equation with the period doubled to 4π is solved.

3.5 The elliptic integral of the first kind and its inverse

We close this chapter with a short intermezzo giving a quick overview of relevant features of the elliptic integral of the first kind and its inverse, the Jacobian elliptic sine function. Although perhaps considered archaic by some today, elliptic function theory is an important part of classical mathematics and was the focus of much attention at the turn of the last century. Consequently, there are a vast number of treatises in the literature. These range from classic texts by Hancock [1910] and Neville [1951] to more contemporary treatments such as those of Lawden [1989] and McKean and Moll [1999]. Common references like Abramowitz and Stegun [1964] and Whittaker and Watson [1952] also provide good coverage of the salient points. The material covered here does not give justice to the rich and fascinating theory of elliptic integrals and functions, we only provide a small glimpse of items relevant to this study and introduce notation to simplify forthcoming analysis.

We first examine one of the standard representations. In Legendre's standard form, the elliptic integral of the first kind is expressed as

$$F(z,k) = \int_0^z \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \int_0^z \frac{\mathrm{d}\tau}{r(\tau,k)}, \tag{3.64}$$

where k is termed the *modulus* of the integral. It is, in essence, a generalization of a circular inverse function and recovers an inverse sine function when $k = 0$. The root $r(z,k)$ in the denominator has branch points at $z = \pm 1$ and $z = \pm 1/k$ whose cuts are by convention chosen as shown in Figure 3.7. The integral (3.64) is confined to one sheet of the Riemann surface and specific integration paths are specified in the complex plane so as to make it single-valued. Indeed, the square root $r(z,k) = \sqrt{(1-z^2)(1-k^2z^2)}$ exists on a Riemann surface of genus one and its integral is thus multi-valued when taken on the entire Riemann surface. The cyclic periods that arise due to the branch points are related to the so-called *complete integrals* of the first kind which are defined as

$$K = \int_{0^+}^{1^+} \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}}, \tag{3.65}$$

$$\mathrm{j}K' = \int_1^{1/k} \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}}, \tag{3.66}$$

and are as indicated in Figure 3.7. The superscripts in the limits indicate the positive side of the branch cut. The corresponding cyclic periods are therefore $4K$, which loops around the cuts stretching from -1 to 1, and $2\mathrm{j}K'$, which loops between the branch points 1 and $1/k$, going from one sheet to the other. The multi-valued integral of $1/r(z,k)$ on the Riemann surface may be written as

$$\int_{(0,r(0))}^{(z,r(z))} \frac{\mathrm{d}\tau}{r(\tau,k)} = \pm \int_0^z \frac{\mathrm{d}\tau}{r(\tau,k)} + 4K\mathbf{Z}_1 + 2\mathrm{j}K'\mathbf{Z}_2, \tag{3.67}$$

where the integers $\mathbf{Z}_{1,2}$ indicate the capture of an arbitrary number of cyclic periods. The integral $F(z,k)$ is well defined and is finite for all z. If $k \in \mathbb{R}$, then if z is in the upper-right quadrant it follows (see Figure 3.7) that $F(z,k) \in [-K,0] \times [0,\mathrm{j}K']$ and so forth. It must be stressed that when $k \in \mathbb{C}$, a range of $F(z,k) \in [-K,0] \times [0,\mathrm{j}K']$ does not correspond to a quadrilateral domain anymore due to the change in the topology. Conversely, the quadri-

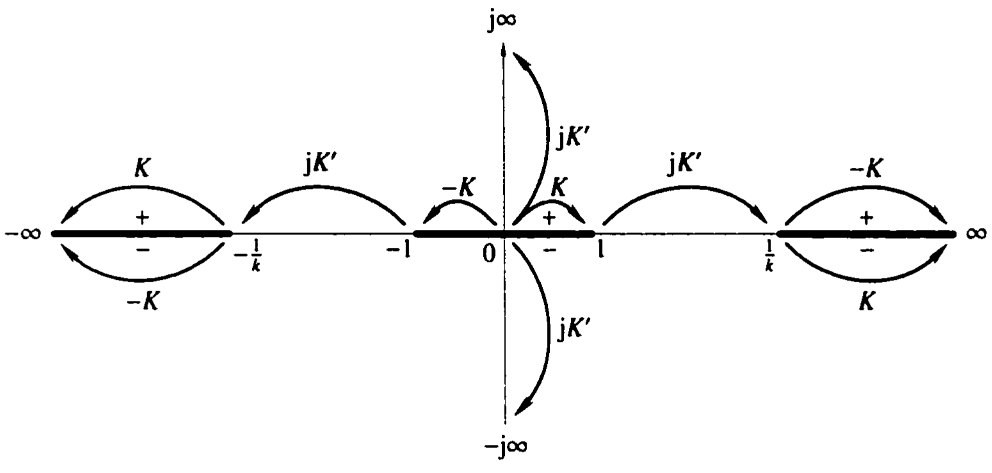


Figure 3.7: The branch points and cuts (thicker lines) associated with the elliptic integral of the first kind. The corresponding periods are as indicated; the + and – signs indicate relative changes in the sign of the root.

lateral domains will not correspond to parallelograms but to deformed curvilinear ranges and this will be explicitly shown below when we consider mappings of the integral of the first kind. This makes it generally not practical to refer to the domain of $F(z,k)$ in terms of quadrants (say, $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$) and we choose instead to refer to the range of $F(z,k)$. The fundamental range of $F(z,k)$ on the top Riemann sheet, regardless of the value of k, is $[-2K, 2K] \times [0, \mathrm{j}K']$ and this is termed the fundamental period parallelogram. If the negative branch of the square root is assumed, then $F(z,k)$ spans $[-2K, 2K] \times [0, -\mathrm{j}K']$.

The inverse of $F(z,k)$ is provided by the Jacobian elliptic sine function sn which, provided $F(z,k)$ lies in the fundamental period parallelogram, satisfies

$$\operatorname{sn}[F(z,k), k] = z$$

and it follows from the domain of $F(z,k)$ that $\operatorname{sn}(z,k)$ will, in any given period, have both a pole and a zero. The elliptic sine is also doubly periodic with periods $4K$ and $2\mathrm{j}K'$ in accordance with the multi-valuedness of $F(z,k)$ when it is defined on the entire Riemann surface. Hence,

$$\operatorname{sn}(z + 4K\mathbf{Z}_1 + 2\mathrm{j}K'\mathbf{Z}_2, k) = \operatorname{sn}(z,k) \tag{3.68}$$

and, following tradition, K and $\mathrm{j}K'$ are also called the quarter-periods of the elliptic sine function.[7] For computation purposes, the elliptic sine function can be evaluated by means

[7] Despite the fact that $\mathrm{j}K'$ is actually a half-period.

of the following Fourier series expansion representation [HANCOCK, 1910; LAWDEN, 1989]

$$\operatorname{sn} z = \frac{\pi}{2kK} \sum_{n=-\infty}^{\infty} \operatorname{cosec} \frac{\pi}{2K} \{z - (2n+1)\mathrm{j}K'\} \tag{3.69}$$

and this is the approach used throughout most of this work. It is appropriate to now examine two transformations which are relevant to the following chapters.

3.5.1 The $\cos\alpha$ transformation

The elliptic integral of the first kind with 2π periodic integrand

$$V_{2\pi}^{1}(\alpha) = \int_{\delta}^{\alpha} \frac{\mathrm{d}\xi}{\sqrt{\cos^2\xi - \cos^2\delta}}, \tag{3.70}$$

where the denominator is the square root (2.75) that arises in the problem of the composite wedge, is required for the elimination of the cyclic periods in both Chapters 4 and 5. It is emphasized that the path integral is taken on the top Riemann sheet dissected by branch cuts and not on the equivalent Riemann surface. Its inversion will be required and, to recast it in terms of Legendre's standard from, the transformation

$$\tau = \frac{\cos\alpha}{\cos\delta} \tag{3.71}$$

is used. Its application to $V_{2\pi}^{1}(\alpha, u)$ yields

$$V_{2\pi}^{1}(\alpha) = \int_{\delta}^{\alpha} \frac{\mathrm{d}\xi}{\sqrt{\cos^2\xi - \cos^2\delta}} = \mathrm{j}\left\{F\left(\frac{\cos\alpha}{\cos\delta}, \cos\delta\right) - K\right\} \tag{3.72}$$

where

$$k = \cos\delta \tag{3.73}$$

and the cyclic periods of $F(x, k)$, $4K$ and $2\mathrm{j}K'$, as verified by direct computation, are related to those of $V_{2\pi}^{1}(\alpha, u)$, $A_{2\pi}^{1}$ and $B_{2\pi}^{1}$, by

$$A_{2\pi}^{1} = 4K', \qquad B_{2\pi}^{1} = 4\mathrm{j}K. \tag{3.74}$$

Function $V_{2\pi}^{1}(\alpha, u)$ can be thought of as providing a mapping from the α plane to the ν plane as indicated in Figure 3.8 (a real valued δ is assumed) where the path integrals along various segments are given. Each of the quadrants $\mathcal{A}_n$ in the α plane are mapped to a distinct period

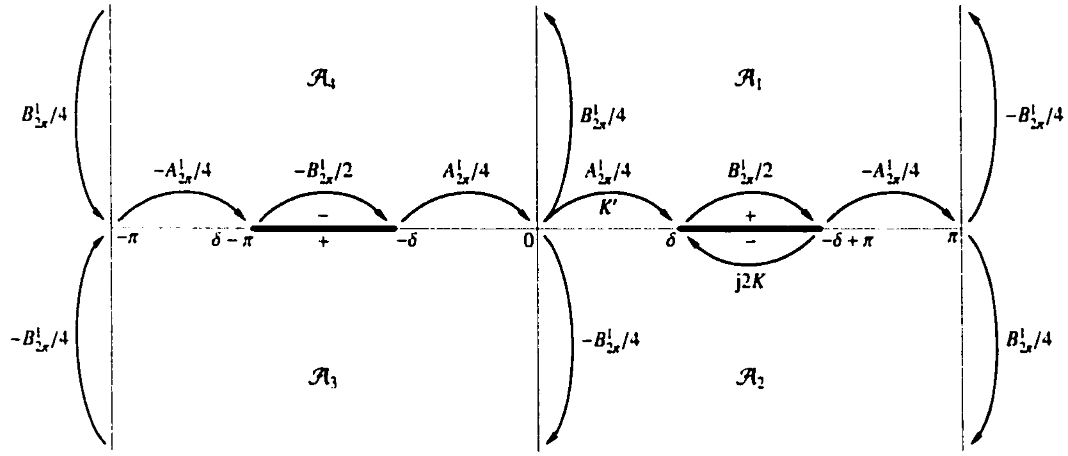


Figure 3.8: The periods of $V^1_{2\pi}(\alpha, u)$ on the top Riemann sheet. It is assumed that the branch points are real and the corresponding cuts are indicated by the thick solid lines.

parallelogram $\mathcal{V}_n$ in the v plane so that

$$\alpha \in \mathcal{A}_n \overset{V^1_{2\pi}}{\longmapsto} v \in \mathcal{V}_n \tag{3.75}$$

and the period parallelograms $\mathcal{V}_n$ are as shown in Figure 3.9. For instance, it is easily verified that for real δ (see Figure 3.8)

$$\alpha \in \mathcal{A}_1 \overset{V^1_{2\pi}}{\longmapsto} \mathcal{V}_1 = \left[0, \frac{B^1_{2\pi}}{2}\right] \times \left[0, -\frac{A^1_{2\pi}}{4}\right] = [0, \mathrm{j}2K] \times [0, -K'] \tag{3.76}$$

and so on for the other domains $\mathcal{A}_n$. These definitions will simplify some of the ensuing analysis. It was mentioned above that it is beneficial, and simpler, to describe the range of $V^1_{2\pi}(\alpha, u)$ as opposed to its domain. This is better understood by considering the inverse mapping of $V^1_{2\pi}(\alpha, u)$ obtained from its representation in terms of $F(z, k)$ (3.72) together with the elliptic sine function given in (3.5). Indeed, they imply

$$\alpha = \arccos\left[k \,\mathrm{sn}\left(K - \mathrm{j}V^1_{2\pi}(\alpha, u)\right)\right] = \left(V^1_{2\pi}\right)^{-1}(v) \tag{3.77}$$

and its application to the period parallelograms[8] $\mathcal{V}_{1,2}$ illustrated in Figure 3.9 produces the original domains $\mathcal{A}_{1,2}$ in the α plane illustrated in Figure 3.10. Note that when δ is real, which is the case for Figure 3.10(a), the recovered domains are the same as the ones illustrated in Figure 3.8. However, when δ is complex, as is the case in Figure 3.10(b), the $\mathcal{A}_{1,2}$ domains are not quadrilaterals anymore. Alternatively, one could choose to operate

[8] It is sufficient to restrict our interest to $\alpha \in \mathcal{V}_{1,2}$.

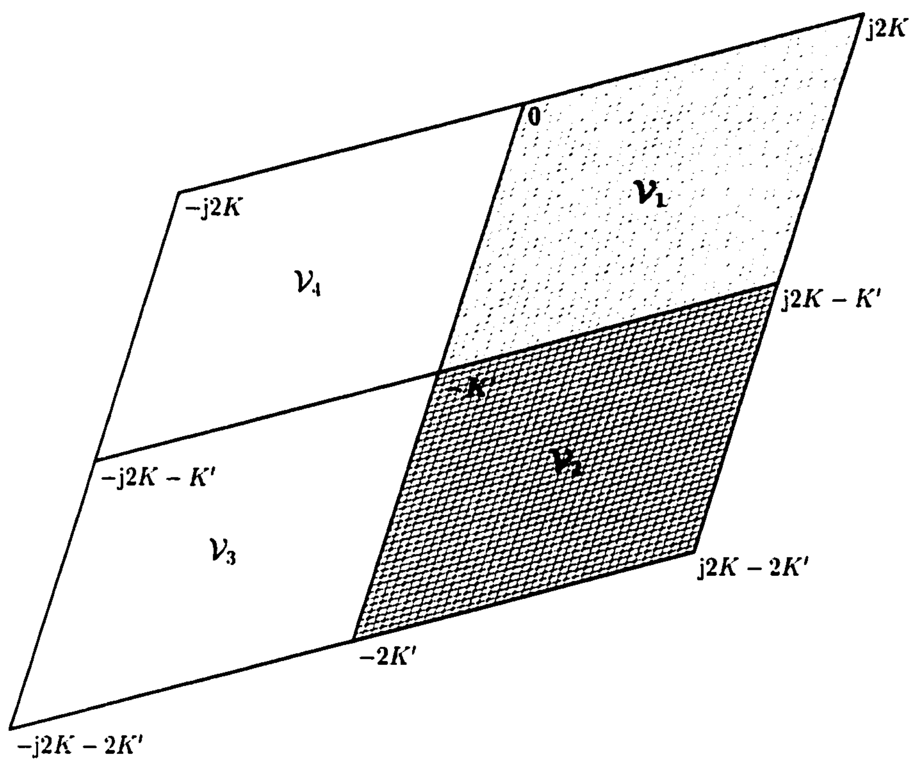
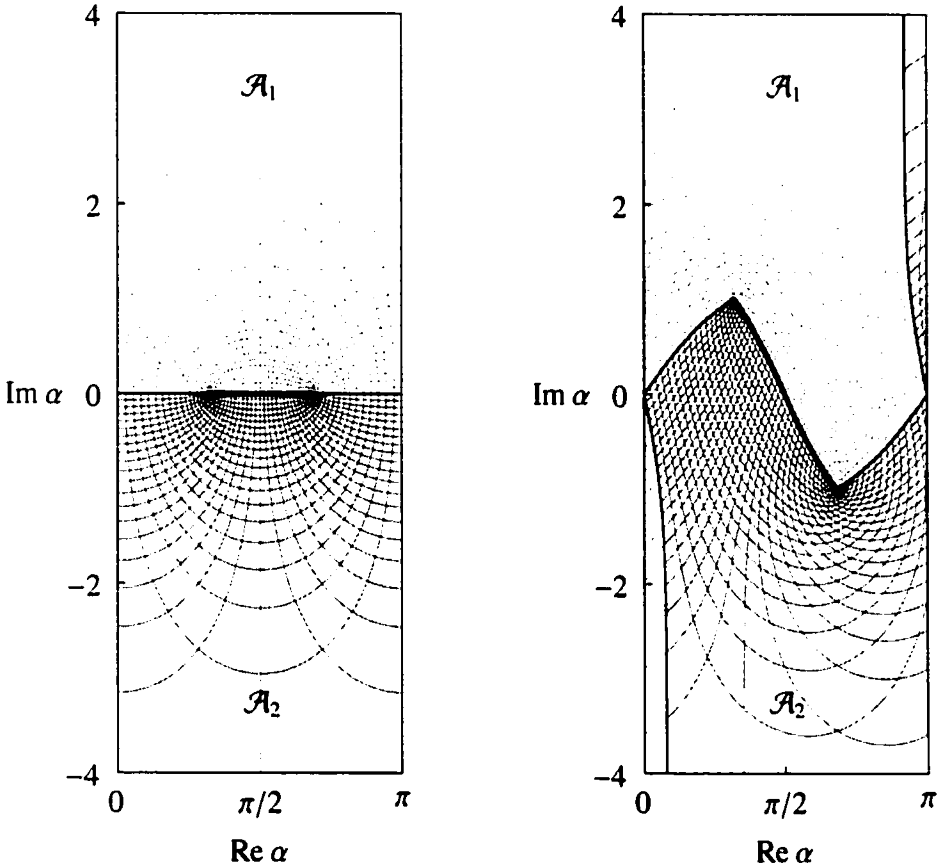


Figure 3.9: The ranges $\mathcal{P}_n$ of $V^1_{2\pi}(\alpha, u)$ corresponding to each of the domain $\mathcal{A}_n$ in Figure 3.8.



(a) $\delta = 1$ (b) $\delta = 1 + \mathrm{j}$

Figure 3.10: The inverse mapping $\left(V_{2\pi}^{1}\right)^{-1}$ of the $\mathcal{P}_1$ (light gray) and $\mathcal{P}_2$ (dark grey) period parallelograms shown in Figure 3.9 back to the domains $\mathcal{A}_1$ and $\mathcal{A}_2$ in the α plane. The cases provided are (a) $\delta = 1$ implying $k = 1$, $K = 1.709$ and $K' = 2.087$; and (b) $\delta = 1 + \mathrm{j}$ implying $k = 0.834 - \mathrm{j}0.989$, $K = 1.272 - \mathrm{j}0.323$ and $K' = 1.290 + \mathrm{j}0.646$. Case (a) reproduces the expected $\mathcal{A}_1$ (light gray) and $\mathcal{A}_2$ (dark gray) quadrilaterals shown in Figure 3.8; case (b) illustrates how the corresponding original domains are deformed when δ is complex.

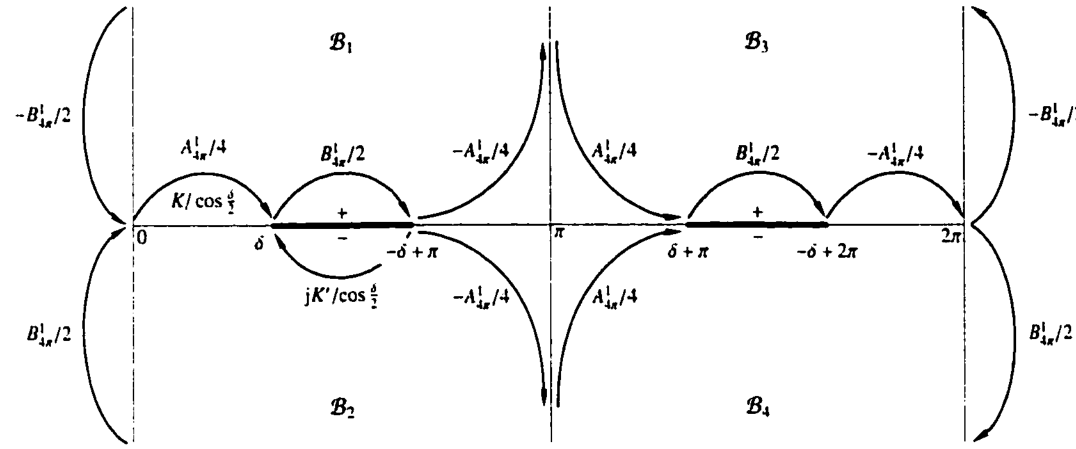


Figure 3.11: The periods of $V^1_{4\pi}(\alpha)$ on the top Riemann sheet. It is assumed that the branch points are real and the corresponding cuts are indicated by the thick solid lines. Contributions along noteworthy paths are indicated in terms of the cyclic periods $A^1_{4\pi}$ and $B^1_{4\pi}$. The region is partitioned in the four quadrants $\mathcal{B}_n$.

with the quadrilateral domains from Figure 3.8 even when δ is complex but this would result in curvilinear period parallelograms $\mathcal{V}_n$.

3.5.2 The $\sin\alpha/2$ transformation

The analysis in Chapter 5 requires two integrals of the first kind, one of which is the 2π periodic $V^1_{2\pi}(\alpha)$ which was just examined. The second one is

$$V^1_{4\pi}(\alpha) = \int_\delta^\alpha \frac{\cos\frac{\xi}{2}}{\sqrt{\cos^2\xi - \cos^2\delta}}\mathrm{d}\xi \tag{3.78}$$

which has a 4π periodic integrand and where the branched denominator remains $u(\alpha)$ given in (2.75). For real-valued δ the dissected α plane is as shown in Figure 3.11. The contributions to the integral along the various paths are indicated and we note the definition of the two periods $A^1_{4\pi}$ and $B^1_{4\pi}$ of the integral. We confine our interest to the strip $\operatorname{Re}\alpha \in [0, 2\pi]$, the values for which we seek to invert $V^1_{4\pi}(\alpha)$. Following our convention for $V^1_{2\pi}(\alpha)$ the cyclic period $B^1_{4\pi}$ loops clockwise around the cuts whereas $A^1_{4\pi}$ loops clockwise between the cuts. The four quadrants $\mathcal{B}_n$, $n \in \{1, 2, 3, 4\}$, are mapped to the v plane by $V^1_{4\pi}(\alpha)$ to four corresponding period parallelograms $\mathcal{W}_n$ such that

$$\alpha \in \mathcal{B}_n \xmapsto{V^1_{4\pi}} v \in \mathcal{W}_n \tag{3.79}$$

and this is illustrated in Figure 3.12. The inversion of $V^1_{4\pi}(\alpha)$ is required in Chapter 5 and

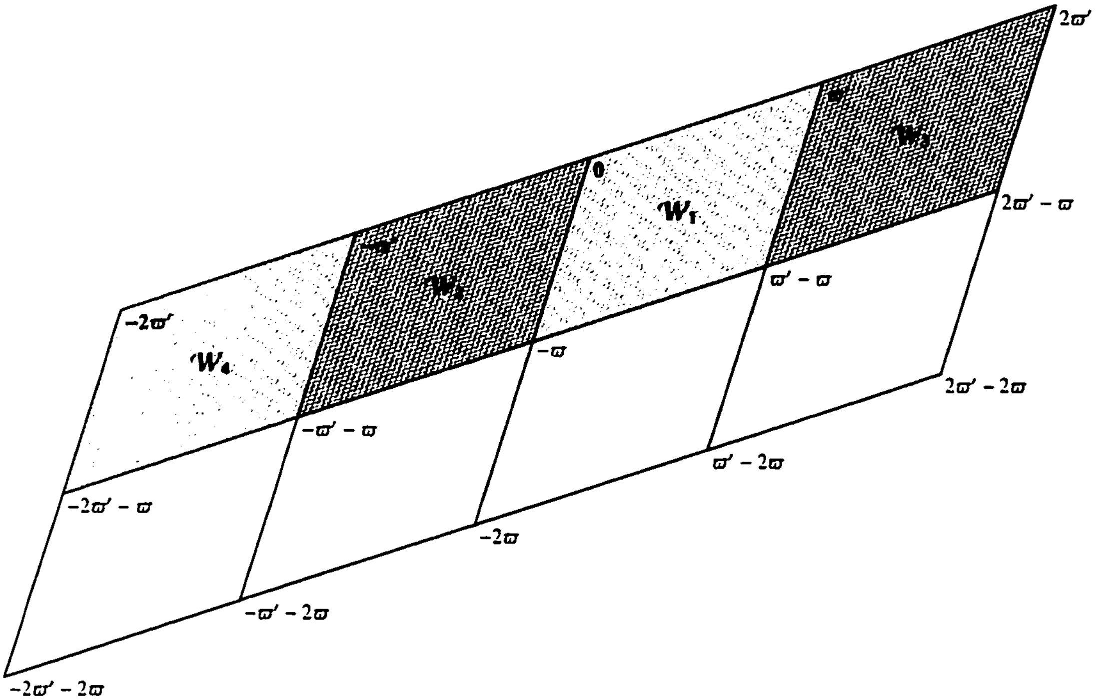


Figure 3.12: The ranges $\mathcal{W}_n$ of $V^1_{4\pi}(\alpha, u)$ corresponding to each of the $\mathcal{B}_n$ subdomains in Figure 3.11 with $\varpi = \sqrt{1+k^2}K$ and $\varpi' = \sqrt{1+k^2}\mathrm{j}K'$. The lower parallelograms correspond to the case where α has a negative real part.

we proceed by first recasting it in terms of Legendre's standard form (3.64) by means of the transformation

$$\tau = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\delta}{2}}. \tag{3.80}$$

Applying it to (3.78),

$$\begin{aligned} V^1_{4\pi}(\alpha) = \int_\delta^\alpha \frac{\cos\frac{\xi}{2}}{\sqrt{\cos^2\xi - \cos^2\delta}}\mathrm{d}\xi &= \sqrt{1+k^2}\left\{F\left(\frac{\sqrt{1+k^2}}{k}\sin\frac{\alpha}{2}, k\right) - K\right\} \\ &= \frac{1}{\cos\frac{\delta}{2}}\left\{F\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\delta}{2}}, \tan\frac{\delta}{2}\right) - \frac{A^1_{4\pi}}{4}\cos\frac{\delta}{2}\right\} \end{aligned} \tag{3.81}$$

where

$$k = \tan\frac{\delta}{2} \tag{3.82}$$

and the cyclic periods in the α plane are related to the complete integrals of the first kind by

$$A^1_{4\pi} = \frac{4K}{\cos\frac{\delta}{2}}, \qquad B^1_{4\pi} = \frac{2jK'}{\cos\frac{\delta}{2}}. \tag{3.83}$$

If $\alpha \in \mathcal{B}_1 \cup \mathcal{B}_2$, the inversion is obtained by solving for F above and applying the elliptic sine function described on page 53. This produces

$$\alpha = 2\arcsin\left\{\frac{k}{\sqrt{1+k^2}}\,\mathrm{sn}\left(K + \frac{1}{\sqrt{1+k^2}}V^1_{4\pi}(\alpha), k\right)\right\} = \left(V^1_{4\pi}\right)^{-1}(v) \tag{3.84}$$

where, corresponding to the range α, the domain of the inverse mapping is $v \in \mathcal{W}_1 \cup \mathcal{W}_2$. This restriction on the value of v arises due to the range of the arcsin function which is limited to Re $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. This is problematic if $v \in \mathcal{W}_3 \cup \mathcal{W}_4$ implying that $\alpha \in \mathcal{B}_3 \cup \mathcal{B}_4$ in which case the above inevitably fails — α then lies outside the range of arcsin. To remedy this, we start off by noting that if Re $\alpha \in [\pi, 2\pi]$ (see Figure 3.11) we can write

$$\begin{aligned} V^1_{4\pi}(\alpha) &= \frac{B^1_{4\pi}}{2} + \frac{B^1_{4\pi}}{2} + \int_{2\pi-\delta}^{\alpha} \frac{\cos\frac{\xi}{2}}{\sqrt{\cos^2\xi - \cos^2\delta}}\mathrm{d}\xi \\ &= B^1_{4\pi} + \int_{\delta}^{2\pi-\alpha} \frac{\cos\frac{\xi}{2}}{\sqrt{\cos^2\xi - \cos^2\delta}}\mathrm{d}\xi \end{aligned} \tag{3.85}$$

where $2\pi - \alpha \in \mathcal{B}_1 \cup \mathcal{B}_2$. Hence, if $v \in \mathcal{W}_3 \cup \mathcal{W}_4$ then

$$\alpha = 2\pi - 2\arcsin\left\{\frac{k}{\sqrt{1+k^2}}\,\mathrm{sn}\left(K - 2jK' + \frac{1}{\sqrt{1+k^2}}V^1_{4\pi}(\alpha), k\right)\right\} \tag{3.86}$$

which correctly produces $\alpha \in \mathcal{B}_3 \cup \mathcal{B}_4$. Equations (3.84) and (3.86) provide the means of inverting $V^1_{4\pi}(\alpha)$ given in (3.78) for $\alpha \in \mathcal{B}_n$. They are also useful to illustrate how the subdomains $\mathcal{B}_n$ are deformed when the branch point δ is complex. This is shown in Figure 3.13 which depicts the subdomains when (a) δ is real ($\delta = 1$) and when (b) δ is complex ($\delta = 1 + j$). The recovered subdomains are seen to correspond to the quadrants of Figure (3.11) when the branch point is real, and the resulting distortion of $\mathcal{B}_n$ for complex branch points is evident.

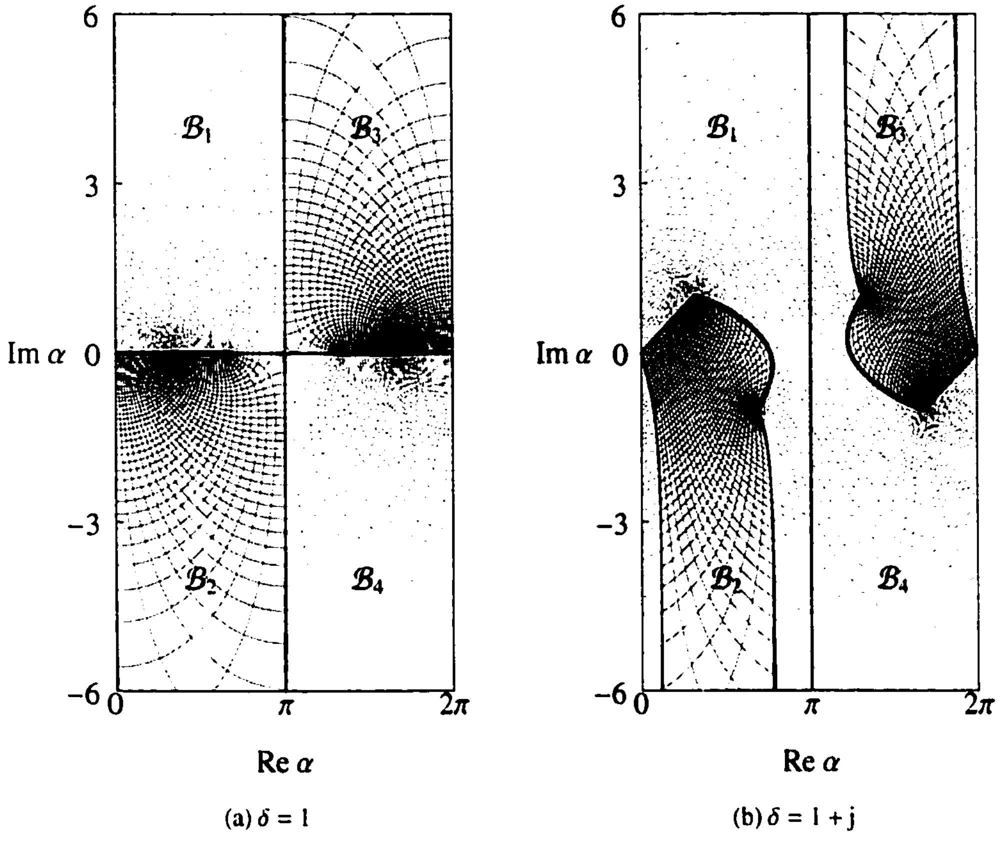


Figure 3.13: The inverse mapping $\left(V^1_{4\pi}\right)^{-1}$ of the $\mathcal{W}_n$ period parallelograms shown in Figure 3.12 back to the domains $\mathcal{B}_n$ in the α plane when (a) $\delta = 1$ and (b) $\delta = 1 + \mathrm{j}$. The thick solid line indicates the branch cut. The corresponding values for the cyclic periods are $K = 1.7130$ and $K' = 2.0777$ when $\delta = 1$; $K = 1.4833+\mathrm{j}0.1436$ and $K' = 1.7991-\mathrm{j}0.8240$ when $\delta = 1 + \mathrm{j}$. Case (a) reproduces the subdomains depicted in Figure 3.11.

CHAPTER 4

THE PENETRABLE WEDGE: EQUATION WITH PERIOD 2π

THE procedure developed in the previous chapter is now applied to the second order functional difference equation (2.73) which, as discussed in section 2.4, is a generalization of the equation obtained by DEMETRESCU *et al.* [1998] in their analysis of a penetrable composite right-angled wedge. Following the prescription given, a logarithmic derivative is first applied to the first order difference equations to obtain branched solutions expressed in terms of a path integral. They are made well-defined by adding appropriate unit periodics to eliminate the singularities of the integrand. Meromorphic solutions of the second order equation follow by taking linear combinations of the branched solutions. Analysis and computations demonstrate that they have the required analytical properties and recover known expressions in proper limits.

4.1 The difference equations

A pair of solutions is sought to equation (2.73),

$$t(\alpha + 3\pi) - 2\left\{1 - 2\frac{\cos^2\delta - \cos^2\theta}{\cos^2\alpha - \cos^2\theta}\right\} t(\alpha + \pi) + t(\alpha - \pi) = 0, \tag{4.1}$$

which is repeated here for convenience. It is easily specialized to the equation obtained by DEMETRESCU *et al.* [1998] using the substitutions given in Section 2.4. Within the context of the problem, the parameters θ and δ are both functions of the resistivity R_e of the wedge and they are such that $\operatorname{Re}\theta \in [0, \pi/2]$ and $\operatorname{Re}\delta \in (0, \pi/2]$. As remarked back on page 26, the parameters θ and δ are allowed to vary independently for the more general equation (4.1). The solutions must be meromorphic, free of poles and zeros in the strip $\mathcal{S}_{2\pi} = \{\alpha : |\operatorname{Re}\alpha| \le 2\pi\}$, and $O(1)$ as $|\operatorname{Im}\alpha| \to \infty$. We proceed to factor the second order difference

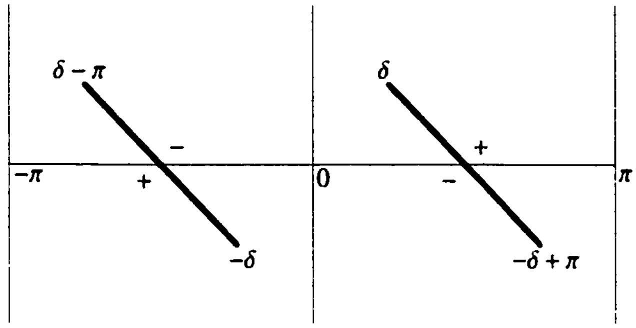


Figure 4.1: The strip of analyticity $S_{2\pi} = \{\alpha : |\text{Re}\,\alpha| \le \pi\}$ with the branch points and associated cuts for the square root $u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta}$ appearing in (4.4) and (4.6). The thick lines indicate the branch cuts of $u(\alpha)$, the positive and negative signs indicate relative changes in sign of $u(\alpha)$ across the different cuts.

operator as in Section 2.1.2 and obtain

$$\left\{\text{D}_\pi - g(\alpha, u)\right\}\left\{\text{D}_\pi - g(\alpha, -u)\right\}t(\alpha) = 0, \tag{4.2}$$

where

$$g(\alpha, \pm u) = \frac{\pm u(\alpha) - u(\theta)}{\pm u(\alpha) + u(\theta)} = \left(\frac{u(\alpha) - u(\theta)}{u(\alpha) + u(\theta)}\right)^{\pm 1}. \tag{4.3}$$

In agreement with its counterpart $r(\alpha)$ discussed on page 16 we note that $g(\alpha, u) = 1/g(\alpha, -u)$ — the roots obtained when factoring the operator are the reciprocals of one another. In this instance

$$u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta} \tag{4.4}$$

which has branch points at $\alpha = \pm\delta, \pm(\pi - \delta)$ in $S_{2\pi}$ and the associated cuts are chosen so that $u(\alpha)$ has the same symmetries as $\cos\alpha$, *viz.*

$$u(\alpha) = u(\alpha + 2\pi) = u(-\alpha) = -u(\alpha + \pi). \tag{4.5}$$

The strip of analyticity $S_{2\pi}$ together with the branch points and the corresponding cuts of $u(\alpha)$ are illustrated in Figure 2.7. From (4.2), the pair of first order equations corresponding to (4.1) are

$$\frac{w(\alpha+\pi,+u)}{w(\alpha-\pi,+u)} = g(\alpha,+u) = \frac{u(\alpha)-u(\theta)}{u(\alpha)+u(\theta)}, \tag{4.6a}$$

$$\frac{w(\alpha+\pi,-u)}{w(\alpha-\pi,-u)} = g(\alpha,-u) = \frac{u(\alpha)+u(\theta)}{u(\alpha)-u(\theta)}, \tag{4.6b}$$

and each of the two branches of $w(\alpha, u)$ corresponds to the solution of one of the first order difference equations; both solutions are embodied in $w(\alpha, u)$. It is sufficient to solve, say, for $w(\alpha, u)$ in (4.6a) as the other branch of its solution $w(\alpha, -u)$ then satisfies (4.6b). Consistent with the fact that $g(\alpha, u) = g(-\alpha, u)$, the function $w(\alpha, u)$ can be constructed such that $w(-\alpha, u) = w(\alpha, -u) = 1/w(\alpha, u)$ since

$$\frac{w(-\alpha+\pi,u)}{w(-\alpha-\pi,u)} = g(-\alpha,u) = g(\alpha,u) = \frac{w(\alpha+\pi,u)}{w(\alpha-\pi,u)}. \tag{4.7}$$

Once $w(\alpha, u)$ is obtained, the branch-free expressions (3.3) provide the fundamental building blocks from which meromorphic solutions to the second order difference equation can be assembled.

4.1.1 Limiting cases of interest

Before proceeding to obtain the general solution $w(\alpha, u)$ to the first order difference equation (4.6a), it is useful to consider the limiting cases — there are two — where the branch points vanish. This information will be useful when characterizing the behavior of the branched solution $w(\alpha, u)$. The first of these is the limit $\delta \to \pi/2$ when $u(\alpha) \to \cos\alpha$ and the second one is the limit $\delta \to 0$ when $u(\alpha) \to \mathrm{j}\sin\alpha$.[1] In both cases the second order difference equations (4.1) can be factored by inspection into a pair of uncoupled first order difference equations whose right-hand sides are meromorphic rational functions of trigonometric polynomials. In these instances, the solution of the second order difference equation is the same as the solution of the first order difference equation since both are meromorphic; there is no need to resort to branch-free linear combinations. These solutions are readily available and are expressed in terms of Maliuzhinets functions.

[1]As shown in Section 4.2.5.2, the case when $\delta \to 0$ is more complicated if one considers the limiting behavior of $u(\alpha)$ with its cuts as defined in Figure 4.1.

4.1.1.1 The limit $\delta \to \pi/2$

In this instance $u(\alpha) \to \cos\alpha$ and the first order equation (4.6a) becomes

$$\frac{w(\alpha+\pi,u)}{w(\alpha-\pi,u)} = \frac{\cos\alpha - \cos\theta}{\cos\alpha + \cos\theta} \tag{4.8}$$

and since (4.6b) goes simply to the reciprocal of (4.8), its solution in that limit is provided by $1/w(\alpha,u)$. From the identity

$$\cos a + \cos b = 2\cos\frac{1}{2}(a-b)\cos\frac{1}{2}(a+b) \tag{4.9a}$$

$$\cos a - \cos b = -2\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a+b) \tag{4.9b}$$

we have

$$\frac{\cos\alpha - \cos\theta}{\cos\alpha + \cos\theta} = \frac{\cos\frac{1}{2}\left(\alpha-\theta+\frac{\pi}{2}+\frac{\pi}{2}\right)}{\sin\frac{1}{2}\left(\alpha-\theta+\frac{\pi}{2}+\frac{\pi}{2}\right)}\,\frac{\sin\frac{1}{2}\left(\alpha+\theta-\frac{\pi}{2}+\frac{\pi}{2}\right)}{\cos\frac{1}{2}\left(\alpha+\theta-\frac{\pi}{2}+\frac{\pi}{2}\right)} \tag{4.10}$$

and it immediately follows, since the Maliuzhinets function $\psi_{2\Phi}(\alpha)$ satisfies [Maliuzhinets, 1958*a*]

$$\frac{\psi_{2\Phi}(\alpha+2\Phi)}{\psi_{2\Phi}(\alpha-2\Phi)} = \frac{\cos\frac{1}{2}(\alpha+\frac{\pi}{2})}{\sin\frac{1}{2}(\alpha+\frac{\pi}{2})}, \tag{4.11}$$

that

$$w(\alpha,u) = \Psi_1(\alpha) = \frac{\psi_{\pi/2}\left(\alpha+\frac{\pi}{2}-\theta\right)}{\psi_{\pi/2}\left(\alpha-\frac{\pi}{2}+\theta\right)} \tag{4.12}$$

which is meromorphic, free of poles and zeros in $S_{2\pi}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. It is useful to consider an integral representation. Since [Maliuzhinets, 1958*a*] (sign errors were corrected in [Senior and Volakis, 1995])

$$\psi_{\pi/2}(\alpha) = \exp\frac{1}{4\pi}\int_0^\alpha \frac{2\xi - \pi\sin\xi}{\cos\xi}\mathrm{d}\xi, \tag{4.13}$$

substitution into (4.12) provides, after manipulation,

$$\Psi_1(\alpha) = \exp\int_0^\alpha \frac{\frac{1}{\pi}(\xi\sin\xi\cos\theta - \theta\sin\theta\cos\xi) + \frac{1}{2}\sin\theta(\cos\xi - \cos\theta)}{\cos^2\xi - \cos^2\theta}\mathrm{d}\xi. \tag{4.14}$$

Solutions such as $\Psi_1(\alpha)$ that are expressed in terms of Maliuzhinets functions can be computed from approximations such as the one provided by HERMAN *et al.* [1983] — a listing of the algorithm is also given in [SENIOR AND VOLAKIS, 1995]. Alternatively, and this is the approach taken by the author, one can proceed by numerical integration of (say) (4.14). The above is particularly useful to verify the behavior of the constructed $w(\alpha, u)$ and it will be shown that our solutions will recover $\Psi_1(\alpha)$ and $1/\Psi_1(\alpha)$ identically as $\delta \to \pi/2$.

4.1.1.2 The limit $\delta \to 0$

In this instance $u(\alpha) \to \mathrm{j}\sin\alpha$ and the first order equation (4.6a) become

$$\frac{w(\alpha+\pi, u)}{w(\alpha-\pi, u)} = \frac{\sin\alpha - \sin\theta}{\sin\alpha + \sin\theta} \tag{4.15}$$

and once again the solution to (4.6b) is provide by $1/w(\alpha, u)$. From the identities

$$\sin a + \sin b = 2\cos\frac{1}{2}(a-b)\sin\frac{1}{2}(a+b) \tag{4.16a}$$

$$\sin a - \sin b = 2\sin\frac{1}{2}(a-b)\cos\frac{1}{2}(a+b) \tag{4.16b}$$

we have

$$\frac{\sin\alpha - \sin\theta}{\sin\alpha + \sin\theta} = -\frac{\sin\frac{1}{2}\left(\alpha+\theta-\frac{\pi}{2}+\frac{\pi}{2}\right)}{\cos\frac{1}{2}\left(\alpha+\theta-\frac{\pi}{2}+\frac{\pi}{2}\right)}\,\frac{\cos\frac{1}{2}\left(\alpha-\theta+\frac{\pi}{2}+\frac{\pi}{2}\right)}{\sin\frac{1}{2}\left(\alpha-\theta+\frac{\pi}{2}+\frac{\pi}{2}\right)} \tag{4.17}$$

and therefore, using (4.11),

$$w(\alpha, u) = \Psi_2(\alpha) = \cos\frac{\alpha}{2}\,\frac{\psi_{\pi/2}^2(\frac{\pi}{2}-\theta)}{\psi_{\pi/2}\left(\alpha+\frac{\pi}{2}-\theta\right)\psi_{\pi/2}\left(\alpha-\frac{\pi}{2}+\theta\right)} \tag{4.18}$$

which is meromorphic, free of poles and zeros in the *open* strip $\mathcal{S}_{2\pi}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. Since

$$\frac{\psi_{\pi/2}\left(\alpha-\pi+\frac{\pi}{2}-\theta\right)\psi_{\pi/2}\left(\alpha-\pi-\frac{\pi}{2}+\theta\right)}{\psi_{\pi/2}\left(\alpha+\pi+\frac{\pi}{2}-\theta\right)\psi_{\pi/2}\left(\alpha+\pi-\frac{\pi}{2}+\theta\right)} = -\frac{\sin\alpha-\sin\theta}{\sin\alpha+\sin\theta}, \tag{4.19}$$

the $\cos\alpha/2$ term provides the required change in sign and exactly cancels the growth of the denominator but inevitably leads to zeros (Ψ_2) or poles ($1/\Psi_2$) at $\alpha = \pm\pi$ on the edges of the strip. It appears that a solution free of poles and zeros in the closed strip and of order unity is unattainable. Using (4.13) and

$$\cos\frac{\alpha}{2} = \exp -\frac{1}{2}\int_0^{\alpha} \frac{1-\cos\xi}{\sin\xi}\mathrm{d}\xi \tag{4.20}$$

in (4.18) results in the integral representation

$$\Psi_2(\alpha) = \exp - \int_0^{\alpha}\left\{\frac{\frac{\xi}{\pi}\sin\theta\cos\xi - \left(\frac{\theta}{2}-\frac{1}{2}\right)\cos\theta\sin\xi - \frac{1}{4}\sin 2\xi}{\cos^2\xi - \cos^2\theta} + \frac{1}{2}\frac{1-\cos\xi}{\sin\xi}\right\}\mathrm{d}\xi. \tag{4.21}$$

4.2 General branched solution

We follow the procedure outlined in Chapter 3 and proceed by first taking the logarithmic derivative and then subsequently add 2π periodic functions to annul the polar and cyclic periods of the integrand. The requirement for vanishing cyclic periods leads to a system of two equations in two unknowns, the solution of which is the focus of subsequent sections.

4.2.1 The logarithmic derivative

Applying a logarithmic derivative to the first order difference equation (4.6a) results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha+\pi,u) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha-\pi,u) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u) \\ &= -\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\end{aligned} \tag{4.22}$$

and the right-hand side conforms to the general form given in (3.15) in the previous chapter's general treatment. Letting $v_0(\alpha,u) = \mathrm{d}/\mathrm{d}\alpha \ln w(\alpha,u)$ enables us to rewrite the above as

$$v_0(\alpha+\pi,u) - v_0(\alpha-\pi,u) = -\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta} \tag{4.23}$$

and, letting

$$v_0(\alpha,u) = -C\alpha\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}, \tag{4.24}$$

substitution in (4.23) yields

$$C(\alpha+\pi)\left(-\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\right)-C(\alpha-\pi)\left(-\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\right)$$
$$=2\pi C\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}. \quad (4.25)$$

Enforcing equality with the right hand-side of (4.23) implies $C=-1/2\pi$. Therefore,

$$v_0(\alpha)=\frac{u(\theta)}{u(\alpha)}\frac{\alpha}{\pi}\frac{\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}=-\frac{\alpha}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u). \quad (4.26)$$

Since $v_0(\alpha,u)$ corresponds to the logarithmic derivative of $w(\alpha,u)$, the latter can be written as

$$w(\alpha,u)=\exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi,u)+v_{2\pi}(\xi,u)\right\}\mathrm{d}\xi, \quad (4.27)$$

where $v_{2\pi}(\alpha,u)$ consists of, as yet, undefined 2π periodic function chosen to obtain a well-defined — single-valued with respect to the choice of the path of integration — path integral. The function $v_0(\alpha,u)$ is even, aperiodic and vanishes as $|\mathrm{Im}\,\alpha|\to\infty$. Within the strip $\mathcal{S}_{2\pi}$ it has branch points associated with $1/u(\alpha)$ and poles at the zeros of $\cos^2\alpha-\cos^2\theta$. It will be shown below that $v_{2\pi}(\alpha,u)$ consists of three functions: $v_1(\alpha,u)$ to eliminate the polar periods, and $v_{2\pi}^1(\alpha,u)$, $v_{2\pi}^3(\alpha,u)$ to eliminate the cyclic periods.

Since the logarithmic derivative of $g(\alpha,u)$ is identical to that of $Cg(\alpha,u)$ where C is some constant, we must resort to the technique proposed in Section 3.2.1 on page 33 and examine if $w(\alpha,u)$ satisfies the first order equation (4.6a) when $|\mathrm{Im}\,\alpha|\to\infty$. In order to do so, we first assume that $v_{2\pi}(\alpha,u)$ is such that the integral is single-valued and that $v_{2\pi}(\alpha,u)\to 0$ as $|\mathrm{Im}\,\alpha|\to\infty$. Under those circumstance, the right-hand side of (4.6a) becomes

$$\frac{w(\alpha+\pi,u)}{w(\alpha-\pi,u)}=\exp\int_{\alpha-\pi}^{\alpha+\pi}\left\{v_0(\xi,u)+v_{2\pi}(\xi,u)\right\}\mathrm{d}\xi\xrightarrow{|\mathrm{Im}\,\alpha|\to\infty}1 \quad (4.28)$$

as both integrands vanish exponentially fast as $|\mathrm{Im}\,\alpha|\to\infty$. The left-hand side becomes

$$g(\alpha,\pm u)=\frac{\pm u(\alpha)-u(\theta)}{\pm u(\alpha)+u(\theta)}\xrightarrow{|\mathrm{Im}\,\alpha|\to\infty}1 \quad (4.29)$$

which puts in evidence the contribution of an eventual multiplicative constant. The agreement of both of the above expressions suggests that $v_0(\alpha,u)+v_{2\pi}(\alpha,u)$ is indeed the solution sought provided $v_{2\pi}(\alpha,u)$ annuls the periods the $v_0(\alpha,u)$. The cancellation of the polar periods is now examined in detail.

4.2.2 Elimination of polar periods

The function $v_0(\alpha, u)$, besides the branch points of $u(\alpha)$, has poles at the zeros of $\cos^2\alpha - \cos^2\theta$ in the strip $\mathcal{S}_{2\pi}$, that is, when $\alpha = \pm\theta, \pm(\pi - \theta)$. Adding to the fact that they must be eliminated to fulfill the requirement for a solution free of poles in $\mathcal{S}_{2\pi}$, these poles have generally non-integer residues

$$\text{Res } v_0(\alpha, u) = -\frac{\alpha}{2\pi}\frac{u(\theta)}{u(\alpha)}. \tag{4.30}$$

These have an adverse effect on the path integral of $w(\alpha, u)$ in (4.27): their contributions are non-integer multiples of $2\pi\mathrm{j}$ which make the integral ill-defined.

The 2π periodic function $v_1(\alpha, u)$ introduced to eliminate the polar periods must have the same even parity as $v_0(\alpha, u)$, vanish as $|\text{Im }\alpha| \to \infty$ and provide the two required degrees of freedom to eliminate the offending poles. The two expressions fulfilling these requirements are

$$v_1'(\alpha, u) = \frac{u(\theta)}{u(\alpha)}\frac{\varphi_0' + \varphi_1' \cos\alpha}{\cos^2\alpha - \cos^2\theta} \tag{4.31}$$

and

$$v_1''(\alpha, u) = \frac{u(\theta)}{u(\alpha)}\frac{\cos\alpha(\varphi_0'' + \varphi_1'' \cos\alpha)}{\cos^2\alpha - \cos^2\theta}. \tag{4.32}$$

In both instances the $u(\theta)$ term is added for convenience. The fact that we can apparently arbitrarily select one or the other is perplexing, but it turns out that within the confines of the approach they are fully equivalent expressions. Prior to demonstrating this we first determine the coefficients φ_n' and φ_n'', a rather simple exercise compared to the forthcoming analysis for the cyclic periods. Given that $v_0(\alpha, u)$, $v_1'(\alpha, u)$ and $v_1''(\alpha, u)$ have the same parity, it is sufficient to enforce

$$\text{Res }\left\{v_0(\alpha, u) + v_1'(\alpha, u)\right\}\Big|_{\alpha=\theta,\pi-\theta} = 0 \tag{4.33}$$

and similarly for $v_1''(\alpha, u)$. This produces a set of two equations in the two unknowns φ_n' for $v_1'(\alpha, u)$, or φ_n'' for $v_1''(\alpha, u)$. Carrying out the algebra, it can be shown that

$$\varphi_0' = \left(\frac{1}{2} - \frac{\theta}{\pi}\right)\sin\theta\cos\theta, \qquad \varphi_1' = -\frac{1}{2}\sin\theta, \tag{4.34}$$

and

$$\varphi_0'' = -\frac{1}{2}\sin\theta, \qquad \varphi_1'' = \left(\frac{1}{2} - \frac{\theta}{\pi}\right)\frac{\sin\theta}{\cos\theta}. \tag{4.35}$$

The resulting expressions are therefore

$$v_1'(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{-\frac{\theta}{\pi}\sin\theta\cos\theta + \frac{1}{2}\sin\theta(\cos\theta - \cos\alpha)}{\cos^2\alpha - \cos^2\theta} \tag{4.36}$$

and

$$v_1''(\alpha, u) = \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta} \frac{-\frac{\theta}{\pi}\sin\theta\cos\alpha + \frac{1}{2}\sin\theta(\cos\alpha - \cos\theta)}{\cos^2\alpha - \cos^2\theta}, \tag{4.37}$$

with either being equally valid for the elimination of the poles. It was hinted at above that they were actually equivalent in the context of the solution procedure. This interesting property is due to the introduction of the 2π periodic functions $v_{2\pi}^{1,3}(\alpha, u)$ related to the elimination of the cyclic periods. While a detailed discussion is presented in the next section, the demonstration of this equivalence between (4.36) and (4.37) warrants the immediate introduction of one of these functions, namely

$$v_{2\pi}^1(\alpha, u) = \frac{1}{u(\alpha)}. \tag{4.38}$$

Two factors are responsible for the equivalence. The first is that (see next section) the final solution will involve in the integrand the function $v_{2\pi}^1(\alpha, u)$ multiplied by a coefficient to be determined. The second is that the functions $v_1'(\alpha, u)$ and $v_1''(\alpha, u)$ actually differ by a constant times $v_{2\pi}^1(\alpha, u)$, *i.e.*,

$$v_1''(\alpha, u) = v_1'(\alpha, u) + \left(\frac{1}{2} - \frac{\theta}{\pi}\right)\frac{\sin\theta}{\cos\theta}\frac{u(\theta)}{u(\alpha)}. \tag{4.39}$$

There is therefore equivalence between the two up to an additive multiple of $1/u(\alpha)$ so that $v_1'(\alpha, u)$ and $v_1''(\alpha, u)$ produce the same expression once the solution process is completed. We choose to use

$$v_1(\alpha, u) = v_1''(\alpha, u) = \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta} \frac{-\frac{\theta}{\pi}\sin\theta\cos\alpha + \frac{1}{2}\sin\theta(\cos\alpha - \cos\theta)}{\cos^2\alpha - \cos^2\theta} \tag{4.40}$$

in our solution. The main reason for this choice lies in its simple nature when $\delta = \pi/2$ since the leading coefficients then go to unity. However, another benefit can be discerned by examining more carefully the sum

$$v_0(\alpha,u)+v_1(\alpha,u)=\frac{u(\theta)}{u(\alpha)}\frac{\cos\alpha}{\cos\theta}\frac{\frac{1}{\pi}(\alpha\sin\alpha\cos\theta-\theta\sin\theta\cos\alpha)+\frac{1}{2}\sin\theta(\cos\alpha-\cos\theta)}{\cos^2\alpha-\cos^2\theta},\tag{4.41}$$

and comparison with the integral representation for $\Psi_1(\alpha)$ (4.14) reveals

$$v_0(\alpha,u)+v_1(\alpha,u)=\frac{u(\theta)}{u(\alpha)}\frac{\cos\alpha}{\cos\theta}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_1(\alpha)\xrightarrow{\delta\to\pi/2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_1(\alpha),\tag{4.42}$$

explicitly recovering the known exact solution in that limit. A similar equality holds if $v_1'(\alpha,u)$ is instead used but its demonstration requires the added consideration of the function $v^1_{2\pi}(\alpha,u)$. In a nutshell, $v_1'(\alpha,u)$ has poles arising at $\alpha=\pm\pi/2$ in the limit which are cancelled by those of $v^1_{2\pi}(\alpha,u)$. This will be reexamined in more detail in the next section when the behavior of the completed solution $w(\alpha,u)$ is considered when $\delta\to\pi/2$. It is however recognized that in the simpler case where the right-hand side of (4.6) is meromorphic, and (4.27) thus free of branch points, the known solutions expressed in terms of Maliuzhinets are recovered by simply following this procedure of pole elimination, an approach that naturally leads to the construction of the cancellation term $v_1''(\alpha,u)$.

At this stage, the partial solution to the branched first order difference equation (4.6a) is

$$w(\alpha,u)=\exp\int_{\alpha_0}^{\alpha}\Big\{v_0(\xi,u)+v_1(\xi,u)+v_{2\pi}(\xi,u)\Big\}\mathrm{d}\xi,\tag{4.43}$$

and the sum $v_0(\alpha,u)+v_1(\alpha,u)$ is now free of poles in the strip of analyticity $\mathcal{S}_{2\pi}$. The multi-valuedness associated with the presence of branch points common to both $v_0(\alpha,u)$ and $v_1(\alpha,u)$ is now dealt with by eliminating the cyclic periods of the integrand.

4.2.3 Elimination of cyclic periods

In order to obtain a well-defined solution such that continuous branch-free solutions can be constructed from $w(\alpha,u)$ by means of the symmetric meromorphic combinations (3.3), it was shown in Section 3.2.3 that all branch point to branch point integrals within the strip of analyticity have to be eliminated identically. This is equivalent to the elimination of the cyclic periods of the integrand in the two-sheeted strip of width 2π. In this instance, as shown in Figure 4.2, the contribution from the two cycles a and b only must be eliminated if the even parity of the integrand is taken advantage of. To do so, even 2π periodic functions are sought that provide two degrees freedom with which the cyclic

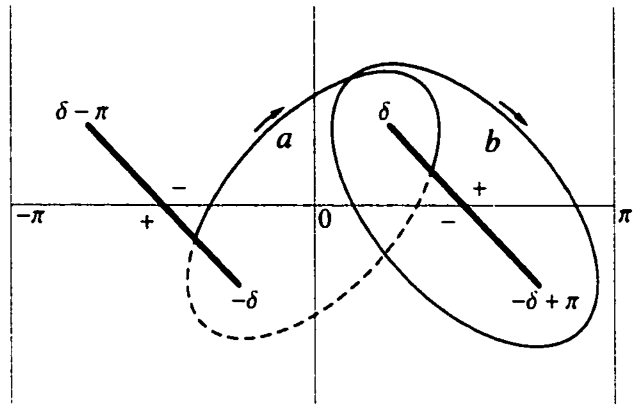


Figure 4.2: The clockwise cyclic paths a and b used to define the cyclic periods occurring in the strip $S_{2\pi}$. Note that cycle a crosses from the upper Riemann sheet (solid line) to the lower Riemann sheet (dashed line) whereas b is confined to the upper sheet.

periods can be eliminated. A number of efforts were originally made in a vain attempt to avoid the introduction of additional poles in the strip $S_{2\pi}$ by relying on functions of form $\vartheta_n \cos^n \alpha / u(\alpha)$ where ϑ_n are constants to be determined and $n \geq 0$. Such an undertaking was doomed to fail since it requires the use of terms which either tend to some arbitrary (parameter dependent) constant or which increase super-exponentially as $|\mathrm{Im}\,\alpha| \to \infty$. This presents two insurmountable problems. First, an arbitrary non-vanishing behavior when $|\mathrm{Im}\,\alpha| \to \infty$ will generally, as discussed in Section 3.2.1, compromise the viability of the solution. Secondly, in both instances the order of the solution as $|\mathrm{Im}\,\alpha| \to \infty$ is altered and its correction, at least in the case of super-exponential growth, is far from obvious. This led to the inevitable conclusion that some of the functions used to eliminate the cyclic periods, in order not to compromise the order requirement, must reintroduce poles in $S_{2\pi}$. Acknowledging this, the two 2π periodic functions used to eliminate the cyclic periods are

$$v^1_{2\pi}(\alpha, u) = \frac{1}{u(\alpha)}, \qquad v^3_{2\pi}(\alpha, u) = \frac{u(\zeta)}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta} \frac{\sin\zeta}{\cos\alpha - \cos\zeta}. \tag{4.44}$$

They are both even, vanish as $|\mathrm{Im}\,\alpha| \to \infty$ — hence they are acceptable 2π periodics — and both give rise to elliptic integrals upon integration; the superscripts indicate the type of elliptic integral produced. The first is particularly attractive since it gives rise to an integral of the first kind, as discussed in Section 3.5.1, and is free of singularities in the strip of analyticity. The second gives rise to an elliptic integral of the third kind and has poles (the integral has logarithmic singularities) at $\alpha = \pm\zeta$ in $S_{2\pi}$. In contrast to $v_0(\alpha, u)$, see (4.30), the poles of $v^3_{2\pi}(\alpha, u)$ have residues ± 1 implying polar periods of $\pm 2\pi \mathrm{j}$ which *do not* compromise the single-valuedness of the path integral. However, the satisfaction of the requirement that the solution be free of poles in $S_{2\pi}$ requires the subsequent elimination of the ζ poles and this will be carried out when the branch-free solutions are constructed in Section 4.3. It is convenient to define the cyclic periods

$$A_{2\pi}^{1,3} = \int_a v_{2\pi}^{1,3}(\alpha, u)\mathrm{d}\alpha, \qquad B_{2\pi}^{1,3} = \int_b v_{2\pi}^{1,3}(\alpha, u)\mathrm{d}\alpha. \tag{4.45}$$

Inspection of (4.45) reveals that the periods $A_{2\pi}^3$ and $B_{2\pi}^3$ associated with the integrals of the third kind are functions of the pole ζ, providing one of two degrees of freedoms required to annul the periods of $v_0(\alpha, u) + v_1(\alpha, u)$. In contrast, the periods $A_{2\pi}^1$ and $B_{2\pi}^1$ associated with the integral of the first kind are constant and a multiplicative constant κ must be introduced to produce the additional degree of freedom. The solution to the first order equation (4.6a) is then

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_1(\xi, u) + \kappa v_{2\pi}^1(\xi, u) + \sigma v_{2\pi}^3(\xi, u) \right\} \mathrm{d}\xi \tag{4.46}$$

where the two unknowns to be determined are the explicit κ and the implicit ζ. The quantity $\sigma = \pm 1$ has been introduced to avoid loss of generality in the definition of the term associated with the integral of the third kind. It accounts for the eventuality where the sign of the residues of $v_{2\pi}^3(\alpha, u)$ must be changed, thereby swapping poles and zeros of $w(\alpha, u)$ between the two Riemann sheets. Its proper definition will be determined in the course of the analysis and, to reduce clutter, it is omitted in what follows pending its reintroduction when appropriate.

An equation system made up of two equations with two unknowns is now obtained by enforcing vanishing cyclic periods on the cycles a and b or, equivalently, vanishing branch point to branch point integrals. Mathematically, we have

$$\int_a (v_0 + v_1 + \kappa v_{2\pi}^1 + v_{2\pi}^3)\mathrm{d}\alpha = 0 \iff \int_{-\delta}^{\delta} (v_0 + v_1 + \kappa v_{2\pi}^1 + v_{2\pi}^3)\mathrm{d}\alpha = 0, \tag{4.47a}$$

$$\int_b (v_0 + v_1 + \kappa v_{2\pi}^1 + v_{2\pi}^3)\mathrm{d}\alpha = 0 \iff \int_{\delta^+}^{\pi-\delta^+} (v_0 + v_1 + \kappa v_{2\pi}^1 + v_{2\pi}^3)\mathrm{d}\alpha = 0, \tag{4.47b}$$

with the superscripted positive sign in the limits indicating the corresponding side of the branch cut (see Figure 4.2) along which to integrate. Defining

$$A_{0+1} = \int_a \left\{ v_0(\alpha, u) + v_1(\alpha, u) \right\} \mathrm{d}\alpha, \qquad B_{0+1} = \int_b \left\{ v_0(\alpha, u) + v_1(\alpha, u) \right\} \mathrm{d}\alpha, \tag{4.48}$$

together with the periods defined in (4.45) enables the rewriting of (4.47) as

$$A_{0+1} + \kappa A^1_{2\pi} + A^3_{2\pi} = 0, \tag{4.49a}$$

$$B_{0+1} + \kappa B^1_{2\pi} + B^3_{2\pi} = 0, \tag{4.49b}$$

with the explicit unknown κ and the implied unknown ζ on which depend $A^3_{2\pi}$ and $B^3_{2\pi}$. This non-linear system can be solved exactly and the manner in which this is achieved becomes apparent by eliminating κ from (4.49) to produce

$$A^1_{2\pi}B^3_{2\pi} - B^1_{2\pi}A^3_{2\pi} = B^1_{2\pi}A_{0+1} - A^1_{2\pi}B_{0+1}, \tag{4.50}$$

where the only unknowns are $A^3_{2\pi}$ and $B^3_{2\pi}$. The left-hand side should look familiar: it has the same form as the left-hand side in (3.60) and (3.63), relationships obtained when we studied the bilinear relations of Riemann. The technique used in Section 3.3 can therefore by employed here to rewrite the left-hand side of (4.50) to provide a more explicit dependence on the unknown ζ. This is the focus of the next section.

4.2.4 Determination of κ and ζ

The key to inverting for ζ in (4.50) lies in following an approach akin to the one presented in Section 3.3 but it must be slightly modified since the functions involved are now 2π periodic. Consider therefore the application of the residue theorem on the Riemann surface delimited by the contour C_1 shown in Figure 4.3. In essence, the periodicity of the integrands now being considered is exploited to confine the analysis to a strip of width 2π in the complex plane. Alternatively, one could proceed by mapping this strip to the entire plane to obtain an equivalent Riemann surface. Our objective is to carry out the analysis presented in Section 3.3.2, with the modified path given above, to derive an alternative expression for the left-hand side of (4.50) in which the unknown ζ appears more explicitly as the argument of an elliptic integral of the first kind, paving the way for its inversion by means of the Jacobian elliptic sine function. To achieve this, we seek to evaluate

$$\int_{C_1} V^1_{2\pi}(\alpha, u)\mathrm{d}V^3_{2\pi}(\alpha, u) = 2\pi \mathrm{j} \sum \mathrm{Res}, \tag{4.51}$$

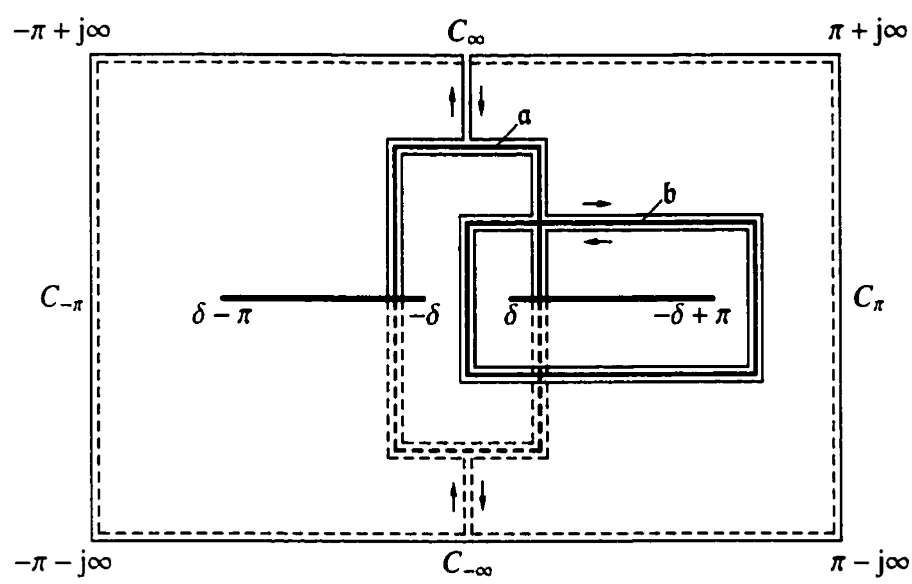


Figure 4.3: The contour $C = C_{\mathfrak{a}\cup\mathfrak{b}} \cup C_{\pm\pi} \cup C_{\pm\infty}$ on the upper (solid line) and lower (dashed line) sheets of the Riemann surface. The thicker inner lines are the canonical dissections $\mathfrak{a}$ and $\mathfrak{b}$ introduced to make the Riemann surface simply connected. The path $C_{\mathfrak{a}\cup\mathfrak{b}}$ denotes the portion of the contour enclosing the dissecting cycles $\mathfrak{a}$ and $\mathfrak{b}$.

where the elliptic integral $V^1_{2\pi}(\alpha, u)$ of the first kind and $V^3_{2\pi}(\alpha, u)$ of the third kind are defined[2] as

$$V^n_{2\pi}(\alpha, u) = \int_{(\delta,0)}^{(\alpha,u)} v^n_{2\pi}(\xi, u)\mathrm{d}\xi, \qquad n \in \{1, 3\}.$$

The path of integration C_1, shown in Figure 4.3, delimits a strip of width 2π centered at the origin of both Riemann sheets and encloses the dissections and branch cuts contained therein. For 2π periodic functions the enclosed surface is topologically equivalent to a torus — a handlebody of genus one — as shown in Figure 4.4. The canonical dissections $\mathfrak{a}$ and $\mathfrak{b}$ are introduced to make the surface simply connected, a key requirement in order for the Cauchy-Goursat theorem to apply and this is more easily appreciated from the dissected torus in Figure 4.4. Examination of the integral in (4.51) shows that only $C_{\mathfrak{a}\cup\mathfrak{b}}$, the portion of the path enclosing the branch cuts and dissections, provides a contribution. The rest of the integral vanishes either by symmetry, as for the parts along $\mathrm{Re}\,\alpha = \pm\pi$ on $C_{\pm\pi}$, or identically, as in the case where $|\mathrm{Im}\,\alpha| = \pm\infty$ on $C_{\pm\infty}$. Evaluation of the integral along $C_{\mathfrak{a}\cup\mathfrak{b}}$

[2]Recall that the inclusion of the branched $u(\alpha)$ in the limits indicates integrals taken on the Riemann surface.

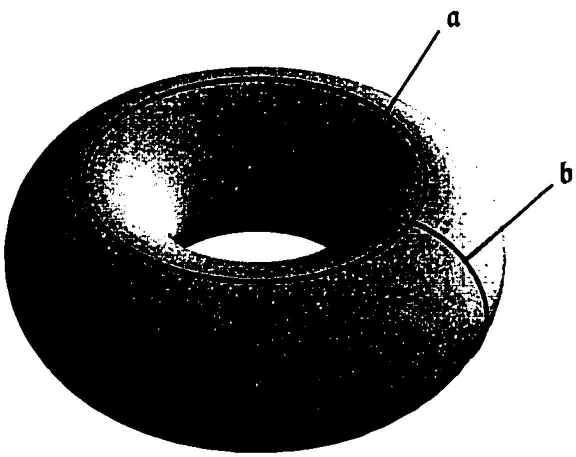


Figure 4.4: The torus, handlebody of genus one, is for 2π periodic functions topologically equivalent to the Riemann surface enclosed by the path C in Figure 4.3. It has been made simply connected by the dissections $\mathfrak{a}$ and $\mathfrak{b}$.

produces, see (3.60),

$$\int_{C_{\mathfrak{a}\cup\mathfrak{b}}} V^1_{2\pi}(\alpha,u)\mathrm{d}V^3_{2\pi}(\alpha,u) = \mathfrak{A}^1_{2\pi}\mathfrak{B}^3_{2\pi} - \mathfrak{B}^1_{2\pi}\mathfrak{A}^3_{2\pi}, \tag{4.52}$$

where following our convention capitalized letters denote cyclic periods on cycles identified by the corresponding lower case letters. The cycles defined by the dissections $\mathfrak{a}$ and $\mathfrak{b}$ in Figure 4.3 are the same as the cycles a and b shown in Figure 4.2 so that $\mathfrak{A}^n_{2\pi} = A^n_{2\pi}$ and $\mathfrak{B}^n_{2\pi} = B^n_{2\pi}$. We therefore obtain, in light of (4.50), the remarkable result

$$\int_{C_1} V^1_{2\pi}(\alpha,u)\mathrm{d}V^3_{2\pi}(\alpha,u) = \mathfrak{A}^1_{2\pi}\mathfrak{B}^3_{2\pi} - \mathfrak{B}^1_{2\pi}\mathfrak{A}^3_{2\pi} = A^1_{2\pi}B^3_{2\pi} - B^1_{2\pi}A^3_{2\pi}, \tag{4.53}$$

which, by virtue of (4.51), can be expressed as a sum of residues. On the Riemann surface, the integrand in (4.51) has residues at the zeroes of $\cos\alpha - \cos\zeta$ on both Riemann sheets at $\alpha = (\zeta, \pm u(\zeta))$ and $\alpha = (-\zeta, \pm u(\zeta))$. They are given by

$$-\frac{u(\zeta)}{u(\alpha)}\,\frac{\cos\alpha}{\cos\zeta}\,\frac{\sin\zeta}{\sin\alpha}\int_{(\delta,0)}^{(\alpha,u)} v^1_{2\pi}(\xi,u)\mathrm{d}\xi\Bigg|_{\alpha=(\pm\zeta,\pm u)} = \begin{cases} \mp\displaystyle\int_{(\delta,0)}^{(\alpha,u)} v^1_{2\pi}(\xi,u)\mathrm{d}\xi, & \alpha = (\zeta,\pm u) \\ \pm\displaystyle\int_{(\delta,0)}^{(\alpha,0)} v^1_{2\pi}(\xi,u)\mathrm{d}\xi, & \alpha = (-\zeta,\pm u) \end{cases} \tag{4.54}$$

and one must mindful of the dissections when carrying out these path integrals on the simply connected Riemann surface.

The key to obtaining simple expressions lies in exploiting the symmetry between the location of the four poles in order to express the residues in terms of $V^1_{2\pi}(\zeta)$ where, since $u(\alpha)$ is omitted, the integral is taken on the upper Riemann sheet dissected with branch cuts. From the symmetry of the poles, the analysis can be restricted without loss of generality to the case where ζ is located in the $0 \leq \operatorname{Re}\alpha \leq \pi$ strip, corresponding to the domains $\mathcal{A}_1$ and $\mathcal{A}_2$ introduced in Section 3.5.1. The cases where $\zeta \in \mathcal{A}_1$ (or $V^1_{2\pi}(\zeta, u) \in \mathcal{V}_1$) and $\zeta \in \mathcal{A}_2$ (or $V^1_{2\pi}(\zeta, u) \in \mathcal{V}_2$) are considered separately. The associated paths of integration are respectively given in Figures 4.5 and 4.6. Consider for example the path required for the evaluation of the residue at $\alpha = (\zeta, u(\zeta))$ depicted in Figure 4.5(a). The path of integration starts at $\alpha = \delta$ on the lower Riemann sheet in order to avoid crossing the canonical dissection $\mathfrak{b}$ situated on the upper sheet. It then goes from δ to the branch point at $\alpha = \delta - \pi$ where it crosses to the upper Riemann sheet and doubles back to $\alpha = (\zeta, u(\zeta))$. Accounting for the contribution to the integral from the different loops as indicated, the path integral on the dissected surface is

$$\begin{aligned}\int_{(\delta,0)}^{(\zeta,u(\zeta))} v^1_{2\pi}(\xi, u)\mathrm{d}\xi &= \frac{A^1_{2\pi}}{2} - \frac{B^1_{2\pi}}{2} + \frac{A^1_{2\pi}}{2} - \frac{A^1_{2\pi}}{2} + \int_\delta^\zeta v^1_{2\pi}(\xi, u)\mathrm{d}\xi \\ &= A^1_{2\pi} - B^1_{2\pi} + V^1_{2\pi}(\zeta)\end{aligned} \tag{4.55}$$

where the last integral is the direct path integral from δ to ζ on the upper Riemann sheet as indicated by the removal of $u(\alpha)$ from the limits. All of the other cases have similar representations due to the symmetry of the location of the residues with respect to the branch points. In the instance where $\zeta \in \mathcal{A}_1$, one obtains the residues

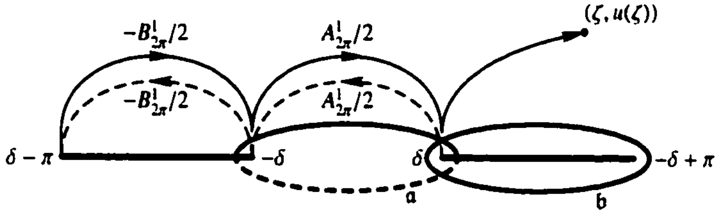
$$\operatorname{Res}\big|_{\alpha=(\zeta,u(\zeta))} = -\left(A^1_{2\pi} - B^1_{2\pi} + V^1_{2\pi}(\zeta)\right), \tag{4.56a}$$

$$\operatorname{Res}\big|_{\alpha=(-\zeta,u(\zeta))} = +\left(\frac{A^1_{2\pi}}{2} - V^1_{2\pi}(\zeta)\right), \tag{4.56b}$$

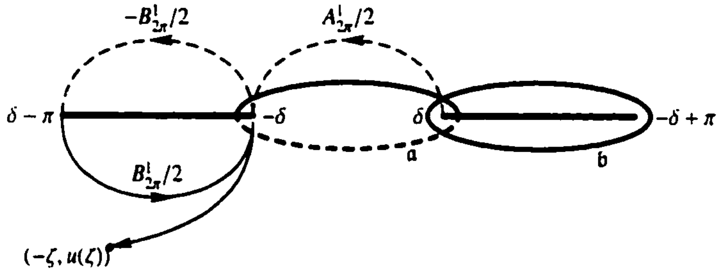
$$\operatorname{Res}\big|_{\alpha=(\zeta,-u(\zeta))} = +\left(-V^1_{2\pi}(\zeta)\right), \tag{4.56c}$$

$$\operatorname{Res}\big|_{\alpha=(-\zeta,-u(\zeta))} = -\left(\frac{A^1_{2\pi}}{2} - B^1_{2\pi} + V^1_{2\pi}(\zeta)\right), \tag{4.56d}$$

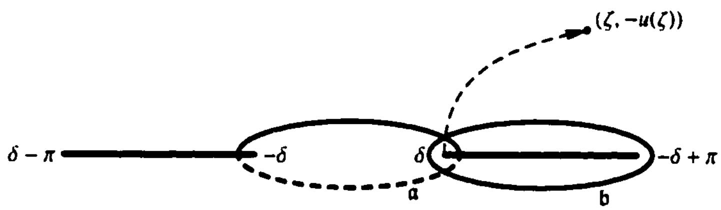
where use has been made of the fact that $V^1_{2\pi}(\zeta, u(\zeta)) = -V^1_{2\pi}(\zeta, -u(\zeta))$. The $\pm$ signs in front of the parentheses distinguish the contribution of the integral of the third kind from the path integral as shown in (4.54). Other paths could be chosen when carrying out the integration but, since the surface is now simply connected, the final result would not differ. The sum



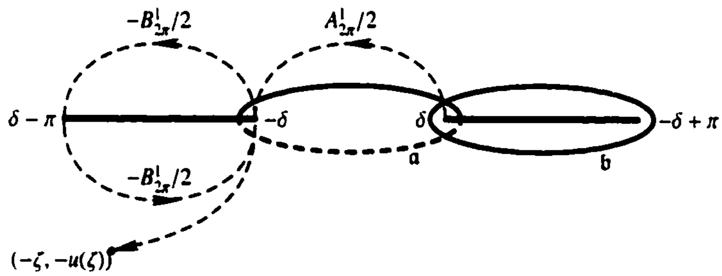
(a)



(b)



(c)



(d)

Figure 4.5: Depiction of the path used for the evaluation of the residues in (4.54) when $\zeta \in \mathcal{A}_1$. The thick straight lines represent the branch cuts and the elliptic lines the canonical dissections. The thin lines indicate the path followed on the upper (solid lines) and lower (dashed lines) sheets, avoiding the crossing of the dissections, from $\alpha = \delta$ to the desired pole. The four cases above account for the four possible poles at (a) $\alpha = (\zeta, u(\zeta))$, (b) $\alpha = (-\zeta, u(\zeta))$ (c) $\alpha = (\zeta, -u(\zeta))$ and (d) $\alpha = (-\zeta, -u(\zeta))$.

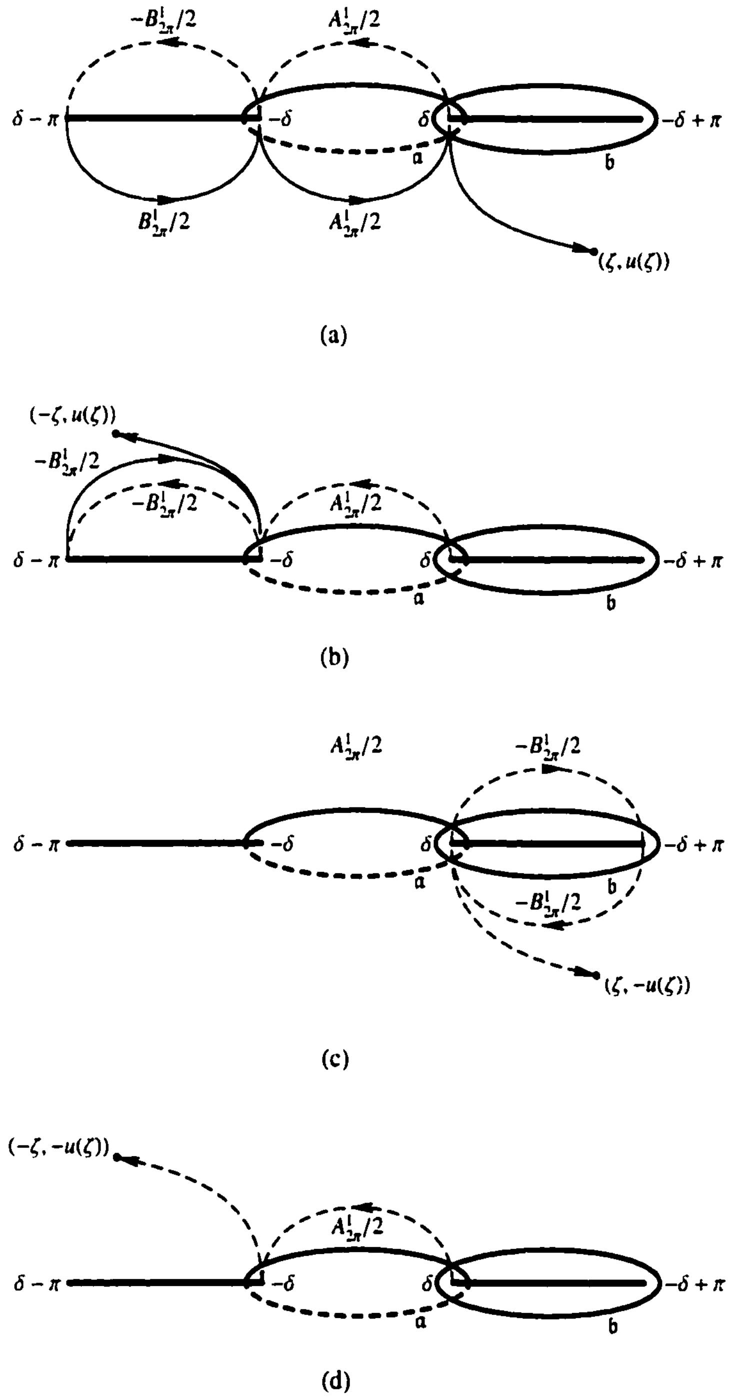


Figure 4.6: Depiction of the path used for the evaluation of the residues in (4.54) when $\zeta \in \mathcal{A}_2$. The thick straight lines represent the branch cuts and the elliptic lines the canonical dissections. The thin lines indicate the path followed on the upper (solid lines) and lower (dashed lines) sheets, avoiding the crossing of the dissections, from $\alpha = \delta$ to the desired pole. The four cases above account for the four possible poles at (a) $\alpha = (\zeta, u(\zeta))$, (b) $\alpha = (-\zeta, u(\zeta))$ (c) $\alpha = (\zeta, -u(\zeta))$ and (d) $\alpha = (-\zeta, -u(\zeta))$.

of residues is therefore

$$\sum \text{Res} = -A^1_{2\pi} + 2B^1_{2\pi} + 4V^1_{2\pi}(\zeta), \qquad \zeta \in \mathcal{A}_1. \tag{4.57}$$

In the case where $\zeta \in \mathcal{A}_2$ we have, from Figure 4.6,

$$\text{Res}\Big|_{\alpha=(\zeta,u(\zeta))} = -\left(A^1_{2\pi} + V^1_{2\pi}(\zeta)\right), \tag{4.58a}$$

$$\text{Res}\Big|_{\alpha=(-\zeta,u(\zeta))} = +\left(\frac{A^1_{2\pi}}{2} - B^1_{2\pi} - V^1_{2\pi}(\zeta)\right), \tag{4.58b}$$

$$\text{Res}\Big|_{\alpha=(\zeta,-u(\zeta))} = +\left(-B^1_{2\pi} - V^1_{2\pi}(\zeta)\right), \tag{4.58c}$$

$$\text{Res}\Big|_{\alpha=(-\zeta,-u(\zeta))} = -\left(\frac{A^1_{2\pi}}{2} + V^1_{2\pi}(\zeta)\right), \tag{4.58d}$$

and the sum of residues is now

$$\sum \text{Res} = -A^1_{2\pi} - 2B^1_{2\pi} + 4V^1_{2\pi}(\zeta), \qquad \zeta \in \mathcal{A}_2. \tag{4.59}$$

The case when ζ lies between $\mathcal{A}_1$ and $\mathcal{A}_2$ is not considered explicitly here as either (4.57) or (4.59) will apply, depending on how the elliptic integral and the elliptic sine function are defined numerically. From the discussion of the elliptic integral of the first kind $V^1_{2\pi}(\alpha, u)$ given in Section 3.5.1, $V^1_{2\pi}(\alpha) \in \mathcal{V}_{1,2}$ if $\alpha \in \mathcal{A}_{1,2}$ and we therefore write, combining (4.57) and (4.59),

$$\sum \text{Res} = -A^1_{2\pi} \pm 2B^1_{2\pi} - 4V^1_{2\pi}(\zeta), \qquad V^1_{2\pi}(\zeta) \in \mathcal{V}_{1,2}. \tag{4.60}$$

with the upper (lower) sign corresponding to $\mathcal{V}_1$ ($\mathcal{V}_2$). Alternatively we could have written instead $\zeta \in \mathcal{A}_{1,2}$, a less convenient choice perhaps for the forthcoming analysis. Substitution of the result of the path integral (4.53) and the sum of residues (4.60) in the residue theorem (4.51) yields

$$V^1_{2\pi}(\zeta) = -\frac{A^1_{2\pi}}{4} \pm \frac{B^1_{2\pi}}{2} + \sigma \mathrm{j}\Lambda, \qquad V^1_{2\pi}(\zeta) \in \mathcal{V}_{1,2}. \tag{4.61}$$

where ζ is the quantity sought. We have defined above for convenience

$$\Lambda = \frac{1}{8\pi}\left(A^1_{2\pi}B_{0+1} - B^1_{2\pi}A_{0+1}\right), \tag{4.62}$$

and have also reintroduced σ from equation (4.46). One may suspect that the sign change associated with the $B_{2\pi}/2$ term might likely be unnecessary since it corresponds to a change of an entire cyclic period $B_{2\pi}$; this will become more obvious once ζ is expressed in terms of an elliptic sine function. Before doing so, (4.61) is used to find the corresponding range in which Λ must lie given the range of $V^1_{2\pi}(\zeta)$. Using the case when $V^1_{2\pi}(\zeta) \in \mathcal{V}_1$ as an example, and recalling (see (3.74)) that

$$k = \cos\delta, \qquad A^1_{2\pi} = 4K', \qquad B^1_{2\pi} = 4\mathrm{j}K, \tag{4.63}$$

the cyclic periods are first expressed in terms of the periods K and K' so that $V^1_{2\pi}(\zeta) \in [-2K, 0] \times [-\mathrm{j}K', 0]$. Equation (4.61) implies that $\mathrm{j}K' + 2K + \sigma\Lambda \in [0, 2\mathrm{j}K] \times [0, K']$ so that $\sigma\Lambda \in [0, -2K] \times [0, -\mathrm{j}K']$. Similarly, for the case where $V^1_{2\pi}(\zeta) \in \mathcal{V}_2$, we have $\sigma\Lambda \in [0, 2K] \times [0, -\mathrm{j}K']$. These results are summarized as follows:

$$V^1_{2\pi}(\zeta) \in \mathcal{V}_1 \implies \Lambda \in \begin{cases} \mathcal{P}^+_1 = [0, -2K] \times [0, -\mathrm{j}K'] & \sigma = +1, \\ \mathcal{P}^-_1 = [0, 2K] \times [0, \mathrm{j}K'] & \sigma = -1, \end{cases} \tag{4.64}$$

and

$$V^1_{2\pi}(\zeta) \in \mathcal{V}_2 \implies \Lambda \in \begin{cases} \mathcal{P}^+_2 = [0, 2K] \times [0, -\mathrm{j}K'] & \sigma = +1, \\ \mathcal{P}^-_2 = [0, -2K] \times [0, \mathrm{j}K'] & \sigma = -1. \end{cases} \tag{4.65}$$

The period parallelograms $\mathcal{P}^\sigma_{1,2}$ are illustrated in Figure 4.7.

We are now in a position to obtain closed-form expressions for ζ in terms of Λ and its range. We proceed as in Section 3.5.1, and obtain a closed-form expression for ζ using the Jacobian elliptic sine function sn. Taking into account the ranges defined above and substituting the expression for $V^1_{2\pi}(\alpha, u)$ (4.61) into the one for its inversion (3.77), we obtain after some algebraic manipulations

$$\zeta = \begin{cases} \arccos\left[k \,\mathrm{sn}\,(\mathrm{j}K' + 3K + \sigma\Lambda, k)\right], & \Lambda \in \mathcal{P}^\sigma_1 \\ \arccos\left[k \,\mathrm{sn}\,(\mathrm{j}K' - K + \sigma\Lambda, k)\right], & \Lambda \in \mathcal{P}^\sigma_2 \end{cases} \tag{4.66}$$

which is an explicit expression for ζ. The correct expression to use in (4.66) as well as the correct definition for $\sigma = \pm 1$ follow from locating Λ in the appropriate $\mathcal{P}$ parallelogram in Figure 4.7. For instance, if $\Lambda \in \mathcal{P}^-_2$ then $\sigma = -1$ and $\zeta = \arccos\left[k \,\mathrm{sn}\,(\mathrm{j}K' - K - \Lambda, k)\right]$. The multiplicative constant κ follows immediately from (4.49a) or (4.49b). It is apparent that (4.66) can be simplified since the $+3K$ and $-K$ in the argument of the Jacobian function sn represent a difference of $4K$, one period of the elliptic sine function. Letting $\mathcal{P}^\sigma = \mathcal{P}^\sigma_1 \cup \mathcal{P}^\sigma_2$,

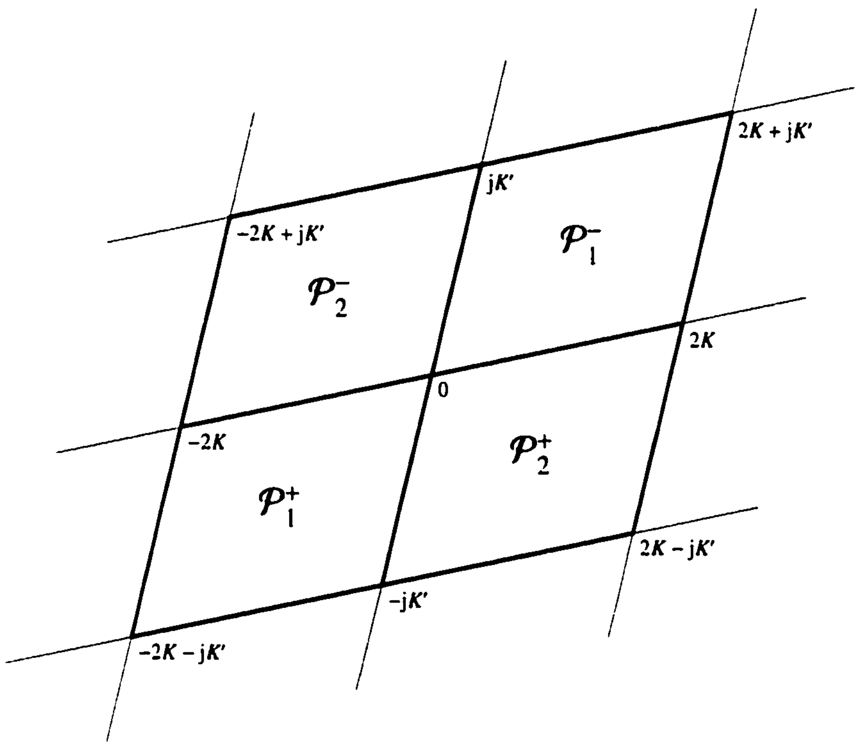


Figure 4.7: The regions $\mathcal{P}_1^\pm$ and $\mathcal{P}_2^\pm$ in terms of the complete integrals of the first kind K and K' with $k = \cos\delta$. The parallelograms $\mathcal{P}$ indicate the various ranges in which Λ must lie when carrying out the inversion for $\zeta_{2\pi}$ with equation (4.66). We also define $\mathcal{P}^\pm = \mathcal{P}_1^\pm \cup \mathcal{P}_2^\pm$.

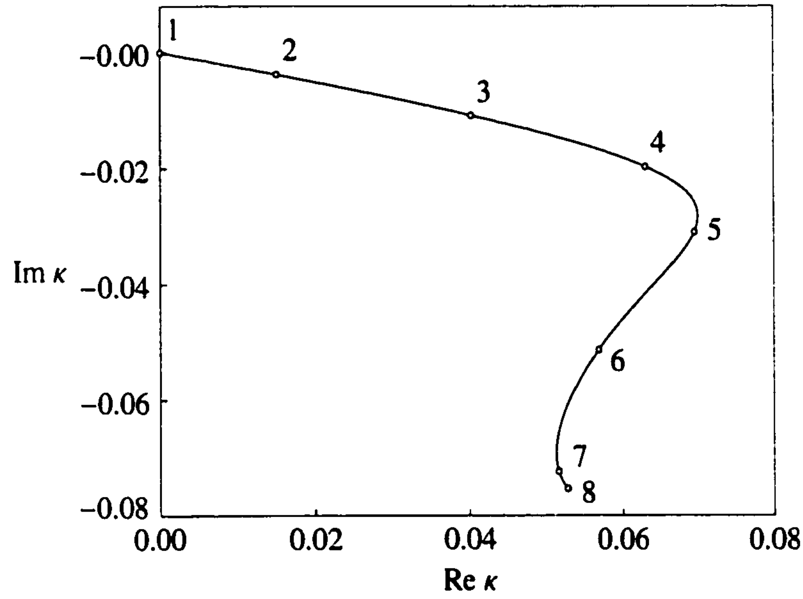


Figure 4.8: The behavior of κ when $\theta = 0.25(1 + \mathrm{j})$ as a function of δ. The range of δ is from 1.57 at point 1 to 0.01 at point 8. The corresponding values of κ are given in Table 4.1.

one obtains the slightly more compact result

$$\zeta = \arccos\left[k \operatorname{sn}(\mathrm{j}K' + 3K + \sigma\Lambda, k)\right], \qquad \Lambda \in \mathcal{P}^{\sigma}. \tag{4.67}$$

It is interesting to examine the behavior of κ and ζ as a function of δ for a fixed value of θ. A plot of the values assumed by κ in the plane as a function of δ is provided in Figure 4.8. The values for the eight sample points indicated on the figure are given in Table 4.1. Note that κ tends to zero as $\delta \to \pi/2$, this will be demonstrated analytically shortly. The corresponding plot for ζ is provided in Figure 4.9 and the values at the sample points are given in Table 4.2. Note that ζ apparently tends to π as $\delta \to 0$, but this has yet to be proved analytically. Once ζ and κ are known (and the auxiliary $\sigma = \pm 1$), $w(\alpha, u)$ constitutes a well-defined branched solution to (4.6a). Its properties, in particular its behaviors when the branch points vanish, are now examined.

4.2.5 Properties of the branched solution

A solution to the first order difference equation (4.6a) has now been constructed and has the form

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_1(\xi, u) + \kappa v_{2\pi}^{1}(\xi, u) + \sigma v_{2\pi}^{3}(\xi, u) \right\} \mathrm{d}\xi. \tag{4.68}$$

Point	δ	ζ
1	1.57	$9.94055 \cdot 10^{-8} - j2.45286 \cdot 10^{-8}$
2	1.25	$1.49975 \cdot 10^{-2} - j3.79812 \cdot 10^{-3}$
3	1.00	$4.02549 \cdot 10^{-2} - j1.08545 \cdot 10^{-2}$
4	0.75	$6.29989 \cdot 10^{-2} - j1.97355 \cdot 10^{-2}$
5	0.50	$6.94261 \cdot 10^{-2} - j3.09402 \cdot 10^{-2}$
6	0.25	$5.68014 \cdot 10^{-2} - j5.13024 \cdot 10^{-2}$
7	0.05	$5.15650 \cdot 10^{-2} - j7.23819 \cdot 10^{-2}$
8	0.01	$5.26927 \cdot 10^{-2} - j7.53512 \cdot 10^{-2}$

Table 4.1: Corresponding values of δ and ζ for the points indicated in Figure 4.8.

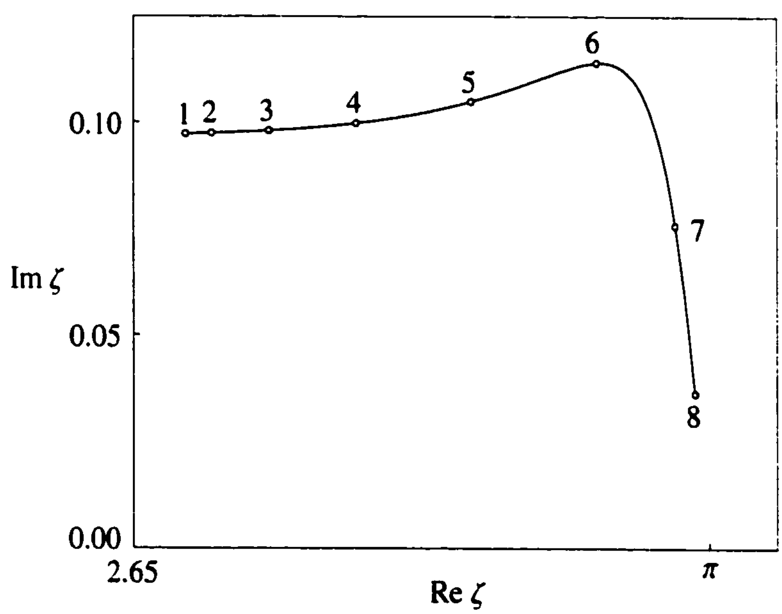


Figure 4.9: The behavior of ζ when $\theta = 0.25(1 + j)$ as a function of δ. The range of δ is from 1.57 at point 1 to 0.01 at point 8. The corresponding values of ζ are given in Table 4.2.

Point	δ	ζ
1	1.57	$2.69458 + j9.72932 \cdot 10^{-2}$
2	1.25	$2.71672 + j9.74614 \cdot 10^{-2}$
3	1.00	$2.76460 + j9.80415 \cdot 10^{-2}$
4	0.75	$2.83888 + j9.78570 \cdot 10^{-2}$
5	0.50	$2.93723 + j1.04880 \cdot 10^{-1}$
6	0.25	$3.04405 + j1.14004 \cdot 10^{-1}$
7	0.05	$3.11198 + j7.57454 \cdot 10^{-2}$
8	0.01	$3.12925 + j3.62956 \cdot 10^{-2}$

Table 4.2: Corresponding values of δ and κ for the points indicated in Figure 4.9.

where

$$v_0(\alpha) = \frac{u(\theta)}{u(\alpha)} \frac{\alpha}{\pi} \frac{\sin\alpha\cos\alpha}{\cos^2\alpha - \cos^2\theta}, \tag{4.69}$$

$$v_1(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{-\frac{\theta}{\pi}\sin\theta\cos\alpha + \frac{1}{2}\sin\theta(\cos\alpha - \cos\theta)}{\cos^2\alpha - \cos^2\theta}, \tag{4.70}$$

$$v_{2\pi}^1(\alpha, u) = \frac{1}{u(\alpha)}, \tag{4.71}$$

and

$$v_{2\pi}^3(\alpha, u) = \frac{u(\zeta)}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta} \frac{\sin\zeta}{\cos\alpha - \cos\zeta}. \tag{4.72}$$

The pole ζ and the sign σ are determined from (4.67); the coefficient κ from (4.49). The lower limit α_0 may be any branch point in the strip $\mathcal{S}_{2\pi}$, *i.e.* $\alpha_0 \in \{\pm\delta, \pm(\pi - \delta)\}$. Furthermore, since the integrand has an even parity, the elimination of the cyclic period on cycle a looping from $-\delta$ to δ is equivalent to enforcing a null integral from 0 to δ, implying that $\alpha_0 = 0$ is also an allowable choice. This signifies that the solution obtained is normalized to unity at $\alpha = 0$. The solution is multi-valued in the sense of the branch $u(\alpha)$ and each of these two branches corresponds to a solution of one of the equations in (4.6). Despite the fact that the integrand has branch points commensurate with those of $u(\alpha)$, the path integral in (4.68) is well-defined following the process of polar and cyclic period elimination carried out in the previous section. A byproduct of the procedure is that $w(\alpha, u)$ has poles and zeros in $\mathcal{S}_{2\pi}$ associated with the integrand of the third kind $v_{2\pi}^3(\alpha, u)$. Indeed, from (4.44)

$v_{2\pi}^3(\alpha, u)$ has residues[3], taking the positive branch of $u(\alpha)$,

$$\text{Res}\; \frac{u(\zeta)}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta} \frac{\sin\zeta}{\cos\alpha - \cos\zeta}\bigg|_{\alpha=\pm\zeta} = \pm 1 \tag{4.73}$$

and in the strip $S_{2\pi}$ $w(\alpha, u)$ therefore has the poles and zeros

$$w(\alpha, u) \sim \frac{\alpha + \zeta}{\alpha - \zeta}, \tag{4.74a}$$

$$w(\alpha, -u) \sim \frac{\alpha - \zeta}{\alpha + \zeta}, \tag{4.74b}$$

which, of course, have periodically occurring counterparts in the complex α plane outside the strip. Similarly, the solution is free of θ related poles in the strip but, much like Maliuzhinets's functions and as inferred by the difference equation (4.6), it has periodically occurring θ poles and zeros outside the strip whose order increases with distance from the strip. The solution goes to its reciprocal when $\alpha \to -a$ or when $u(\alpha) \to -u(\alpha)$ and hence $w(\alpha, u) = 1/w(-a, u) = 1/w(\alpha, -u)$. Since the integrand vanishes as $|\text{Im}\,\alpha| \to \infty$ then, as required,

$$w(\alpha, u) \sim O(1) \quad \text{as } |\text{Im}\,\alpha| \to \infty. \tag{4.75}$$

4.2.5.1 The limit $\delta \to \pi/2$

As $\delta \to \pi/2$, $w(\alpha, u)$ recovers the known solution $\Psi_1(\alpha)$ given in (4.12) with an added multiplicative contribution from the integral of the third kind associated with $v_{2\pi}^3(\alpha, u)$. The key step in understanding the limiting behavior of $w(\alpha, u)$ is the determination of both κ and ζ in that limit. This can be achieved by solely considering the equation system (4.49). As $\delta \to \pi/2$, (4.49b) becomes, with the use of the representation for $v_0 + v_1$ in terms of $\Psi_1(\alpha, \theta)$ (4.42) together with the fact that $u(\alpha) \to \cos\alpha$,

$$\lim_{\varepsilon\to\pi/2} \int_{\varepsilon}^{\pi-\varepsilon} \left\{ \frac{d}{d\alpha} \ln \Psi_1(\alpha) + \frac{\kappa}{\cos\alpha} + \frac{\sin\zeta}{\cos\alpha - \cos\zeta} \right\} d\alpha = 0, \tag{4.76}$$

and since both the first and the last member of the integrand are analytic in the neighborhood of $\pi/2$ in the limit, their contribution vanishes and we are left with only a contribution of $\pi j\kappa$ (a half-residue from a Cauchy principal value) from the middle term. Hence, in order

[3] In what follows it is assumed for simplicity that $\sigma = 1$. The reader should bear in mind that expressions involving $v_{2\pi}^3(\alpha, u)$ have to be changed accordingly if instead $\sigma_1 = -1$.

for equality to hold,

$$\kappa \xrightarrow{\delta\to\pi/2} 0, \tag{4.77}$$

in agreement with the behavior of κ in Figure 4.8. It now follows that (4.49a), in the limit, becomes

$$\lim_{\varepsilon\to\pi/2}\int_0^{\varepsilon}\left\{\frac{d}{d\alpha}\ln\Psi_1(\alpha)+\frac{\sin\zeta}{\cos\alpha-\cos\zeta}\right\}d\alpha = 0 \tag{4.78}$$

where the middle term has been removed in accordance with (4.77). The two remaining terms can be evaluated explicitly. We have, taking the limit,

$$\int_0^{\pi/2}\frac{d}{d\alpha}\ln\Psi_1(\alpha)d\alpha = \ln\Psi_1(\alpha)\Big|_0^{\pi/2} = \ln\Psi_1(\pi/2) \tag{4.79}$$

since $\Psi_1(0) = 1$. The other term gives

$$\int_0^{\pi/2}\frac{\sin\zeta}{\cos\alpha-\cos\zeta}d\alpha = \ln\frac{\tan\frac{\zeta}{2}+\tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2}-\tan\frac{\alpha}{2}}\Bigg|_0^{\pi/2} = \ln\frac{\tan\frac{\zeta}{2}+1}{\tan\frac{\zeta}{2}-1} \tag{4.80}$$

and (4.78) then implies

$$\zeta \xrightarrow{\delta\to\pi/2} 2\tan^{-1}\frac{1+\Psi_1(\pi/2)}{1-\Psi_1(\pi/2)}. \tag{4.81}$$

When $\theta = 0.25(1+j)$ this yields $\zeta = 2.69458 + j9.72932\cdot 10^{-2}$ which is identical to the value given in Table 4.2 for $\delta = 1.57$. Since κ vanishes when $\delta = \pi/2$, the solution $w(\alpha, u)$ in (4.68) assumes the simple form

$$w(\alpha,u) = \frac{\tan\frac{\zeta}{2}+\tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2}-\tan\frac{\alpha}{2}}\Psi_1(\alpha), \tag{4.82}$$

and, similarly,

$$w(\alpha,-u) = \frac{\tan\frac{\zeta}{2}-\tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2}+\tan\frac{\alpha}{2}}\frac{1}{\Psi_1(\alpha)}, \tag{4.83}$$

where the rational term is the contribution from $v_{2\pi}^3(\alpha,u)$ in the limit. The recovery of the known solution $\Psi_1(\alpha)$ is made possible by appropriately constructing the branch-free solutions, the topic of the next section.

It can also be shown that if $v_1'(\alpha, u)$ given in (4.36), as opposed to $v_1''(\alpha, u)$, had been used to construct the solution the same limit (4.82) is recovered. Indeed, using (4.39) and (4.42),

$$v_0(\alpha, u) + v_1'(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \Psi_1(\alpha) - \left(\frac{1}{2} - \frac{\theta}{\pi}\right) \frac{\sin\theta}{\cos\theta} \frac{u(\theta)}{u(\alpha)} \tag{4.84}$$

and (4.49b) becomes

$$\lim_{\varepsilon\to\pi/2} \int_{\varepsilon}^{\pi-\varepsilon} \left\{ \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \Psi_1(\alpha) - \left(\frac{1}{2} - \frac{\theta}{\pi}\right) \frac{\sin\theta}{\cos\alpha} + \frac{\kappa}{\cos\alpha} + \frac{\sin\zeta}{\cos\alpha - \cos\zeta} \right\} d\alpha = 0 \tag{4.85}$$

implying that

$$\kappa \to \left(\frac{1}{2} - \frac{\theta}{\pi}\right) \sin\theta. \tag{4.86}$$

Using this value of κ in conjunction with $v_1'(\alpha, u)$ in (4.49a) now reproduces (4.78) and the limiting value of ζ remains unchanged. This only serves to underline the previously made statement back on page 70 that $v_1'(\alpha, u)$ and $v_1''(\alpha, u)$ are equivalent within the framework of this construction process.

4.2.5.2 The limit $\delta \to 0$

This is considerably more complicated than the case $\delta \to \pi/2$ considered above. Firstly, the function $u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta} \to \sqrt{\cos^2\alpha - 1}$ and the cuts now span the real axis. The function in that limit *is* free of branch cuts but it becomes a dissected version of $\pm \mathrm{j} \sin\alpha$. This is inevitable as the function $u(\alpha)$, as defined, is even and the end result is therefore a version of $\sin\alpha$ suitably dissected so as to obtain an even function. The resulting dissections and the value taken by $u(\alpha)$ in that limit are shown in Figure 4.10. Secondly, singularities arise at $\alpha = \pm\pi$ which further complicate matters. This limit, in particular the behavior of the pole ζ, is not well understood and only a partial treatment is given here. The multiplicative constant κ is itself determined using (4.49a). If it is assumed[4] that $u(\alpha) \to -\mathrm{j}\sin\alpha$ as

[4]This may seem unjustifiable given the previous remarks on the behavior of $u(\alpha)$. It is however valid in the context of the branch-free solutions.

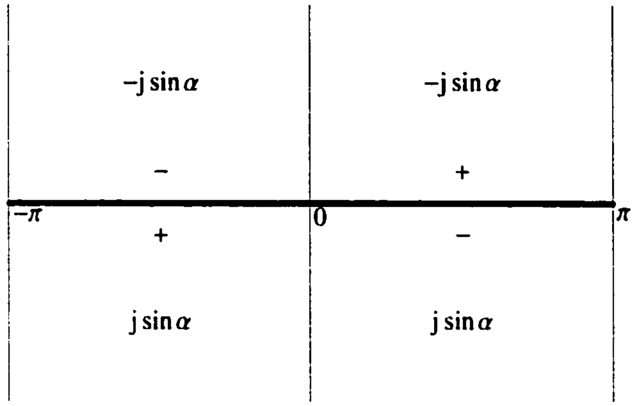


Figure 4.10: Configuration of the 2π strip of analyticity when $\delta \to 0$ when the cuts then stretch across the real axis. The values assumed by $u(\alpha)$ in the plane are indicated. The positive and negative signs indicate relative changes in sign of $u(\alpha)$ across the different cuts.

$\delta \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left\{-\frac{\sin(\theta)}{\sin\alpha}\frac{\alpha}{\pi}\frac{\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\right.$$
$$-\frac{\sin(\theta)}{\sin\alpha}\frac{\cos\alpha}{\cos\theta}\frac{-\frac{\theta}{\pi}\sin\theta\cos\alpha+\frac{1}{2}\sin\theta(\cos\alpha-\cos\theta)}{\cos^2\alpha-\cos^2\theta} \tag{4.87}$$
$$\left.+\kappa\frac{\mathrm{j}}{\sin\alpha}-\frac{\sin(\zeta)}{\sin\alpha}\frac{\cos\alpha}{\cos\zeta}\frac{\sin\zeta}{\cos\alpha-\cos\zeta}\right\}\mathrm{d}\alpha=0$$

which has half-residue contributions at $\alpha = 0$ (or, alternatively, taking the Cauchy principal value) from all terms save the first one yielding

$$\kappa \to -\mathrm{j}\left\{-\frac{1}{2}+\left(\frac{1}{2}-\frac{\theta}{\pi}\right)\frac{1}{\cos\theta}+\frac{1+\cos\zeta}{\cos\zeta}\right\} \tag{4.88}$$

where ζ is undetermined. Using the above value of κ the expression for $w(\alpha, u)$ becomes, after some algebra and still assuming the above limit,

$$w(\alpha,u)=\exp\int_0^{\alpha}\left\{-\frac{\frac{\xi}{\pi}\sin\theta\cos\xi-\left(\frac{\theta}{2}-\frac{1}{2}\right)\cos\theta\sin\xi-\frac{1}{4}\sin 2\xi}{\cos^2\xi-\cos^2\theta}-\frac{1}{2}\frac{1-\cos\xi}{\sin\xi}\right.$$
$$\left.+\frac{1+\cos\zeta}{\cos\zeta}\frac{1}{\sin\xi}+\frac{\sin\zeta\cos\xi}{\cos\zeta\sin\xi}\frac{\sin\zeta}{\cos\alpha-\cos\zeta}\right\}\mathrm{d}\xi \tag{4.89}$$

which, after comparing with (4.21), becomes

$$w(\alpha,u) = \exp\int_0^\alpha \left\{\frac{d}{d\xi}\ln\Psi_2(\xi) + \frac{1+\cos\zeta}{\cos\zeta}\frac{1}{\sin\xi} + \frac{\sin\zeta\cos\xi}{\cos\zeta\sin\xi}\frac{\sin\zeta}{\cos\alpha-\cos\zeta}\right\}d\xi. \quad (4.90)$$

The limiting value of ζ should be obtainable by enforcing vanishing cyclic periods on the b cycle going from 0 to π but the presence of a pole in the integrand at $\alpha = \pi$ makes this non-trivial. As shown in Figure 4.9, numerical computations suggest that ζ tends to a value in the neighborhood of π as $\delta \to 0$. If this is the case, the above neatly tends to the known solution $\Psi_2(\alpha)$ as the extra terms involving ζ then vanish. However, this appears incorrect due to the fact that $\mathrm{d}/\mathrm{d}\alpha \ln\Psi_2(\xi)$, while satisfying the requirement for a vanishing cyclic period (or polar period in the limit) on the loop a, does not obviously satisfy a similar requirement from 0 to π.

4.3 Branch-free solutions

We now proceed to construct a pair of branch-free solution having the desired analytical properties in $S_{2\pi}$. The means to do so were introduced back in Section 3.1.1 where the branch-free constructs (3.3)

$$t_\Sigma(\alpha) = w(\alpha,u(\alpha)) + w(\alpha,-u(\alpha)), \quad (4.91a)$$

$$t_\Delta(\alpha) = \frac{w(\alpha,u(\alpha)) - w(\alpha,-u(\alpha))}{u(\alpha)}. \quad (4.91b)$$

were discussed. It was shown that provided the branched function $w(\alpha,u)$ is well defined, the resulting expressions are branch-free. As such, (4.91) constitute meromorphic solutions but, as pointed out in (4.74), $w(\alpha,u)$ has poles and zeroes in the 2π strip of analyticity. Consider therefore the linear combination

$$t_1(\alpha) = Q_1(\alpha)\left\{t_\Sigma(\alpha) + r_1(\alpha)t_\Delta(\alpha)\right\} \quad (4.92)$$

where $Q_1(\alpha)$ and $r_1(\alpha)$ are both unit periodics. In order to satisfy the order requirement, $Q_1(\alpha)$ and $r_1(\alpha)/u(\alpha)$ are $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. The trick here is to choose $r_1(\alpha)$ such that the poles can be eliminated analytically without violating the order requirement. The approach used is best shown by rewriting $t_1(\alpha)$ in terms of $w(\alpha,\pm u)$, *viz.*

$$t_1(\alpha) = Q_1(\alpha)\left\{\left(1+\frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha,u) + \left(1-\frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha,-u)\right\}. \quad (4.93)$$

The benefit is that the poles and zeroes of $w(\alpha, \pm u)$ are known from (4.74), and, if $r_1(\alpha)$ is chosen such that a *double* zero is introduced at a pole of, say, $w(a, -u)$, then the expression in curly braces has a known pole-zero pair. Consider therefore the case where

$$1 - \frac{r_1(\alpha)}{u(\alpha)} \sim (\alpha + \zeta)^2 \tag{4.94}$$

which implies that the second term in braces in (4.93) has a simple zero coincident with that of $w(\alpha, u)$ and the term in curly braces has a pole at $\alpha = \zeta$ due to $w(\alpha, u)$. Explicitly,

$$\left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right) w(\alpha, u) + \left(1 - \frac{r_1(\alpha)}{u(\alpha)}\right) w(\alpha, -u) \sim \left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right) \frac{\alpha + \zeta}{\alpha - \zeta} + (\alpha + \zeta)^2 \frac{\alpha - \zeta}{\alpha + \zeta}. \tag{4.95}$$

The function $r_1(\alpha)$ is constructed by assuming a 2π periodic trigonometric form

$$r_1(\alpha) = \rho_0 + \rho_1 \cos\alpha + \rho_2 \sin\alpha \tag{4.96}$$

and enforcing the requirement (4.94). It can thus be shown that

$$\begin{aligned} \rho_1 &= \cos\zeta \left(-\rho_0 + \frac{\sin^2\zeta}{u(\alpha)}\right), \\ \rho_2 &= \sin\zeta \left(\rho_0 + \frac{\cos^2\zeta}{u(\alpha)}\right), \end{aligned} \tag{4.97}$$

with ρ_0 arbitrary. Choosing $\rho_0 = 0$, then

$$r_1(\alpha) = \frac{\cos\zeta \sin^2\delta \cos\alpha + \sin\zeta \cos^2\delta \sin\alpha}{u(\zeta)}. \tag{4.98}$$

It is now only required to choose the unit periodic $Q_1(\alpha)$ to eliminate the pole at $\alpha = \zeta$ and the zero at $\alpha = -\zeta$ of the term in curly braces in (4.93), leading to the choice

$$Q_1(\alpha) = \frac{\tan\frac{\zeta}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2} + \tan\frac{\alpha}{2}} \tag{4.99}$$

which is $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$, as required. The solution is therefore

$$t_1(\alpha) = \frac{\tan\frac{\zeta}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2} + \tan\frac{\alpha}{2}} \left\{\left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right) w(\alpha, u) + \left(1 - \frac{r_1(\alpha)}{u(\alpha)}\right) w(\alpha, -u)\right\} \tag{4.100}$$

with $r_1(\alpha)$ given in (4.98). This is free of poles and zeros in $S_{2\pi}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. Since $r_1(\alpha) \to \cos\alpha$ and $u(\alpha) \to \cos\alpha$ as $\delta \to \pi/2$, then

$$t_1(\alpha) \xrightarrow{\delta\to\pi/2} Q_1(\alpha)\left\{\left(1+\frac{\cos\alpha}{\cos\alpha}\right)\frac{\Psi_1(\alpha)}{Q_1(\alpha)}+\left(1-\frac{\cos\alpha}{\cos\alpha}\right)\frac{Q_1(\alpha)}{\Psi_1(\alpha)}\right\} = \Psi_1(\alpha) \tag{4.101}$$

where (4.82) and (4.83) (together with (4.99)) have also been used. The solution $t_1(\alpha)$ therefore recovers $\Psi_1(\alpha)$, given in (4.14), the known solution when $\delta = \pi/2$. The other limit of interest where $\delta \to 0$ is, as mentioned in the previous section, much less obvious. Despite this there are a number of similarities with the $\delta \to \pi/2$ case. Indeed, the factors $1 \pm r_1(\alpha)/u(\alpha)$ once again assume limits of 0 or 2 depending on the branch of $u(\alpha)$, as before. However, in this instance the limiting value is also a function of α. Indeed, now $r_1(\alpha) \to \sin\zeta\sin\alpha/u(\zeta)$ so that,

$$1-\frac{r_1(\alpha)}{u(\alpha)} \xrightarrow{\delta\to0} 1-\frac{\sin\zeta\sin\alpha}{u(\zeta)u(\alpha)} = 1\pm1 \tag{4.102}$$

depending on *both* the value of α and ζ. Assuming ζ fixed, which is certainly the case in the context of a given problem, then $r_1(\alpha)/u(\alpha)$ will go to 1 on the upper (lower) half of the plane, and -1 on the lower (upper) half of the plane. This relatively complicated behavior does have its benefits. Indeed, since the $u(\alpha)$ terms appearing in the definition of $w(\alpha, u)$ will have the same behavior, it implies that, within the framework of the branch-free solutions, it will be allowable to consider $u(\alpha)$ as behaving as, say, $-\mathrm{j}\sin\alpha$ in that limit. It is however impossible to complete the analysis without determining the limiting value of ζ and this, as discussed in the previous section, is currently unknown. All these results do however suggest that the solution is closely related (if not exactly equal) to the $\Psi_2(\alpha)$ solution. This is supported by numerical computations which suggest

$$t_1(\alpha) \xrightarrow{\delta\to0} \Psi_2(\alpha). \tag{4.103}$$

The technique was implemented in Fortran 77 and typical results are shown in Figure 4.11 which gives $|t_1(\alpha)|$ computed along the real axis for a particular θ and a variety of δ. We observe that $t_1(\alpha) \to \Psi_1(\alpha)$ as $\delta \to \pi/2$, and to $\Psi_2(\alpha)$ as $\delta \to 0$. As required, the solution has continuous real and imaginary parts with the corresponding plots given in Figure 4.12.

A second solution obtained by following a similar process is

$$t_2(\alpha) = Q_2(\alpha)\left\{\left(1-\frac{r_2(\alpha)}{u(\alpha)}\right)w(\alpha,u)+\left(1+\frac{r_2(\alpha)}{u(\alpha)}\right)w(\alpha,-u)\right\}, \tag{4.104}$$

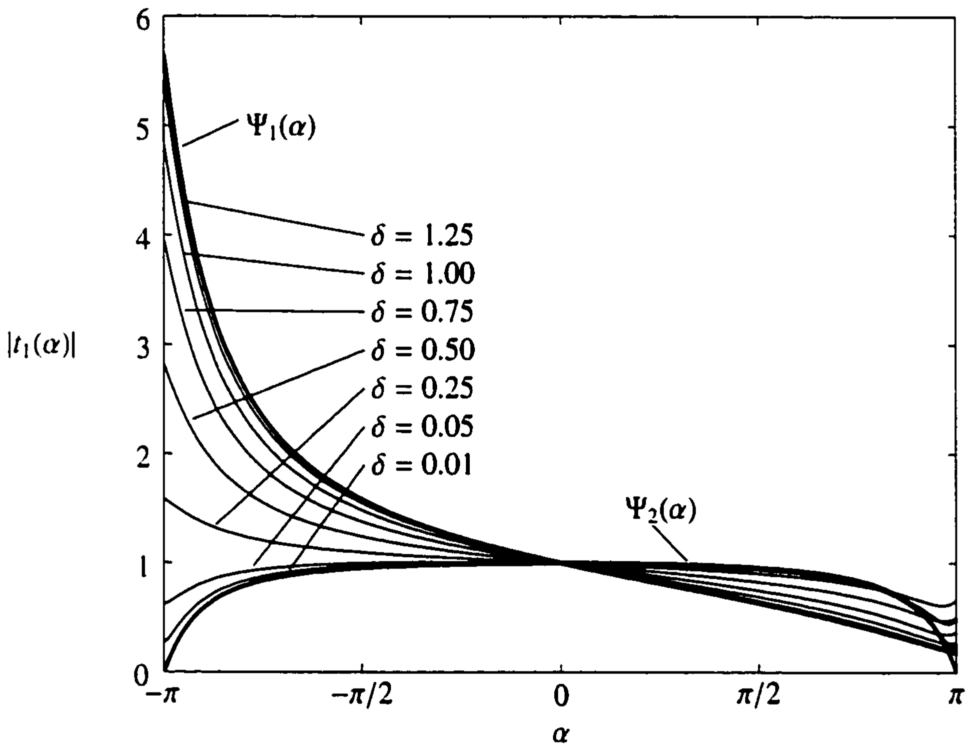
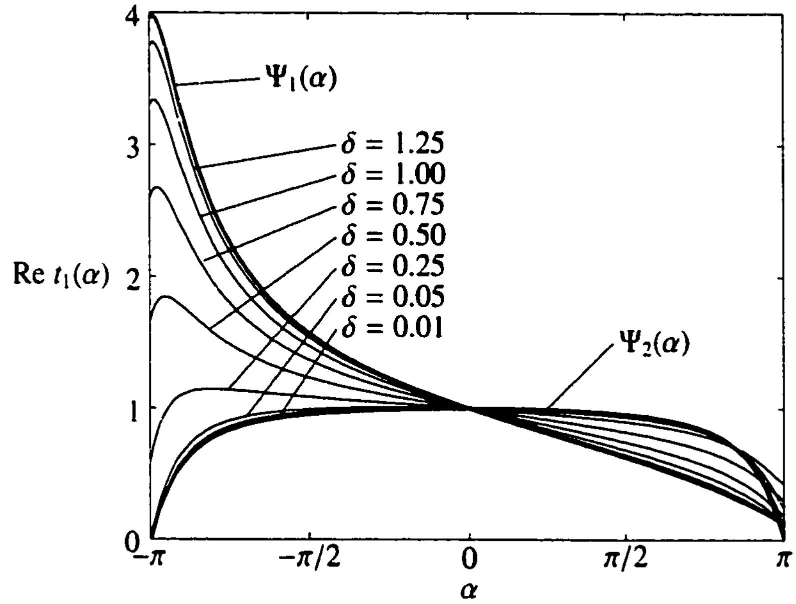
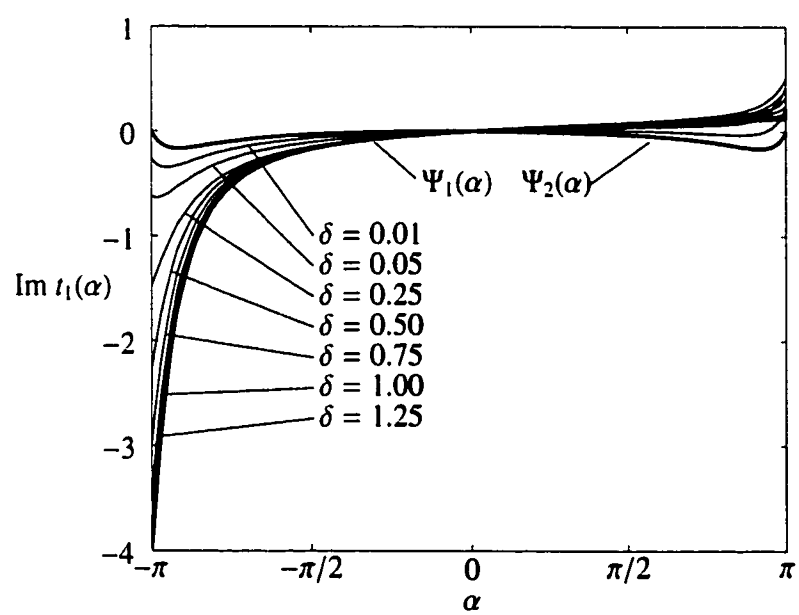


Figure 4.11: Magnitude of the branch free solution $t_1(\alpha)$ given in (4.100) when $\theta = 0.25(1 + j)$ for various values of δ. The thicker lines correspond to the known limiting functions $\Psi_1(\alpha)$ per (4.12) for $\delta = \pi/2$ and $\Psi_2(\alpha)$ per (4.18) for $\delta = 0$.



(a)



(b)

Figure 4.12: The (a) real and (b) imaginary part of the branch free solution $t_1(\alpha)$ given in (4.100) when $\theta = 0.25(1 + \mathrm{j})$ for various values of δ. The thicker lines correspond to the known limiting functions $\Psi_1(\alpha)$ for $\delta = \pi/2$ and $\Psi_2(\alpha)$ for $\delta = 0$.

and the elimination of the pole at $\alpha = \zeta$ is now sought instead. This necessitates

$$1 - \frac{r_2(\alpha)}{u(\alpha)} \sim (\alpha - \zeta)^2 \tag{4.105}$$

and the first term in braces in (4.104) will therefore now have a simple zero at $\alpha = \zeta$ coincident with that of $w(\alpha, -u)$ and the only pole is that of $w(\alpha, -u)$ at $\alpha = -\zeta$. Once again, write

$$r_2(\alpha) = \varrho_0 + \varrho_1 \cos\alpha + \varrho_2 \sin\alpha, \tag{4.106}$$

and enforcing the requirement (4.105) with $\varrho_0 = 0$ leads to

$$\begin{aligned} \varrho_1 &= \frac{\cos\zeta \sin^2\delta}{u(\alpha)}, \\ \varrho_2 &= -\frac{\sin\zeta \cos^2\delta}{u(\alpha)}, \end{aligned} \tag{4.107}$$

so that

$$r_2(\alpha) = \frac{\cos\zeta \sin^2\delta \cos\alpha - \sin\zeta \cos^2\delta \sin\alpha}{u(\zeta)}. \tag{4.108}$$

Elimination of the pole at $\alpha = \zeta$ and the zero at $\alpha = -\zeta$ in (4.104) leads to

$$Q_2(\alpha) = \frac{\tan\frac{\zeta}{2} + \tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2} - \tan\frac{\alpha}{2}} = \frac{1}{Q_1(\alpha)}, \tag{4.109}$$

and

$$t_2(\alpha) = \frac{\tan\frac{\zeta}{2} + \tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2} - \tan\frac{\alpha}{2}} \left\{ \left(1 - \frac{r_2(\alpha)}{u(\alpha)}\right) w(\alpha, u) + \left(1 + \frac{r_2(\alpha)}{u(\alpha)}\right) w(\alpha, -u) \right\} \tag{4.110}$$

with $r_2(\alpha)$ given in (4.108). This is free of poles and zeros in $S_{2\pi}$ and $O(1)$ as $|\text{Im}\,\alpha| \to \infty$. Notice however that, since $w(-\alpha, u) = w(\alpha, -u)$, $Q_1(-\alpha) = Q_2(\alpha)$ and $r_1(-\alpha) = r_2(\alpha)$,

$$t_2(\alpha) = t_1(-\alpha); \tag{4.111}$$

the solution $t_2(\alpha)$ is the reflection across the origin of $t_1(\alpha)$. Accordingly,

$$t_2(\alpha) \xrightarrow{\delta \to \pi/2} \frac{1}{\Psi_1(\alpha)}, \tag{4.112}$$

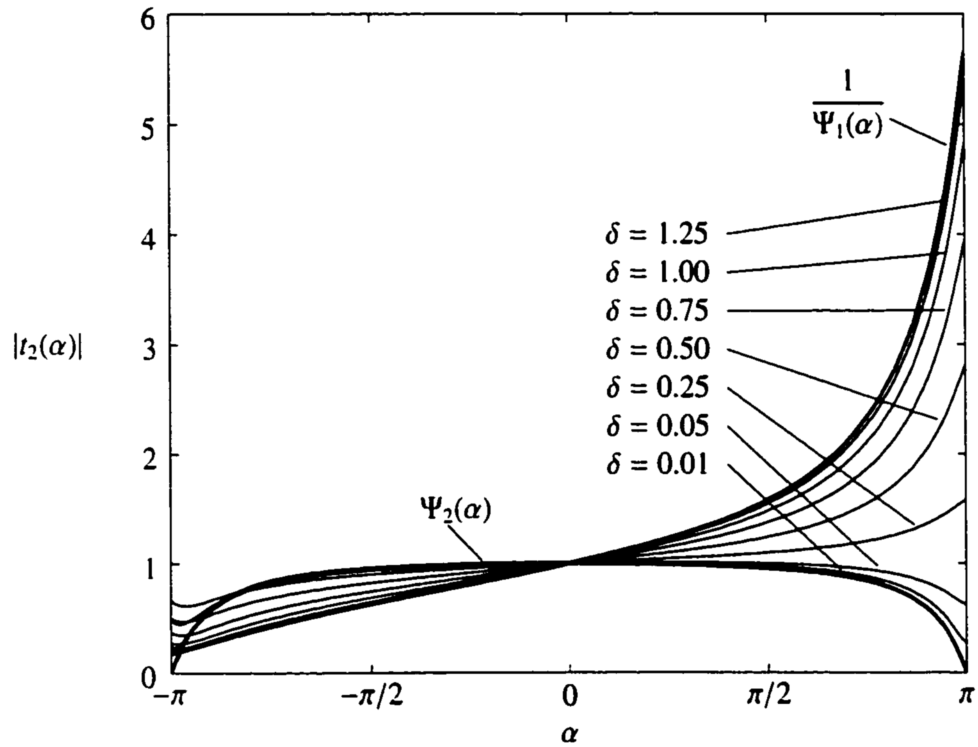


Figure 4.13: Magnitude of the branch free solution $t_2(\alpha)$ given in (4.110) when θ = 0.25(1 + j) for various values of δ. The thicker lines correspond to the known limiting functions $1/\Psi_1(\alpha)$ for $\delta = \pi/2$ and $\Psi_2(\alpha)$ for $\delta = 0$.

whereas, since $t_1(\alpha)$ is even as $\delta \to 0$, $t_2(\alpha)$ shares this limit with $t_1(\alpha)$ and

$$t_2(\alpha) \xrightarrow{\delta \to 0} \Psi_2(\alpha). \tag{4.113}$$

However, a second solution can be obtained in that limit and it is given by $1/\Psi_2(\alpha)$. The behavior of $t_2(\alpha)$ is shown in Figure 4.13, an expected result given (4.111) and the behavior of $t_1(\alpha)$ shown in Figure 4.11. We have therefore succeeded in constructing a pair of branch-free solutions to the second order difference equations (4.1). They have the desired analyticity properties and recover known solutions in the limits when the branch points vanish or, equivalently, when the second order equation trivially factors into a pair of uncoupled meromorphic first order equations.

4.4 Generalizations

A measure of the complexity of the equations under study is provided by the width of the strip of analyticity and the number of singularities, both poles and zeros, found therein.

The configuration of concern throughout this chapter is a strip of width 2π containing four poles, due to $v_0(\alpha, u)$, and four branch points, as pictured in Figure 4.1. This is significantly simpler than the topology associated with the anisotropic half-plane illustrated in Figure 2.4 which has width 4π and contains no less than 16 poles (on each sheet) as well as 16 branch points. To gain insight into how to attack the problem of the half-plane, two variations are examined. The first is for the case of a strip of width 2π where the number of singularities populating the strip is doubled to eight. This is, in essence, the case of the anisotropic half-plane with a strip of analyticity reduced from 4π to 2π. For the second case, the width of the strip of analyticity is doubled to obtain a strip of width 4π enclosing eight poles and eight branch points. In both instances, the equations considered are mathematical extensions of the physical equations.

4.4.1 Generalization I: Doubled number of singularities

We consider a first order equation of the form

$$\frac{w(\alpha+\pi,u)}{w(\alpha-\pi,u)} = \frac{u(\alpha) \mp \gamma \sin\alpha}{u(\alpha) \pm \gamma\sin\alpha} \tag{4.114}$$

where $\gamma \in \mathbb{C}$ and

$$u(\alpha) = \sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)}. \tag{4.115}$$

This is the same as equation (2.57) obtained in the case of the anisotropic half-plane but has a reduced period 2π. The cuts of the branched function $u(\alpha)$ are such that it is even symmetric and

$$u(\alpha) = u(-\alpha) = u(\alpha+\pi). \tag{4.116}$$

The resulting configuration is as shown in Figure 4.14. We proceed by taking the logarithmic derivative of (4.114) and since

$$\frac{d}{d\alpha}\ln g(\alpha,u) = \frac{k(\alpha)}{u(\alpha)}\left(\frac{2\sin\theta_1\sin\alpha}{\sin^2\alpha-\sin^2\theta_1} - \frac{2\sin\theta_2\sin\alpha}{\sin^2\alpha-\sin^2\theta_2}\right) \tag{4.117}$$

where

$$k(\alpha) = \gamma\frac{\sin\theta_1\sin\theta_2 - \sin^2\alpha}{\sin\theta_1 - \sin\theta_2}, \tag{4.118}$$

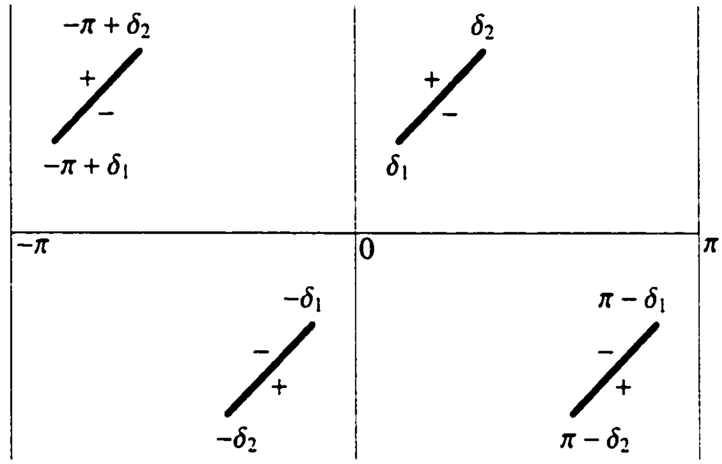


Figure 4.14: The configuration of the strip of analyticity of width 2π in the case of the first generalization. This is, in essence, the configuration shown in Figure 2.4.

a solution of the form

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_{2\pi}(\xi, u) \right\} d\xi \tag{4.119}$$

is sought where

$$v_0(\alpha, u) = -\frac{\alpha}{2\pi} \frac{d}{d\alpha} \ln g(\alpha, u) = -\frac{\alpha}{2\pi} \frac{k(\alpha)}{u(\alpha)} \left(\frac{2 \sin\theta_1 \sin\alpha}{\sin^2\alpha - \sin^2\theta_1} - \frac{2 \sin\theta_2 \sin\alpha}{\sin^2\alpha - \sin^2\theta_2} \right). \tag{4.120}$$

This has poles at $\alpha = \pm\theta_{1,2}, \pm(\pi - \theta_{1,2})$. To eliminate them, choose

$$v_1(\alpha, u) = -\frac{k(\alpha)}{u(\alpha)} \left(\frac{f(\alpha, \theta_1)}{\sin^2\alpha - \sin^2\theta_1} - \frac{f(\alpha, \theta_2)}{\sin^2\alpha - \sin^2\theta_2} \right) \tag{4.121}$$

where, to maintain the required odd parity,

$$f(\alpha, \theta) = \left(\frac{1}{2} - \frac{\theta}{\pi} \right) \cos\theta \sin\alpha - \frac{1}{2} \sin\alpha \cos\alpha. \tag{4.122}$$

It is observed that while the individual members of (4.121) are non-vanishing as $|\mathrm{Im}\,\alpha| \to \infty$, the difference of the two does since they tend to the same limit. Summing up $v_0(\alpha, u)$

and $v_1(\alpha, u)$, we have

$$v_0(\alpha, u) + v_1(\alpha, u) = \frac{k(\alpha)}{u(\alpha)} \sum_{n=1}^{2} \frac{(-1)^n \frac{\alpha}{\pi} \sin\theta_n \cos\alpha - \frac{\theta_n}{\pi} \cos\theta_n \sin\alpha + \frac{1}{2}(\cos\theta_n - \cos\alpha)\sin\alpha}{\sin^2\alpha - \sin^2\theta_n}. \quad (4.123)$$

In the limit $\theta_1 = \theta_2$, (4.114) becomes

$$\frac{w(\alpha + \pi)}{w(\alpha - \pi)} = \frac{\sin\alpha + \sin\theta_1}{\sin\alpha - \sin\theta_1} \frac{\sin\alpha - \sin\theta_2}{\sin\alpha + \sin\theta_2} \quad (4.124)$$

which, in terms of Maliuzhinets functions, has a solution

$$w(\alpha) = \frac{\Psi_3(\alpha, \theta_1)}{\Psi_3(\alpha, \theta_2)} \quad (4.125)$$

with

$$\Psi_3(\alpha, \theta) = \frac{\psi_{\pi/2}\left(\alpha + \frac{\pi}{2} - \theta\right) \psi_{\pi/2}\left(\alpha - \frac{\pi}{2} + \theta\right)}{\psi_{\pi/2}^2\left(\frac{\pi}{2} - \theta\right)}. \quad (4.126)$$

Comparison with (4.18) and (4.21) shows

$$v_0(\alpha, u) + v_1(\alpha, u) = \frac{k(\alpha)}{u(\alpha)} \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \frac{\Psi_3(\alpha, \theta_1)}{\Psi_3(\alpha, \theta_2)} \quad (4.127)$$

and since $k(\alpha)/u(\alpha) \to 1$ as $\theta_1 \to \theta_2$, the known solution is then recovered.

The elimination of the cyclic periods requires, at first glance at least, three degrees of freedom since we are dealing with functions having an odd parity. The contribution from $-\delta_1$ to δ_1 therefore vanishes, leaving us to deal with the elimination of the branch point to branch point contributions (see Figure 4.14) from δ_1 to δ_2, δ_2 to $\pi - \delta_2$ and $\pi - \delta_1$ to $\pi - \delta_2$. The requirement for an odd parity requires the use of elliptic integrands of form

$$v_{2\pi}^1(\alpha, u) = \frac{\sin\alpha}{u(\alpha)}, \quad v_{2\pi}^3(\alpha, u) = \frac{u(\zeta)}{u(\alpha)} \frac{k(\alpha)}{k(\zeta)} \frac{2\cos\zeta \sin\alpha}{\sin^2\alpha - \sin^2\zeta} \quad (4.128)$$

which provide two degrees of freedom: one from a multiplicative constant associated with $v_{2\pi}^1(\alpha, u)$, and one from the pole ζ of $v_{2\pi}^3(\alpha, u)$. Interestingly enough, it turns out that two degrees of freedom are sufficient to annul all of the cyclic periods due to the symmetry properties of the the expressions involved. It can be shown that if

$$\int_{\delta_1}^{\delta_2} \left\{ v_0(\xi, u) + v_1(\xi, u) + \kappa v_{2\pi}^1(\xi, u) + v_{2\pi}^3(\xi, u) \right\} \mathrm{d}\xi = 0 \tag{4.129}$$

then

$$\int_{\pi-\delta_1}^{\pi-\delta_2} \left\{ v_0(\xi, u) + v_1(\xi, u) + \kappa v_{2\pi}^1(\xi, u) + v_{2\pi}^3(\xi, u) \right\} \mathrm{d}\xi =$$

$$- \int_{\delta_1}^{\delta_2} \left\{ \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\xi} \ln g(\xi, u) + \frac{k(\xi)}{u(\xi)} \sum_{n=1}^{2} (-1)^n \frac{\sin \xi \cos \xi}{\sin^2 \xi - \sin^2 \theta} \right\} \mathrm{d}\xi = 0. \tag{4.130}$$

It is therefore sufficient to enforce (4.129) as well as

$$\int_{\delta_1}^{\pi-\delta_1} \left\{ v_0(\xi, u) + v_1(\xi, u) + \kappa v_{2\pi}^1(\xi, u) + v_{2\pi}^3(\xi, u) \right\} \mathrm{d}\xi = 0 \tag{4.131}$$

in order to eliminate all three cyclic periods in the strip of analyticity. This is the same level of complexity as for the problem of the penetrable wedge which also required only two degrees of freedom.

While the elimination of the cyclic periods is apparently straightforward, the construction of branch-free solutions will be more complicated than for the more elementary case solved earlier. Indeed, the term associated with the integral of the third kind now has double the number of poles due to the $\sin^2 \alpha - \sin^2 \zeta$ term in the denominator required to obtain the desired symmetry.

4.4.2 Generalization II: Doubled periodicity

We now consider a more direct extension of the equation solved previously. The increase in complexity now comes from doubling the periodicity of equation (4.6a),

$$\frac{w(\alpha + 2\pi, u)}{w(\alpha - 2\pi, u)} = g(\alpha, u) = \frac{u(\alpha) - u(\theta)}{u(\alpha) + u(\theta)}, \tag{4.132}$$

with, as in (4.4),

$$u(\alpha) = \sqrt{\cos^2 \alpha - \cos^2 \delta}, \tag{4.133}$$

and is defined such that $u(\alpha) = u(\alpha + 2\pi) = u(-\alpha) = -u(\alpha + \pi)$. This yields the familiar configuration found in Figure 4.1 but extended to a 4π strip as shown in Figure 4.15. Initially at least, the construction of the solution follows closely the analysis presented earlier in this

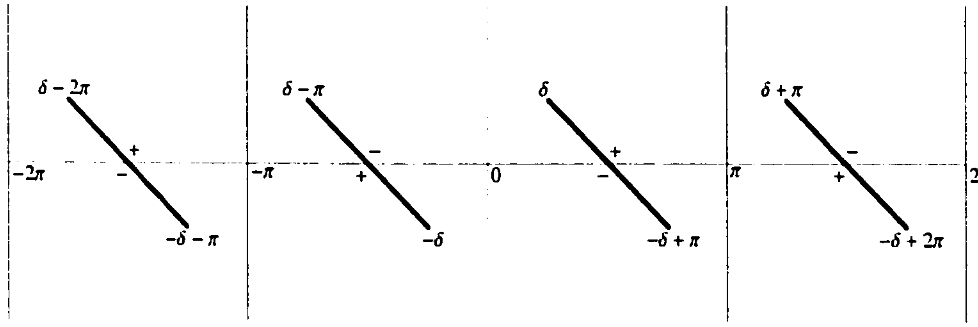


Figure 4.15: The configuration of the strip of analyticity of width 4π in the case of the second generalization. This is, in essence, the configuration shown in Figure 4.1 with the period doubled.

chapter. Application of a logarithmic derivative to obtain $v_0(\alpha, u)$ is straightforward and so is the elimination of the polar periods. However, the elimination of the cyclic periods is considerably more complicated since the analysis must now be carried out on a Riemann surface of genus three. This complication warrants a thorough investigation, and is carried out in the next chapter.

4.5 Summary

The technique developed in Chapter 3 was applied to solve a second order functional difference equation which is a generalized version of the one obtained for the problem of the penetrable composite right-angled wedge. A pair of meromorphic solutions were obtained which are free of poles and zeros in the strip $S_{2\pi}$ and which, as required, are $O(1)$ as $|\text{Im}\,\alpha| \to \infty$. The solutions tend to known solutions in the limits where the second order equations trivially factors into a pair of uncoupled meromorphic first order difference equations. This is shown explicitly for the limit $\delta \to \pi/2$ and numerically when $\delta \to 0$.

Two generalizations were also examined at the end of the chapter; they correspond to natural mathematical extensions of the problem solved. In the first one, the first order difference equation considered leads to a strip of analyticity of width 2π but with double the number of branch points. This corresponds to the case of the anisotropic impedance half-plane with the period reduced to 2π. It was shown that the process of cyclic period elimination is of the same complexity as the one for the equation of the penetrable wedge. The second generalization doubles the period of the equation for the penetrable wedge. This effectively doubles the width of the strip of analyticity and the number of singularities that must be addressed. The elimination of cyclic periods is now more complicated and is the focus of the next chapter.

CHAPTER 5

Mathematical Extension of the Penetrable Wedge: Equation with Period 4π

SOLUTIONS of the second order equation for the problem of a penetrable composite right-angled wedge were derived in the previous chapter. A mathematical extension of that equation, obtained by doubling its period from 2π to 4π, is now considered. This is the second generalization mentioned at the end of the last chapter and it provides a test of the applicability of the method in scenarios of increased complexity. Whereas the analysis up to and including the elimination of the polar periods follows closely that of the original equation, significant differences arise in the elimination of the cyclic periods since part of the analysis must now be carried out on a Riemann surface of genus three. Furthermore, the scarcity of degrees of freedom makes the construction of meromorphic solutions having the desiring analyticity properties difficult, and both analytical and partially numerical approaches are presented.

5.1 The difference equations

The second order functional difference equation is the same as the one for the penetrable wedge (4.1) save that its period is now doubled to 4π, *viz.*

$$t(\alpha + 5\pi) - 2\left\{1 - 2\frac{\cos^2\delta - \cos^2\theta}{\cos^2\alpha - \cos^2\theta}\right\} t(\alpha + \pi) + t(\alpha - 3\pi) = 0. \tag{5.1}$$

A pair of meromorphic solutions free of poles and zeros in the strip $S_{4\pi} = \{\alpha : |\mathrm{Re}\,\alpha| \le 2\pi\}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$ is now sought. Recall that, within the scope of the generalized equation, the parameters θ and δ are allowed to vary independently — see page 26 for more details. Factoring the difference operator, as done in Section 2.1.2, produces the expected

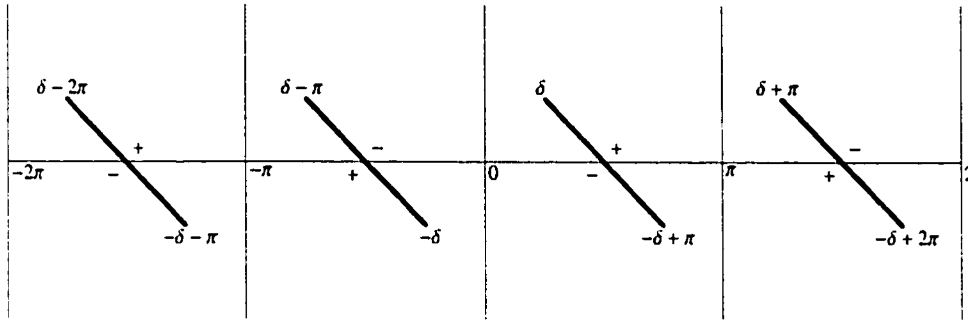


Figure 5.1: The strip of analyticity $S_{4\pi} = \{\alpha : |\text{Re } \alpha| \leq 2\pi\}$. The thick lines indicate the branch cuts of $u(\alpha)$; the positive and negative signs indicate relative changes in sign of $u(\alpha)$ across the different cuts.

first order equation pair

$$\frac{w(\alpha + 2\pi, +u)}{w(\alpha - 2\pi, +u)} = g(\alpha, +u) = \frac{u(\alpha) - u(\theta)}{u(\alpha) + u(\theta)}, \tag{5.2a}$$

$$\frac{w(\alpha + 2\pi, -u)}{w(\alpha - 2\pi, -u)} = g(\alpha, -u) = \frac{u(\alpha) + u(\theta)}{u(\alpha) - u(\theta)}, \tag{5.2b}$$

of period 4π where

$$u(\alpha) = \sqrt{\cos^2 \alpha - \cos^2 \delta}. \tag{5.3}$$

The same definition for $u(\alpha)$ is kept: the branches of $u(\alpha)$ are chosen to have the same symmetry as $\cos \alpha$, hence $u(\alpha) = u(-\alpha) = -u(\alpha + \pi)$. The resulting 4π wide strip of analyticity is illustrated in Figure 5.1. Comparison with Figure 3.3 shows, as expected, that the strip of analyticity has been doubled and now contains eight branch points at $\alpha = \pm\delta, \pm(\delta - \pi), \pm(\delta + \pi), \pm(-\delta + 2\pi)$. From the symmetry of the equation pair, it is sufficient to restrict our attention to (5.2a) and, as before, we construct $w(\alpha, u)$ such that $w(-\alpha, u) = w(\alpha, -u) = 1/w(\alpha, u)$.

5.1.1 Limiting cases of interest

The limiting cases of interest are those where the branch points vanish, they are the same as those in Chapter 4 since $u(\alpha)$ remains unchanged. These correspond to $\delta \to \pi/2$ when $u(\alpha) \to \cos \alpha$ and $\delta \to 0$ when $u(\alpha) \to \mathrm{j} \sin \alpha$. In both instances the second order difference equations (4.1) can be factored by inspection into a pair of first order difference equations whose right-hand sides are meromorphic rational functions of trigonometric polynomials. Their solutions are readily available and expressed in terms of Maliuzhinets functions [Maliuzhinets, 1958*a*].

5.1.1.1 The limit $\delta \to \pi/2$

In this instance $u(\alpha) \to \cos\alpha$ and the first order equation (5.2a) becomes

$$\frac{w(\alpha+2\pi,u)}{w(\alpha-2\pi,u)} = \frac{\cos\alpha-\cos\theta}{\cos\alpha+\cos\theta} \tag{5.4}$$

and since (5.2b) is the reciprocal of (5.4), its solution in the limit is $1/w(\alpha,u)$. From Section 4.1, it follows that

$$w(\alpha,u) = \Psi_4(\alpha) = \frac{\psi_\pi\left(\alpha+\frac{\pi}{2}-\theta\right)}{\psi_\pi\left(\alpha-\frac{\pi}{2}+\theta\right)} \tag{5.5}$$

which is meromorphic, free of poles and zeros in $S_{4\pi}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. This should be compared with $\Psi_1(\alpha)$ in (4.12) which has the same form but involves the $\psi_{\pi/2}(\alpha)$ function. An integral representation is obtained by making use of (see [Maliuzhinets, 1958*a*] and, for corrections, [Senior and Volakis, 1995])

$$\psi_\pi(\alpha) = \exp -\frac{1}{8\pi}\int_0^\alpha \frac{\pi\sin\xi - 2\sqrt{2}\pi\sin\frac{\xi}{2} + 2\xi}{\cos\xi}\mathrm{d}\xi. \tag{5.6}$$

Substitution in (5.5) provides, after manipulation,

$$\Psi_4(\alpha) = \exp\int_0^\alpha \frac{-\frac{\xi}{2\pi}\cos\theta\sin\xi + \varphi_1 + \varphi_2\cos\frac{\xi}{2} + \varphi_3\cos\xi + \varphi_4\cos\frac{3\alpha}{2}}{\cos^2\xi - \cos^2\theta}\mathrm{d}\xi \tag{5.7}$$

where

$$\begin{aligned}
\varphi_1 &= -\frac{1}{4}\sin\theta\cos\theta,\\
\varphi_2 &= \frac{1}{4}\left(-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right) + \frac{1}{2}\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right),\\
\varphi_3 &= \left(\frac{\theta}{2\pi}-\frac{1}{4}\right)\sin\theta,\\
\varphi_4 &= \frac{1}{4}\left(-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right).
\end{aligned} \tag{5.8}$$

As for the case of the 2π periodic equation considered in the previous chapter, the above will enable us to demonstrate that the general solution recovers the limiting functions $\Psi_4(\alpha)$ and $1/\Psi_4(\alpha)$ identically as $\delta \to \pi/2$.

5.1.1.2 The limit $\delta \to 0$

Now $u(\alpha) \to \mathrm{j}\sin\alpha$ and the first order equation (4.6a) become

$$\frac{w(\alpha + 2\pi, u)}{w(\alpha - 2\pi, u)} = \frac{\sin\alpha - \sin\theta}{\sin\alpha + \sin\theta} \tag{5.9}$$

and once again the solution to (5.2b) is provided by $1/w(\alpha, u)$. From the material in Section 4.1.1, it follows that

$$w(\alpha, u) = \Psi_5(\alpha) = \cos\frac{\alpha}{4}\,\frac{\psi_\pi^2\left(\frac{\pi}{2} - \theta\right)}{\psi_\pi\left(\alpha + \frac{\pi}{2} - \theta\right)\psi_\pi\left(\alpha - \frac{\pi}{2} + \theta\right)} \tag{5.10}$$

which is meromorphic, free of poles and zeros in the *open* strip $S_{4\pi}$ and $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. It invites comparison with (4.18) which is the corresponding solution when the period is 2π. Since

$$\frac{\psi_\pi\left(\alpha - 2\pi + \frac{\pi}{2} - \theta\right)\psi_\pi\left(\alpha - 2\pi - \frac{\pi}{2} + \theta\right)}{\psi_\pi\left(\alpha + 2\pi + \frac{\pi}{2} - \theta\right)\psi_\pi\left(\alpha + 2\pi - \frac{\pi}{2} + \theta\right)} = -\frac{\sin\alpha - \sin\theta}{\sin\alpha + \sin\theta}, \tag{5.11}$$

the $\cos\alpha/4$ term provides the required change in sign and also exactly cancels the growth of the denominator. However, it inevitably leads to zeros (Ψ_5) or poles ($1/\Psi_5$) at $\alpha = \pm 2\pi$ on the edges of the strip. Based on the properties of the Maliuzhinets functions, it is apparently impossible to construct a solution to (5.9) that is $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$ and simultaneously free of poles and zeros in the closed strip $S_{4\pi}$.

5.2 General branched solution

The approach followed is the same as in Chapter 4 though it must now be adapted to the strip $S_{4\pi}$ instead of $S_{2\pi}$ and this will have ramifications on the logarithmic derivative as well as the elimination of the polar periods. Furthermore, the requirement for vanishing cyclic periods now leads to a system of four equations in four unknowns which can be decoupled into two systems of two equations in two unknowns.

5.2.1 The logarithmic derivative

Applying a logarithmic derivative to the first order equation (5.2a) results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha+2\pi,u)-\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln w(\alpha-2\pi,u)&=\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u)\\&=-\frac{u(\theta)}{u(\alpha)}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\end{aligned}\tag{5.12}$$

which is exactly the same as in (4.22) save for the period occurring in the left-hand side. This distinction will alter the form of $v_0(\alpha,u)$. If we let $v_0(\alpha,u) = \mathrm{d}/\mathrm{d}\alpha \ln w(\alpha,u)$ and proceed as in Section 3.2.1, we obtain

$$v_0(\alpha,u)=\frac{\alpha}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g(\alpha,u)=-\frac{\alpha}{2\pi}\frac{u(\theta)}{u(\alpha)}\frac{\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}\tag{5.13}$$

and this differs by a minus sign from the corresponding expression (4.26) obtained in the previous chapter. A tentative solution to (5.2a)

$$w(\alpha,u)=\exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi,u)+v_{4\pi}(\xi,u)\right\}\mathrm{d}\xi,\tag{5.14}$$

immediately follows where $v_{4\pi}(\alpha,u)$ is a yet undefined 4π periodic function which eliminates the periods of $v_0(\alpha,u)$ to obtain a well-defined path integral. The function $v_0(\alpha,u)$ is even, aperiodic and vanishes as $|\mathrm{Im}\,\alpha| \to \infty$. Within the strip $S_{4\pi}$ it has branch points commensurate with those of $1/u(\alpha)$ and poles at the zeros of $\cos^2\alpha - \cos^2\theta$. The function $v_{4\pi}(\alpha,u)$ now consists of five functions: $v_1(\alpha,u)$ to eliminate the polar periods, and $v^1_{2\pi}(\alpha,u)$, $v^3_{2\pi}(\alpha,u)$, $v^1_{4\pi}(\alpha,u)$, $v^3_{4\pi}(\alpha,u)$ to eliminate the four cyclic periods that arise. This should be contrasted to the solution in the last chapter where only two functions were required to annul the cyclic periods. At the end of Section 4.2.1, it was showed that examination of the limit as $|\mathrm{Im}\,\alpha| \to \infty$ was sufficient to dismiss uncertainty with respect to what is seemingly an arbitrary coefficient when taking the logarithmic derivative. The same analysis shows that (5.14) indeed satisfies (5.2a).

5.2.2 Elimination of polar periods

In addition to the branch points of $u(\alpha)$ the function $v_0(\alpha,u)$ has poles in the strip $S_{4\pi}$ at the zeros of $\cos^2\alpha - \cos^2\theta$ at $\alpha = \pm\theta, \pm(\pi-\theta), \pm(\pi+\theta), \pm(2\pi-\theta)$. The poles have residues

$$\mathrm{Res}\; v_0(\alpha,u)=\frac{\alpha}{4\pi}\frac{u(\theta)}{u(\alpha)}\tag{5.15}$$

which, being generally non-integer, jeopardize the single-valuedness of the path integral in (5.14). Of course, since a solution that is free of poles is sought, the mere presence of poles, regardless of their residue, warrants their elimination. The 4π period function $v_1(\alpha, u)$ introduced to eliminate the poles must have the same even parity as $v_0(\alpha, u)$, vanish as $|\text{Im}\,\alpha| \to \infty$ and provide the four required degrees of freedom to eliminate the offending poles. In this instance we are faced with no less than four possible combinations. Indeed, the function $v_1(\alpha, u)$ will generally have the form

$$v_1(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{1}{\cos^2\alpha - \cos^2\theta} \sum_{n=0}^{5} \varphi_n \cos\frac{n\alpha}{2} \tag{5.16}$$

which is even and vanishes as $|\text{Im}\,\alpha| \to \infty$. We require, however, only four degrees of freedom as opposed to the six provided by the constants φ_n in (5.16). Examination of the $\cos n\alpha/2$ terms at the θ poles show that the φ_0 and φ_4 terms are dependent since they share the same symmetry at the poles. Similar consideration shows that the φ_1 and the φ_4 terms also form a dependent pair. Since only one member from both pairs is admissible, the general solution will involve one member of $\{\varphi_0, \varphi_4\}$, one member of $\{\varphi_1, \varphi_5\}$, φ_2 and φ_3 for a total of four possible combinations. Though not explicitly demonstrated here, indications are that each of those four possible choices ultimately leads to the same final solution once all cyclic periods have been eliminated. This stems for the fact that we must now introduce two integrals of the first kind in the course of eliminating the cyclic periods as compared to only one in the 2π case. Moreover, when combined with $v_1(\alpha, u)$, the first leads to terms having the same symmetry as the $\{\varphi_0, \varphi_4\}$ pair above whereas the second leads to terms having the same symmetry as the $\{\varphi_1, \varphi_5\}$ terms. To prove the equivalence, one has to show that the combination of $v_1(\alpha, u)$ and the two terms leading to integrals of the first kind is invariant regardless which of the four combinations of φ_n is chosen above. This may be difficult — or tedious — to show generally but is a simple matter when either $\delta \to \pi/2$ or $\delta \to 0$.

In light of our experience for the equation with period 2π, we choose

$$v_1(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{\varphi_0 + \varphi_1 \cos\frac{\alpha}{2} + \varphi_2 \cos\alpha + \varphi_3 \cos\frac{3\alpha}{2}}{\cos^2\alpha - \cos^2\theta} \tag{5.17}$$

where all the terms in the numerator are even and 4π periodic. This has the benefit of remaining free of poles in the limit as $\delta \to \pi/2$ since the two leading coefficients go to unity. A system of four equations in four unknowns is obtained by enforcing Res $\Big\{v_0(\alpha, u)+$

$v_1(\alpha, u)\Big\} = 0$ and identically recovers the coefficients in (5.8). Comparison with (5.7) shows

$$v_0(\alpha, u) + v_1(\alpha, u) = \frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \Psi_4(\alpha), \tag{5.18}$$

which reduces to $\mathrm{d}/\mathrm{d}\alpha \ln \Psi_4(\alpha)$ when $\delta = \pi/2$. Again, in the simpler case where the right-hand side of the logarithmic derivative of the first order equation (5.12) is meromorphic, so that the integrand of $w(\alpha, u)$ in (5.14) is free of cyclic periods, the known solutions expressed in terms of Maliuzhinets can be recovered by following the above procedure of pole elimination. Note that the requirement for a solution free of poles will automatically lead to the above choice of $v_1(\alpha, u)$ since the three other possibilities, when considered on their own, have poles in the limit as $\delta \to \pi/2$. The solution to the first order equation (5.2a) is now

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + v_{4\pi}(\xi, u) \Big\} \mathrm{d}\xi, \tag{5.19}$$

where the sum $v_0(\alpha, u) + v_1(\alpha, u)$ is free of poles in $\mathcal{S}_{4\pi}$. We now turn our attention to the elimination of the multi-valuedness that arises from the presence of the branch points of $u(\alpha)$.

5.2.3 Elimination of cyclic periods

To mitigate the adverse effect of the branch points on the single-valuedness of the path integral in (5.19), we must enforce vanishing branch point to branch point integrals throughout the strip $\mathcal{S}_{4\pi}$, as discussed in 3.2.3. Exploiting the even parity of the integrands, we are faced with eliminating the contributions from the four cycles a, b, c and d shown in Figure 5.2. Unit periodics, *i.e.* 4π periodic functions, providing four degrees of freedom are therefore required and, following our discussion on the behavior of functions as $|\mathrm{Im}\,\alpha| \to \infty$, they must also vanish in that limit. We introduce the following four even 4π periodic terms:

$$v^1_{2\pi}(\alpha, u) = \frac{1}{u(\alpha)}, \qquad v^3_{2\pi}(\alpha, u) = \frac{u(\zeta_{2\pi})}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta_{2\pi}} \frac{\sin\zeta_{2\pi}}{\cos\alpha - \cos\zeta_{2\pi}}, \tag{5.20}$$

and

$$v^1_{4\pi}(\alpha, u) = \frac{\cos\frac{\alpha}{2}}{u(\alpha)}, \qquad v^3_{4\pi}(\alpha, u) = \frac{u(\zeta_{4\pi})}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta_{4\pi}} \frac{1}{2} \frac{\sin\frac{\zeta_{4\pi}}{2}}{\cos\frac{\alpha}{2} - \cos\frac{\zeta_{4\pi}}{2}}. \tag{5.21}$$

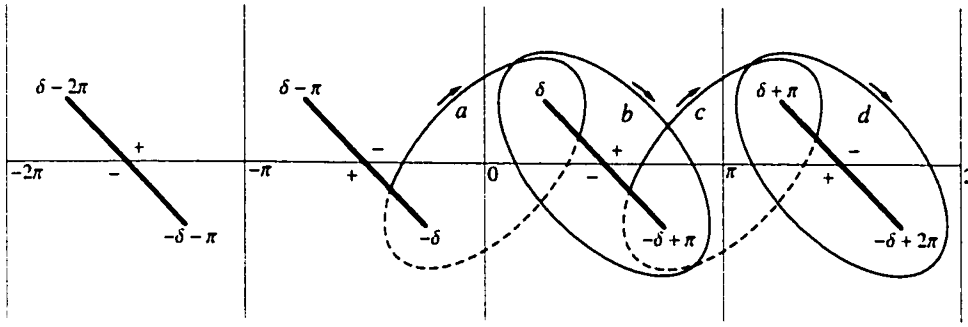


Figure 5.2: The clockwise cyclic paths a, b, c and d used to define the cyclic periods occurring in the strip $S_{4\pi}$. Cycles a and c cross from the upper Riemann sheet (solid line) to the lower Riemann sheet (dashed line) whereas b and d are confined to the upper sheet.

These all give rise to elliptic integrals with the first pair being 2π periodic and the second 4π periodic. The subscript identifies the periodicity of the term while the superscript identifies the type of elliptic integral to which it gives rise. Hence $v^1_{2\pi}(\alpha, u)$ is 2π periodic and gives rise to an elliptic integral of the first kind while $v^3_{2\pi}(\alpha, u)$, also 2π periodic, gives rise to an elliptic integral of the third kind with logarithmic singularities at $\pm\zeta_{2\pi}$, $\pm(\zeta_{2\pi} + 2\pi)$. Likewise, $v^3_{4\pi}(\alpha, u)$ is 4π periodic and gives rise to an elliptic integral of the third kind with logarithmic singularities at $\pm\zeta_{4\pi}$. The use of expressions associated with integrals of the third kind — with poles having non-vanishing residues — results from the impossibility of introducing the required number of degrees of freedom without violating the order requirement. It must be emphasized that the poles of both $v^3_{2\pi}(\alpha, u)$ and $v^3_{4\pi}(\alpha, u)$ have residues ± 1 and their polar periods therefore do not disrupt the single-valuedness of the path integral. For future reference we define the cyclic periods

$$\begin{aligned} A^{1,3}_{2\pi,4\pi} &= \int_a v^{1,3}_{2\pi,4\pi}(\alpha, u)\mathrm{d}\alpha, & B^{1,3}_{2\pi,4\pi} &= \int_b v^{1,3}_{2\pi,4\pi}(\alpha, u)\mathrm{d}\alpha, \\ C^{1,3}_{2\pi,4\pi} &= \int_c v^{1,3}_{2\pi,4\pi}(\alpha, u)\mathrm{d}\alpha, & D^{1,3}_{2\pi,4\pi} &= \int_d v^{1,3}_{2\pi,4\pi}(\alpha, u)\mathrm{d}\alpha, \end{aligned} \tag{5.22}$$

and, as in Chapter 4,

$$A_{0+1} = \int_a \left\{v_0(\alpha, u) + v_1(\alpha, u)\right\}\mathrm{d}\alpha \tag{5.23}$$

with similar definitions on cycles b, c and d. Inspection of (5.20) and (5.21) reveals that the periods associated with the integrals of the third kind are functions of the poles $\zeta_{2\pi}$ and $\zeta_{4\pi}$, providing two of the four degrees of freedoms required to annul the periods of $v_0(\alpha, u) + v_{4\pi}(\alpha, u)$. In contrast, the periods associated with the integrals of the first kind are constant and two multiplicative constants, $\kappa_{2\pi}$ and $\kappa_{4\pi}$, must be introduced to produce the two additional degrees of freedom. The solution to (5.2a) then takes the form

$$w(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + \kappa_{2\pi} v^1_{2\pi}(\xi, u) + \sigma_{2\pi} v^3_{2\pi}(\xi, u) \\ + \kappa_{4\pi} v^1_{4\pi}(\xi, u) + \sigma_{4\pi} v^3_{4\pi}(\xi, u) \Big\} \mathrm{d}\xi \quad (5.24)$$

where the four unknowns to be determined are $\kappa_{2\pi}, \zeta_{2\pi}$ and $\kappa_{4\pi}, \zeta_{4\pi}$. The quantities $\sigma_{2\pi} = \pm 1$ and $\sigma_{4\pi} = \pm 1$ have been introduced to avoid loss of generality in the definition of the terms associated with the integrals of the third kind. They account for the eventuality where the sign of the residues of $v^3_{2\pi}(\alpha, u)$ or $v^3_{4\pi}(\alpha, u)$ must be changed, thereby swapping poles and zeros of $w(\alpha, u)$ between the two Riemann sheets. Their proper definition will be determined in the course of the analysis and, to reduce clutter, they will be omitted in what follows pending their reintroduction when appropriate.

An equation system consisting of four equations in the four unknowns is obtained by enforcing vanishing cyclic periods on the cycles a, b, c and d. Doing so for the cycle d, for example, leads to

$$\int_{\delta+\pi^-}^{-\delta+2\pi^-} \left(v_0 + v_1 + \kappa_{2\pi} v^1_{2\pi} + v^3_{2\pi} + \kappa_{4\pi} v^1_{4\pi} + v^3_{4\pi} \right) \mathrm{d}\alpha = 0 \quad (5.25)$$

with the superscripted negative sign in the limits indicating the corresponding side of the branch cut (see Figure 5.2) along which to integrate. Upon use of the cyclic periods defined in (5.22) this becomes

$$D_{0+1} + \kappa_{2\pi} D^1_{2\pi} + D^3_{2\pi} + \kappa_{4\pi} D^1_{4\pi} + D^3_{4\pi} = 0 \quad (5.26)$$

which is further simplified by exploiting the symmetries $D^1_{2\pi} = -B^1_{2\pi}$, $D^3_{2\pi} = -B^3_{2\pi}$ and $D^1_{4\pi} = B^1_{4\pi}$. We finally obtain

$$D_{0+1} - \kappa_{2\pi} B^1_{2\pi} - B^3_{2\pi} + \kappa_{4\pi} B^1_{4\pi} + D^3_{4\pi} = 0, \quad (5.27)$$

a relationship equivalent to (5.25). Repeating the same process for the a, b and c cycles yields

$$A_{0+1} + \kappa_{2\pi}A^1_{2\pi} + A^3_{2\pi} + \kappa_{4\pi}A^1_{4\pi} + A^3_{4\pi} = 0, \tag{5.28a}$$

$$B_{0+1} + \kappa_{2\pi}B^1_{2\pi} + B^3_{2\pi} + \kappa_{4\pi}B^1_{4\pi} + B^3_{4\pi} = 0, \tag{5.28b}$$

$$C_{0+1} - \kappa_{2\pi}A^1_{2\pi} - A^3_{2\pi} \qquad\qquad + C^3_{4\pi} = 0, \tag{5.28c}$$

$$D_{0+1} - \kappa_{2\pi}B^1_{2\pi} - B^3_{2\pi} + \kappa_{4\pi}B^1_{4\pi} + D^3_{4\pi} = 0, \tag{5.28d}$$

with the explicit unknowns $\kappa_{2\pi}$ and $\kappa_{4\pi}$ and the unknowns $\zeta_{2\pi}$ and $\zeta_{4\pi}$ implied by the presence of cyclic periods associated with the integrals of the third kind. This seemingly intractable system can be fully solved analytically. The quantities associated with the 4π periodic elliptic integrals can be decoupled by adding equation (5.28a) to (5.28c) and (5.28b) to (5.28d) to obtain

$$\kappa_{4\pi}A^1_{4\pi} + A^3_{4\pi} + C^3_{4\pi} = -(A_{0+1} + C_{0+1}), \tag{5.29a}$$

$$2\kappa_{4\pi}B^1_{4\pi} + B^3_{4\pi} + D^3_{4\pi} = -(B_{0+1} + D_{0+1}), \tag{5.29b}$$

and the elimination of $\kappa_{4\pi}$ produces

$$A^1_{4\pi}\left(B^3_{4\pi} + D^3_{4\pi}\right) - 2B^1_{4\pi}\left(A^3_{4\pi} + C^3_{4\pi}\right) = -A^1_{4\pi}(B_{0+1} + D_{0+1}) + 2B^1_{4\pi}(A_{0+1} + C_{0+1}) \tag{5.30}$$

in which the only (implicit) unknown $\zeta_{4\pi}$ determines the periods $A^3_{4\pi}, B^3_{4\pi}, C^3_{4\pi}$ and $D^3_{4\pi}$. Despite appearances, the above equation can be inverted to obtain $\zeta_{4\pi}$ and this is described in Section 5.2.4. Once $\zeta_{4\pi}$ has been obtained, the value of $\kappa_{4\pi}$ immediately follows either from equation (5.29a) or (5.29b). One can then proceed to solve for $\zeta_{2\pi}$ by subtracting equation (5.28c) from (5.28a) and (5.28d) from (5.28b) to obtain, respectively,

$$2\kappa_{2\pi}A^1_{2\pi} + 2A^3_{2\pi} = -\kappa_{4\pi}A^1_{4\pi} - A^3_{4\pi} + C^3_{4\pi} - A_{0+1} + C_{0+1}, \tag{5.31a}$$

$$2\kappa_{2\pi}B^1_{2\pi} + 2B^3_{2\pi} = -B^3_{4\pi} + D^3_{4\pi} - B_{0+1} + D_{0+1}, \tag{5.31b}$$

and the elimination of $\kappa_{2\pi}$ gives

$$A^1_{2\pi}B^3_{2\pi} - B^1_{2\pi}A^3_{2\pi} = \frac{1}{2}\Big\{A^1_{2\pi}\left(-B^3_{4\pi} + D^3_{4\pi} - B_{0+1} + D_{0+1}\right) - B^1_{2\pi}\left(-\kappa_{4\pi}A^1_{4\pi} - A^3_{4\pi} + C^3_{4\pi} - A_{0+1} + C_{0+1}\right)\Big\} \tag{5.32}$$

where the only unknown is now $\zeta_{2\pi}$, the value of which determines the periods $A^3_{2\pi}$ and $B^3_{2\pi}$. This equation is of the same type as (4.50) and the solution procedure is similar.

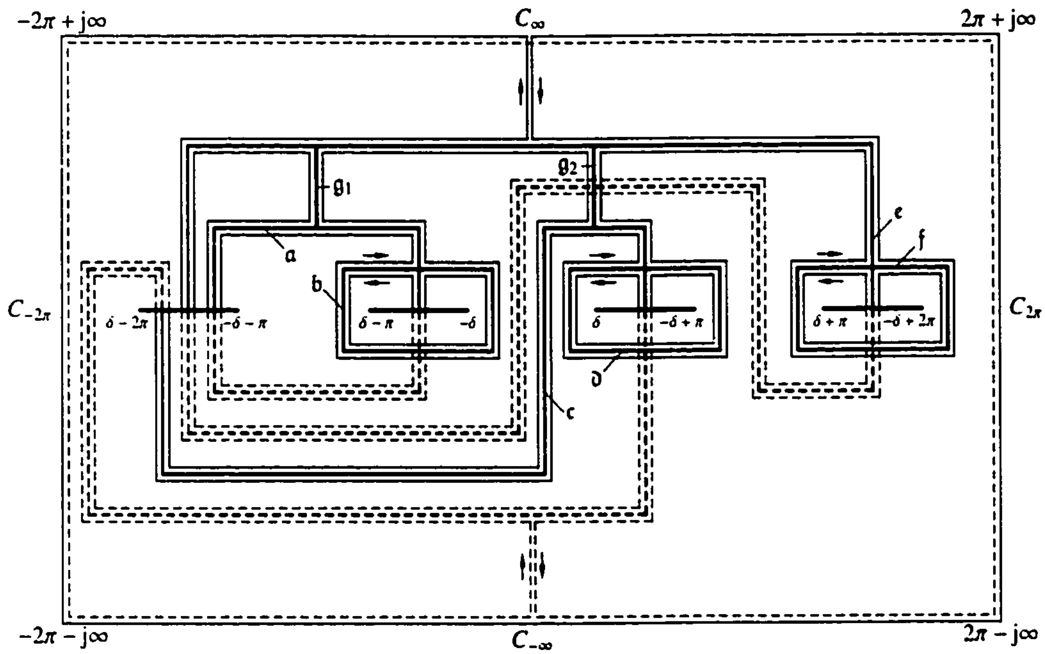


Figure 5.3: The contour $C_2 = C_{\mathfrak{a}\cup\mathfrak{b}} \cup C_{\mathfrak{c}\cup\mathfrak{d}} \cup C_{\mathfrak{e}\cup\mathfrak{f}} \cup C_{\mathfrak{g}_{1,2}} \cup C_{\pm 2\pi} \cup C_{\pm\infty}$ on the upper (solid line) and lower (dashed line) Riemann sheets. The thicker inner lines are the dissections $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}$ and $\mathfrak{g}_{1,2}$ introduced to make the Riemann surface simply-connected. The path $C_{\mathfrak{a}\cup\mathfrak{b}}$ denotes, for example, the portion of the contour enclosing the dissecting cycles $\mathfrak{a}$ and $\mathfrak{b}$.

5.2.4 Determination of $\zeta_{4\pi}$ and $\kappa_{4\pi}$

The procedure required to solve for $\zeta_{4\pi}$ and $\kappa_{4\pi}$ follows the same approach as the one given in Section 4.2.4. However, the cyclic periods appearing on the left-hand side of (5.30) are now related to 4π periodic expressions and the application of the residue theorem must be carried out on the Riemann surface delimited by the contour C_2 of width 4π shown in Figure 5.3. Again, the path of integration is chosen such that it encloses a portion of the strip equal in width to the period of the integrand. We therefore consider

$$\int_{C_2} V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\pi}(\alpha, u) = 2\pi \mathrm{j} \sum \mathrm{Res}, \tag{5.33}$$

with the elliptic integrals defined as

$$V^n_{4\pi}(\alpha, u) = \int_{(\delta,0)}^{(\alpha,u)} v^n_{4\pi}(\xi, u)\mathrm{d}\xi, \qquad n \in \{1, 3\}.$$

We also recall the distinction made between $V^n_{4\pi}(\alpha, u)$ and $V^n_{4\pi}(\alpha)$: the first is defined on the dissected Riemann surface whereas the second is defined only on the top sheet (positive

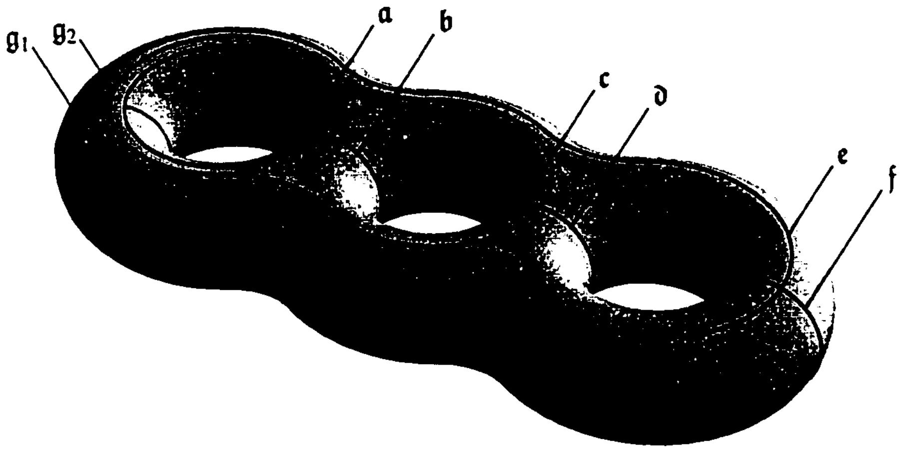


Figure 5.4: Handlebody of genus 3, the topological equivalent of the Riemann surface in Figure 5.3. It has been made simply-connected by introducing the dissections $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}$ and $\mathfrak{g}_{1,2}$.

branch) of the Riemann surface. For 4π periodic functions, the enclosed surface is now the topological equivalent of a handlebody of genus three (a sphere with three handles) as shown in Figure 5.4 and it can be appreciated that making it simply-connected involves a larger number of canonical dissections than the torus of section 4.2.4. To do so [Appell, 1976], three pairs of dissections $\mathfrak{a},\mathfrak{b}$; $\mathfrak{c},\mathfrak{d}$ and $\mathfrak{e},\mathfrak{f}$ are required as well as the two auxiliary dissections $\mathfrak{g}_1$ and $\mathfrak{g}_2$. They are shown in both Figures 5.3 and 5.4; the simple-connectedness is once again better appreciated by examining the handlebody representation of the Riemann surface. To keep the analysis relatively straightforward it is beneficial to draw the dissections such that only members belonging to the same dissection pair, $\mathfrak{a}$ and $\mathfrak{b}$ for example, intersect. This, while simplifying the evaluation of the path integral around the dissections, entails the rather intricate set of dissections shown in Figure 5.3. Carrying out the integration it is seen that only $C_{\mathfrak{a}\cup\mathfrak{b}} \cup C_{\mathfrak{c}\cup\mathfrak{d}} \cup C_{\mathfrak{e}\cup\mathfrak{f}}$, the portion of the path enclosing the dissection pairs, contributes. The rest of the integral vanishes either by symmetry, as for the parts along Re $\alpha = \pm 2\pi$ on $C_{\pm 2\pi}$, or identically, as in the case where $|\mathrm{Im}\ \alpha| = \pm\infty$ on $C_{\pm\infty}$. The contributions from the paths enclosing the three dissection pairs, following the analysis in

Section 3.3 pertaining to equation (3.60), are

$$\int_{C_{\mathfrak{a}\cup\mathfrak{b}}} V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\pi}(\alpha, u) = \mathfrak{A}^1_{4\pi}\mathfrak{B}^3_{4\pi} - \mathfrak{B}^1_{4\pi}\mathfrak{A}^3_{4\pi}, \tag{5.34}$$

$$\int_{C_{\mathfrak{c}\cup\mathfrak{d}}} V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\pi}(\alpha, u) = \mathfrak{C}^1_{4\pi}\mathfrak{D}^3_{4\pi} - \mathfrak{D}^1_{4\pi}\mathfrak{C}^3_{4\pi}, \tag{5.35}$$

$$\int_{C_{\mathfrak{e}\cup\mathfrak{f}}} V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\pi}(\alpha, u) = \mathfrak{E}^1_{4\pi}\mathfrak{F}^3_{4\pi} - \mathfrak{F}^1_{4\pi}\mathfrak{E}^3_{4\pi}. \tag{5.36}$$

Comparing the cycles defined by the dissections with those defined in Figure 5.2, together with symmetry, it is possible to rewrite the above canonical periods in terms of the cyclic periods defined in (5.22). The cyclic periods are defined on intervals between adjacent branch points and extending these definitions to the negative real axis, by means of the even parity of the expressions, the branch point to branch point intervals in $S_{4\pi}$ are then defined as shown in Figure 5.5(a). The canonical cycles $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$,$\mathfrak{e}$ and $\mathfrak{f}$ from Figure 5.3 are then partitioned into branch point to branch point contributions, as shown in Figures 5.5(b), 5.5(c) and 5.5(d). By comparing with Figure 5.5(a), they are then easily expressed in terms of the cyclic periods and it can then be shown that

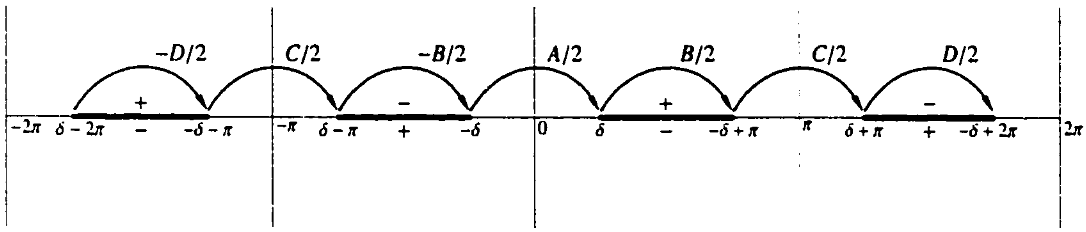
$$\mathfrak{A} = C, \tag{5.37a}$$
$$\mathfrak{B} = -B, \tag{5.37b}$$
$$\mathfrak{C} = A + B + C + D, \tag{5.37c}$$
$$\mathfrak{D} = B, \tag{5.37d}$$
$$\mathfrak{E} = A + B + 2C, \tag{5.37e}$$
$$\mathfrak{F} = D, \tag{5.37f}$$

leading to

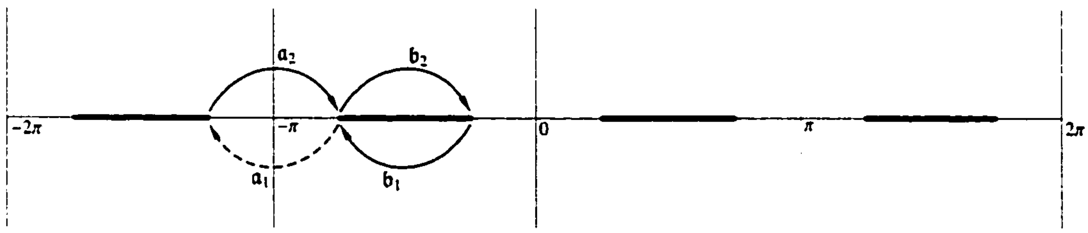
$$\mathfrak{A}^1_{4\pi}\mathfrak{B}^3_{4\pi} - \mathfrak{B}^1_{4\pi}\mathfrak{A}^3_{4\pi} = B^1_{4\pi}C^3_{4\pi} - C^1_{4\pi}B^3_{2\pi}, \tag{5.38a}$$
$$\mathfrak{C}^1_{4\pi}\mathfrak{D}^3_{4\pi} - \mathfrak{C}^1_{4\pi}\mathfrak{D}^3_{4\pi} = A^1_{4\pi}B^3_{4\pi} - B^1_{4\pi}A^3_{4\pi} + C^1_{4\pi}B^3_{4\pi} - B^1_{4\pi}C^3_{4\pi} + D^1_{4\pi}B^3_{4\pi} - B^1_{4\pi}D^3_{4\pi}, \tag{5.38b}$$
$$\mathfrak{E}^1_{4\pi}\mathfrak{F}^3_{4\pi} - \mathfrak{E}^1_{4\pi}\mathfrak{F}^3_{4\pi} = A^1_{4\pi}D^3_{4\pi} - D^1_{4\pi}A^3_{4\pi} + B^1_{4\pi}D^3_{4\pi} - D^1_{4\pi}B^3_{4\pi} + 2(C^1_{4\pi}D^3_{4\pi} - D^1_{4\pi}C^3_{4\pi}). \tag{5.38c}$$

Summing up these contributions,

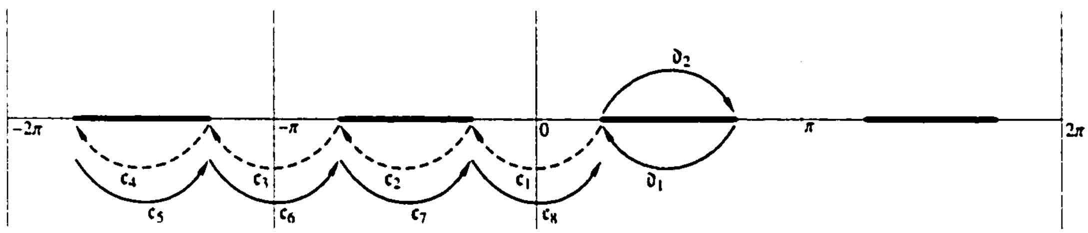
$$\begin{aligned}\int_{C_2} V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\tau}(\alpha, u) &= \left(\int_{C_{\mathfrak{a}\cup\mathfrak{b}}} + \int_{C_{\mathfrak{c}\cup\mathfrak{d}}} + \int_{C_{\mathfrak{e}\cup\mathfrak{f}}}\right) V^1_{4\pi}(\alpha, u)\mathrm{d}V^3_{4\pi}(\alpha, u) \\ &= A^1_{4\pi}B^3_{4\pi} - B^1_{4\pi}A^3_{4\pi} + A^1_{4\pi}D^3_{4\pi} - D^1_{4\pi}A^3_{4\pi} + 2\left(C^1_{4\pi}D^3_{4\pi} - D^1_{4\pi}C^3_{4\pi}\right) \\ &= A^1_{4\pi}\left(B^3_{4\pi} + D^3_{4\pi}\right) - 2B^1_{4\pi}\left(A^3_{4\pi} + C^3_{4\pi}\right)\end{aligned} \tag{5.39}$$



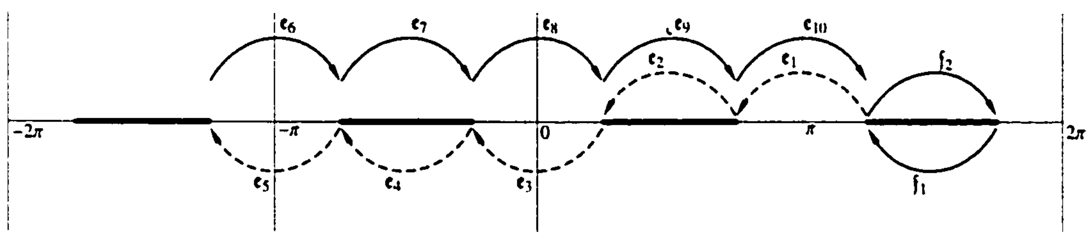
(a) Basic cyclic periods



(b) Canonical cycles a and b



(c) Canonical cycles c and d



(d) Canonical cycles e and f

Figure 5.5: Figures used to express the canonical periods in terms of the basic cyclic periods (*e.g.* $\mathfrak{A}$ in terms of A, B, C, D). The canonical paths, see Figure 5.3, on the (solid lines) upper and (dashed lines) lower sheets in (b), (c) and (d) are written as chains of branch point to branch point segments — $\mathfrak{e}$ into a sequence of $\mathfrak{e}_n$ for example — which are easily expressed in terms of the basic cyclic periods given in (a).

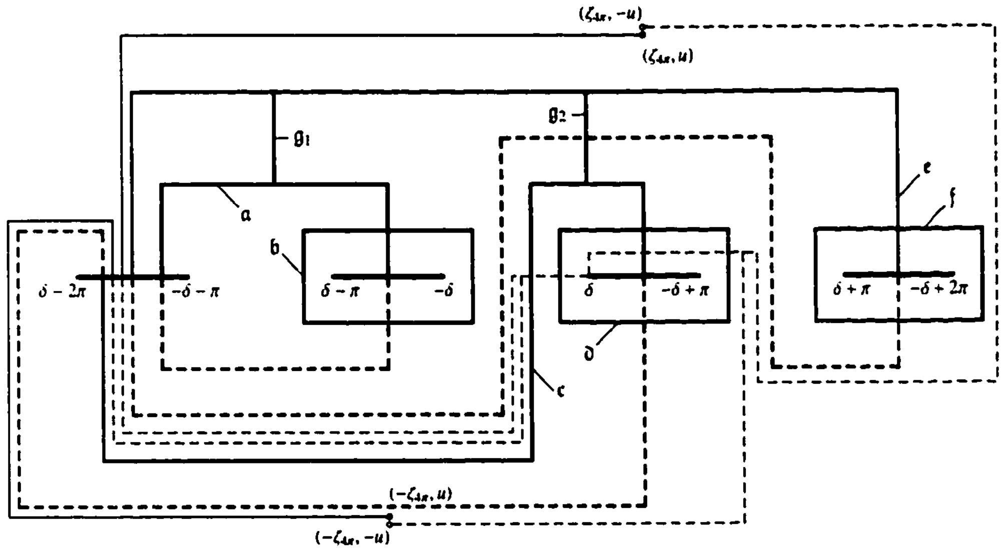


Figure 5.6: The paths used to evaluate the integrals in (5.40) when $V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_1$. The four branch cuts are indicated together with the dissections (thicker lines) illustrated in Figure 5.3. The paths (thinner lines) go from δ to each of the the four symmetric poles of $v^3_{4\pi}(\alpha, u)$ on the Riemann surface. They lie on both the upper (solid lines) and lower (dashed lines) Riemann sheet and do not cross the dissections.

where on the last line we have made use of $D^1_{4\pi} = B^1_{4\pi}$ and $C^1_{4\pi} = 0$. This is remarkable in that it reproduces the left-hand side of (5.30) and can be expressed in terms of residues in accordance with (5.33). The integrand in (5.33) has residues at

$$
-\left.\frac{u(\zeta_{4\pi})}{u(\alpha)}\frac{\cos\alpha}{\cos\zeta_{4\pi}}\frac{\sin\frac{\zeta_{4\pi}}{2}}{\sin\frac{\alpha}{2}}\int_{(\delta,0)}^{(\alpha,u)} v^1_{4\pi}(\xi,u)\mathrm{d}\xi\right|_{\alpha=(\pm\zeta_{4\pi},\pm u)}
$$

$$
= \begin{cases} \mp\displaystyle\int_{(\delta,0)}^{(\alpha,u)} v^1_{4\pi}(\xi,u)\mathrm{d}\xi, & \alpha = (\zeta_{4\pi}, \pm u) \\ \pm\displaystyle\int_{(\delta,0)}^{(\alpha,u)} v^1_{4\pi}(\xi,u)\mathrm{d}\xi, & \alpha = (-\zeta_{4\pi}, \pm u) \end{cases} \tag{5.40}
$$

which are expressed in terms of $V^1_{4\pi}(\zeta_{4\pi})$ and $V^1_{4\pi}(2\pi - \zeta_{4\pi})$ after carrying out the path integrals on the dissected Riemann surface. In order to do so, the cases where $V^1_{4\pi}(\zeta_{4\pi})$ lies in the four subdomains $\mathcal{W}_{1,2,3,4}$, as described in Section 3.5.2, must be considered separately. This is carried out by expressing the paths taken by the integrals in (5.40) in terms of the cyclic periods $A^1_{4\pi}$ and $B^1_{4\pi}$ of the integral of the first kind. The paths are illustrated in Figure 5.6 for the case when $V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_1$ and Figure 5.5(a) is once again useful to find the equivalent path expressed in terms of the cyclic periods. In this instance,

$$\int_{(\delta,0)}^{(\zeta_{4\pi},u)} v_{4\pi}^{1}(\xi,u)\mathrm{d}\xi = A_{4\pi}^{1} + V_{4\pi}^{1}(\zeta_{4\pi}), \tag{5.41a}$$

$$\int_{(\delta,0)}^{(-\zeta_{4\pi},u)} v_{4\pi}^{1}(\xi,u)\mathrm{d}\xi = \frac{A_{4\pi}^{1}}{2} + 2B_{4\pi}^{1} - V_{4\pi}^{1}(\zeta_{4\pi}), \tag{5.41b}$$

$$\int_{(\delta,0)}^{(\zeta_{4\pi},-u)} v_{4\pi}^{1}(\xi,u)\mathrm{d}\xi = B_{4\pi}^{1} - V_{4\pi}^{1}(\zeta_{4\pi}), \tag{5.41c}$$

$$\int_{(\delta,0)}^{(-\zeta_{4\pi},-u)} v_{4\pi}^{1}(\xi,u)\mathrm{d}\xi = \frac{A_{4\pi}^{1}}{2} - B_{4\pi}^{1} + V_{4\pi}^{1}(\zeta_{4\pi}), \tag{5.41d}$$

and if $V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_1$, the sum of the residues is therefore

$$\sum \mathrm{Res} = -A_{4\pi}^{1} + 4B_{4\pi}^{1} - 4V_{4\pi}^{1}(\zeta_{4\pi}). \tag{5.42}$$

Carrying out this procedure for the three other cases, the sum of residues becomes

$$\sum \mathrm{Res} = \begin{cases} -A_{4\pi}^{1} + 4B_{4\pi}^{1} - 4V_{4\pi}^{1}(\zeta_{4\pi}), & V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_1, \\ -A_{4\pi}^{1} - 4B_{4\pi}^{1} - 4V_{4\pi}^{1}(\zeta_{4\pi}), & V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_2, \\ -A_{4\pi}^{1} - 4V_{4\pi}^{1}(2\pi - \zeta_{4\pi}), & V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_3 \cup \mathcal{W}_4. \end{cases} \tag{5.43}$$

Using (5.39) and (5.43) in (5.33), it is possible to express the unknowns $V_{4\pi}^{1}(\zeta_{4\pi})$ and $V_{4\pi}^{1}(2\pi - \zeta_{4\pi})$ in terms of known quantities so that we can invert for $\zeta_{4\pi}$. If we still use the case where $V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_1$ as an example, the residue theorem (5.33) together with (5.39) and (5.43) implies

$$\begin{aligned} V_{4\pi}^{1}(\zeta_{4\pi}) &= -\frac{A_{4\pi}^{1}}{4} + B_{4\pi}^{1} - \frac{\sigma_{4\pi}}{8\pi\mathrm{j}}\left\{A_{4\pi}^{1}(B_{4\pi}^{3} + D_{4\pi}^{3}) - 2B_{4\pi}^{1}(A_{4\pi}^{3} + C_{4\pi}^{3})\right\} \\ &= -\frac{A_{4\pi}^{1}}{4} + B_{4\pi}^{1} + \frac{\sigma_{4\pi}}{8\pi\mathrm{j}}\left\{A_{4\pi}^{1}(B_{0+1} + D_{0+1}) - 2B_{4\pi}^{1}(A_{0+1} + C_{0+1})\right\} \end{aligned} \tag{5.44}$$

where use has been made of (5.30) on the last line and the sign $\sigma_{4\pi}$ from equation (5.24) has been reintroduced. The only unknown in the above equation is $\zeta_{4\pi}$. More generally, we have

$$V_{4\pi}^{1}(\zeta_{4\pi}) = \begin{cases} -\frac{A_{4\pi}^{1}}{4} + B_{4\pi}^{1} + \frac{\sigma_{4\pi}}{\cos\frac{\delta}{2}}\Lambda_{4\pi}, & V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_1 \\ -\frac{A_{4\pi}^{1}}{4} - B_{4\pi}^{1} + \frac{\sigma_{4\pi}}{\cos\frac{\delta}{2}}\Lambda_{4\pi}, & V_{4\pi}^{1}(\zeta_{4\pi}) \in \mathcal{W}_2 \end{cases} \tag{5.45}$$

and

$$V^1_{4\pi}(2\pi - \zeta_{4\pi}) = -\frac{A^1_{4\pi}}{4} + \frac{\sigma_{4\pi}}{\cos\frac{\delta}{2}}\Lambda_{4\pi}, \qquad V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_3 \cup \mathcal{W}_4 \tag{5.46}$$

where we have defined

$$\Lambda_{4\pi} = \frac{\cos\frac{\delta}{2}}{8\pi\mathrm{j}}\left\{A^1_{4\pi}(B_{0+1} + D_{0+1}) - 2B^1_{4\pi}(A_{0+1} + C_{0+1})\right\}. \tag{5.47}$$

Looking ahead to the inversion process, it is convenient to specify from (5.45) and (5.46) the corresponding range of $\Lambda_{4\pi}$ in terms of the periods K and K' from the range of $V^1_{4\pi}(\zeta_{4\pi})$. Recalling from (3.83) the useful relationships

$$k = \tan\frac{\delta}{2}, \qquad A^1_{4\pi} = \frac{4K}{\cos\frac{\delta}{2}}, \qquad B^1_{4\pi} = \frac{2\mathrm{j}K'}{\cos\frac{\delta}{2}}, \tag{5.48}$$

it is a simple task to show that

$$V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_1 \implies \begin{cases} \Lambda_{4\pi} \in Q_1^+ = [0, K] \times [-\mathrm{j}K', -2\mathrm{j}K'], & \sigma_{4\pi} = +1 \\ \Lambda_{4\pi} \in Q_1^- = [0, -K] \times [\mathrm{j}K', 2\mathrm{j}K'], & \sigma_{4\pi} = -1 \end{cases} \tag{5.49a}$$

$$V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_2 \implies \begin{cases} \Lambda_{4\pi} \in Q_2^+ = [0, K] \times [\mathrm{j}K', 2\mathrm{j}K'], & \sigma_{4\pi} = +1 \\ \Lambda_{4\pi} \in Q_2^- = [0, -K] \times [-\mathrm{j}K', -2\mathrm{j}K'], & \sigma_{4\pi} = -1 \end{cases} \tag{5.49b}$$

$$V^1_{4\pi}(\zeta_{4\pi}) \in \mathcal{W}_3 \cup \mathcal{W}_4 \implies \begin{cases} \Lambda_{4\pi} \in Q_0^+ = [0, K] \times [-\mathrm{j}K', \mathrm{j}K'], & \sigma_{4\pi} = +1 \\ \Lambda_{4\pi} \in Q_0^- = [0, -K] \times [-\mathrm{j}K', \mathrm{j}K'], & \sigma_{4\pi} = -1. \end{cases} \tag{5.49c}$$

The inversion is made possible by first solving for $V^1_{4\pi}$ in (5.45) and (5.46). Using the $\sin\alpha/2$ transformation described in Section 3.5.2 to recast it in terms of Legendre's form, the inversion then follows by direct application of a Jacobian elliptic sine function sn. This produces the following explicit expressions for $\zeta_{4\pi}$,

$$\zeta_{4\pi} = \begin{cases} 2\pi - 2\arcsin\left[\sin\frac{\delta}{2}\,\mathrm{sn}\,(\sigma_{4\pi}\Lambda_{4\pi}, k)\right], & \Lambda_{4\pi} \in Q_0^{\sigma_{4\pi}} \\ 2\arcsin\left[\sin\frac{\delta}{2}\,\mathrm{sn}\,(2\mathrm{j}K' + \sigma_{4\pi}\Lambda_{4\pi}, k)\right], & \Lambda_{4\pi} \in Q_1^{\sigma_{4\pi}} \\ 2\arcsin\left[\sin\frac{\delta}{2}\,\mathrm{sn}\,(-2\mathrm{j}K' + \sigma_{4\pi}\Lambda_{4\pi}, k)\right], & \Lambda_{4\pi} \in Q_2^{\sigma_{4\pi}}. \end{cases} \tag{5.50}$$

The proper expression and the sign of $\sigma_{4\pi} = \pm 1$ are selected by using Figure 5.7 to deter-

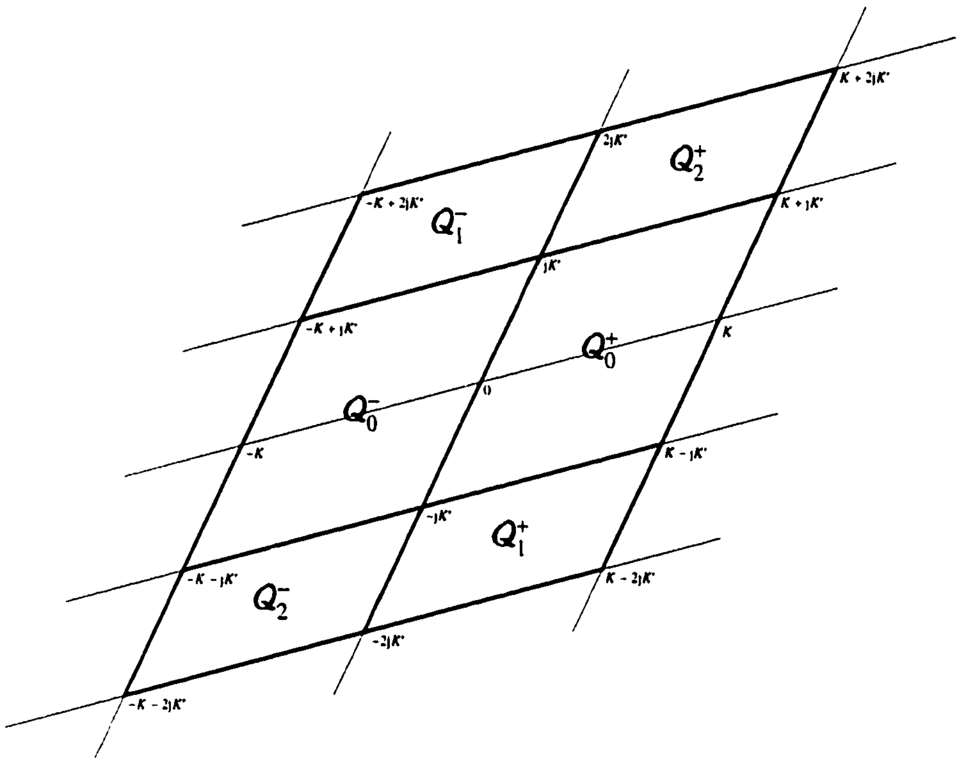


Figure 5.7: The ranges $Q_0^\pm$, $Q_1^\pm$ and $Q_2^\pm$ in terms of the complete integrals of the first kind K and K' with $k = \tan\frac{\delta}{2}$. The parallelograms Q indicate the various ranges in which $\Lambda_{4\pi}$ must lie when carrying out the inversion for $\zeta_{4\pi}$ with equation (5.51) or (5.50).

mine period parallelogram Q in which the quantity $\Lambda_{4\pi}$ lies. The multiplicative constant $\kappa_{4\pi}$ immediately follows from (5.29). While expression (5.50) is useful when implementing the procedure, it can be simplified given that the arguments of the elliptic sine function all differ by a multiple of the fundamental period $2\mathrm{j}K'$. Since $\mathrm{sn}(x) = \mathrm{sn}(x + 2\mathrm{j}K'\mathbb{Z})$, equation (5.50) can be rewritten as

$$\zeta_{4\pi} = \begin{cases} 2\pi - 2\arcsin\left[\sin\frac{\delta}{2}\,\mathrm{sn}\left(\sigma_{4\pi}\Lambda_{4\pi}, k\right)\right], & \Lambda_{4\pi} \in Q_0^{\sigma_{4\pi}} \\ 2\arcsin\left[\sin\frac{\delta}{2}\,\mathrm{sn}\left(\sigma_{4\pi}\Lambda_{4\pi}, k\right)\right], & \Lambda_{4\pi} \in Q_1^{\sigma_{4\pi}} \cup Q_2^{\sigma_{4\pi}}. \end{cases} \tag{5.51}$$

It is insightful to examine the behavior of both $\kappa_{4\pi}$ and $\zeta_{4\pi}$ as functions of the branch point δ for a fixed value of θ. The behavior of $\kappa_{4\pi}$ (and $\kappa_{2\pi}$) as a function of δ is illustrated in Figure 5.8 when $\theta = 0.25(1+\mathrm{j})$. The values for the sample points labeled on the figure are given in Table 5.1. It can be observed that $\kappa_{4\pi}$ tends to zero as $\delta \to \pi/2$, and to some non-zero value as $\delta \to 0$. This will be verified analytically when the properties of the solutions $w(\alpha, u)$ are examined. The corresponding plot for $\zeta_{4\pi}$ is provided in Figure 5.9 and $\zeta_{4\pi}$ apparently tends to 2π when $\delta \to 0$.

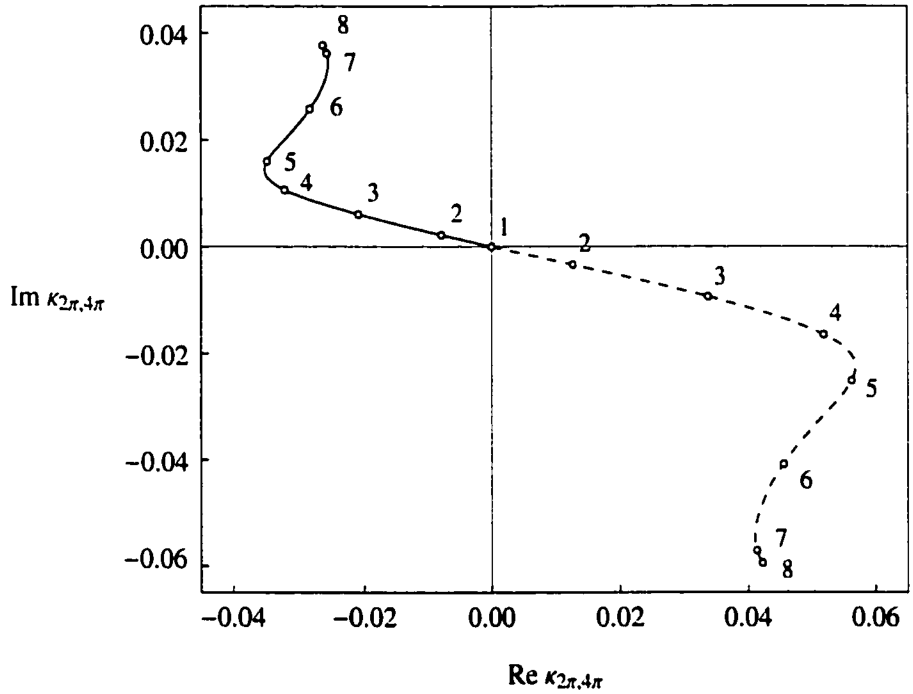


Figure 5.8: The behavior of (solid line) $\kappa_{2\pi}$ and (dashed line) $\kappa_{4\pi}$ when $\theta = 0.25(1 + j)$ as a function δ. The range of δ is from 1.57 at point 1 to 0.01 at point 8. The corresponding values of $\kappa_{2\pi}$ and $\kappa_{4\pi}$ are given in Table 5.1.

Point	δ	$\kappa_{2\pi}$	$\kappa_{4\pi}$
1	1.57	$-5.260 \cdot 10^{-8} + j1.457 \cdot 10^{-8}$	$8.493 \cdot 10^{-8} - j2.214 \cdot 10^{-8}$
2	1.25	$-7.890 \cdot 10^{-3} + j2.208 \cdot 10^{-3}$	$1.272 \cdot 10^{-2} - j3.369 \cdot 10^{-3}$
3	1.00	$-2.089 \cdot 10^{-2} + j6.114 \cdot 10^{-3}$	$3.371 \cdot 10^{-2} - j9.385 \cdot 10^{-3}$
4	0.75	$-3.214 \cdot 10^{-2} + j1.066 \cdot 10^{-2}$	$5.189 \cdot 10^{-2} - j1.652 \cdot 10^{-2}$
5	0.50	$-3.486 \cdot 10^{-2} + j1.601 \cdot 10^{-2}$	$5.625 \cdot 10^{-2} - j2.511 \cdot 10^{-2}$
6	0.25	$-2.824 \cdot 10^{-2} + j2.580 \cdot 10^{-2}$	$4.551 \cdot 10^{-2} - j4.084 \cdot 10^{-2}$
7	0.05	$-2.571 \cdot 10^{-2} + j3.618 \cdot 10^{-2}$	$4.126 \cdot 10^{-2} - j5.715 \cdot 10^{-2}$
8	0.01	$-2.633 \cdot 10^{-2} + j3.767 \cdot 10^{-2}$	$4.217 \cdot 10^{-2} - j5.942 \cdot 10^{-2}$

Table 5.1: Corresponding values of δ and $\kappa_{2\pi,4\pi}$ for the points indicated in Figure 5.8.

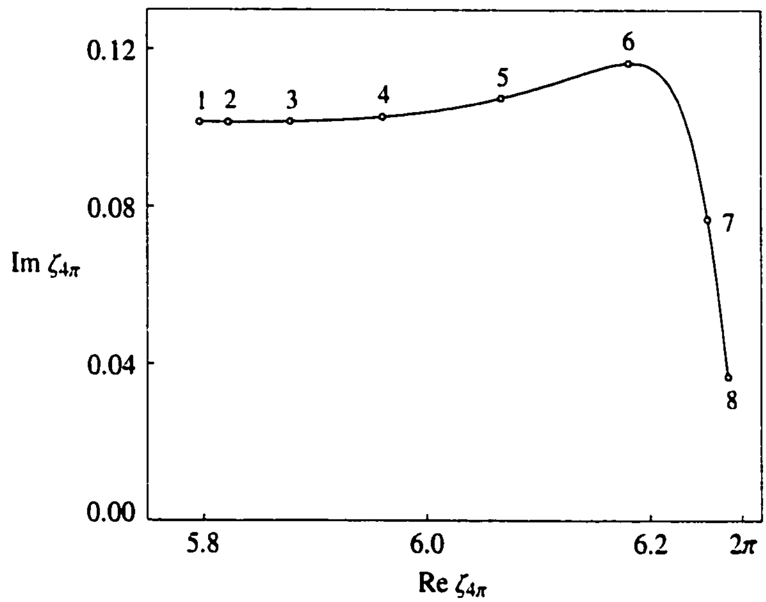


Figure 5.9: The behavior of $\zeta_{4\pi}$ when $\theta = 0.25(1 + j)$ as δ. The range of δ is from 1.57 at point 1 to 0.01 at point 8. The corresponding values of $\zeta_{4\pi}$ are given in Table 5.2.

Point	δ	$\zeta_{4\pi}$
1	1.57	$5.796 + j1.015 \cdot 10^{-1}$
2	1.25	$5.822 + j1.015 \cdot 10^{-1}$
3	1.00	$5.877 + j1.016 \cdot 10^{-1}$
4	0.75	$5.960 + j1.028 \cdot 10^{-1}$
5	0.50	$6.068 + j1.076 \cdot 10^{-1}$
6	0.25	$6.181 + j1.165 \cdot 10^{-1}$
7	0.05	$6.252 + j7.687 \cdot 10^{-2}$
8	0.01	$6.270 + j3.677 \cdot 10^{-2}$

Table 5.2: Corresponding values of δ and $\zeta_{4\pi}$ for the points indicated in Figure 5.9.

5.2.5 Determination of $\zeta_{2\pi}$ and $\kappa_{2\pi}$

Equation (5.32), in which the only unknown is $\zeta_{2\pi}$, has the same form as (4.50) and the same analysis is required to invert for $\zeta_{2\pi}$. Recalling the results obtained in Section 4.2.4, it was shown there that by applying the residue theorem on a Riemann surface of genus one made simply connected, the following relationship can be obtained

$$A^1_{2\pi}B^3_{2\pi} - B^1_{2\pi}A^3_{2\pi} = 2\pi \mathrm{j}\left(-A^1_{2\pi} \pm 2B^1_{2\pi} - 4V^1_{2\pi}(\zeta)\right), \qquad V^1_{2\pi}(\zeta, u) \in \mathcal{V}_{1,2} \tag{5.52}$$

with the positive (negative) sign corresponding to $\mathcal{V}_1$ ($\mathcal{V}_2$). Equation (5.52) together with (5.32) imply

$$V^1_{2\pi}(\zeta) = -\frac{A^1_{2\pi}}{4} \pm \frac{B^1_{2\pi}}{2} + \sigma_{2\pi}\mathrm{j}\Lambda_{2\pi}, \qquad V^1_{2\pi}(\zeta, u) \in \mathcal{V}_{1,2}. \tag{5.53}$$

where, in contrast to (4.62),

$$\Lambda_{2\pi} = \frac{1}{16\pi}\left\{A^1_{2\pi}(-B^3_{4\pi} + D^3_{4\pi} - B_{0+1} + D_{0+1}) - B^1_{2\pi}(-\kappa_{4\pi}A^1_{4\pi} - A^3_{4\pi} + C^3_{4\pi} - A_{0+1} + C_{0+1})\right\}. \tag{5.54}$$

The inversion now follows directly from (4.67) and is given by

$$\zeta_{2\pi} = \arccos\left[k\,\mathrm{sn}\,(\mathrm{j}K' + 3K + \sigma_{2\pi}\Lambda_{2\pi}, k)\right], \qquad \Lambda_{2\pi} \in \mathcal{P}^{\sigma}. \tag{5.55}$$

where the period parallelograms $\mathcal{P}^{\sigma}$, which denote the allowable range for $\Lambda_{2\pi}$, are illustrated in Figure 5.10. The value of $\kappa_{2\pi}$ follows from (5.31). Notice that, with the chosen notation, the analysis is exactly the same as carried out in Chapter 4 save for a different expression for Λ. Before closing this section, we examine the behavior of $\kappa_{2\pi}$ and $\zeta_{4\pi}$ as functions of δ with θ fixed. The behavior of $\kappa_{2\pi}$ is provided in Figure 5.8 (along with $\kappa_{4\pi}$). In the same fashion as $\kappa_{4\pi}$, $\kappa_{2\pi}$ tends to zero as $\delta \to \pi/2$ and to some non-zero value as $\delta \to 0$. A plot of $\zeta_{2\pi}$ is given in Figure 5.11; it apparently tends to a value in the neighborhood of π as $\delta \to 0$.

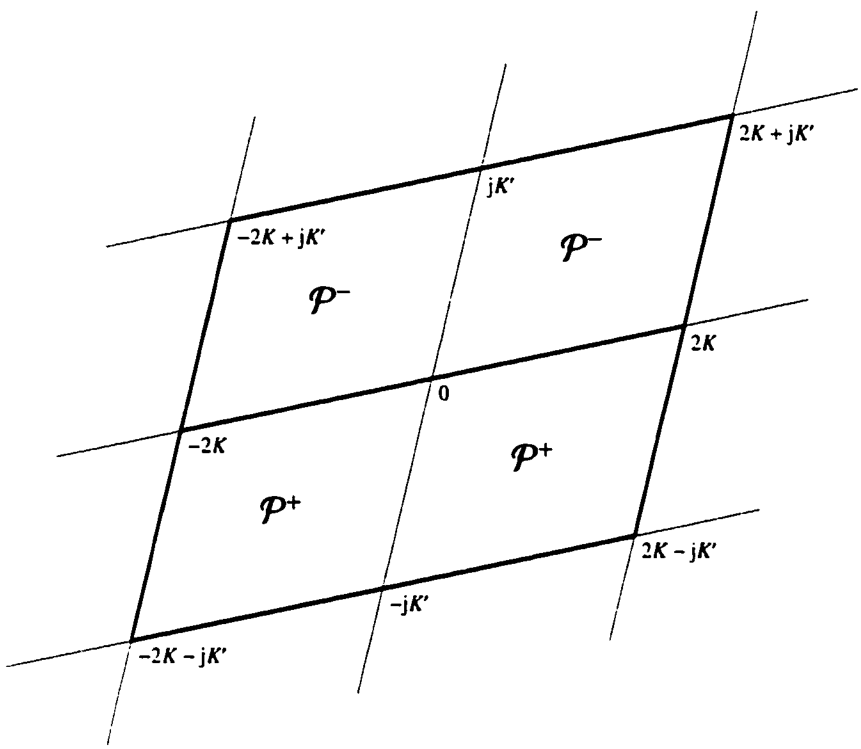


Figure 5.10: The regions $\mathcal{P}^{\pm}$ in terms of the complete integrals of the first kind K and K' with $k = \cos\delta$. The parallelograms $\mathcal{P}$ indicate the various ranges in which $\Lambda_{2\pi}$ must lie when carrying out the inversion for $\zeta_{2\pi}$ with equation (5.55).

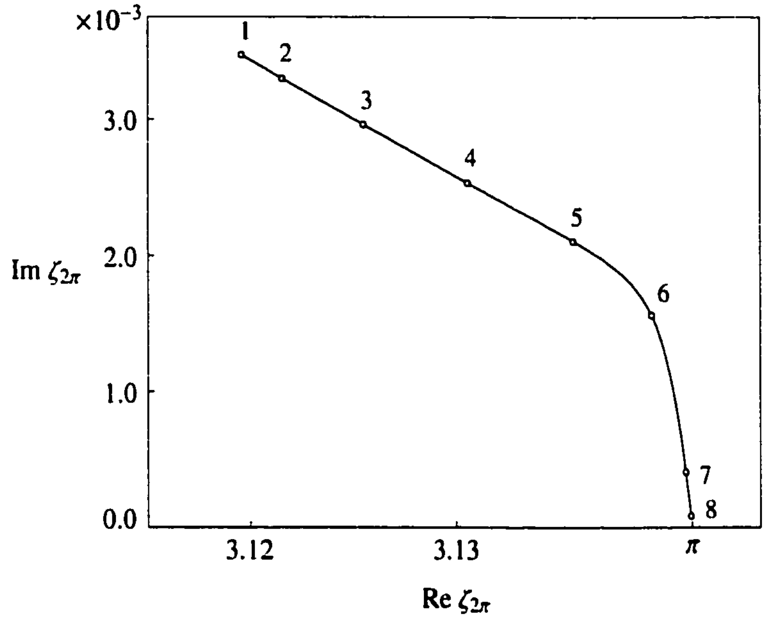


Figure 5.11: The behavior of $\zeta_{2\pi}$ when $\theta = 0.25(1 + j)$ as δ varies from 0.01 (point 7) to 1.57 (point 1). The values of δ and $\zeta_{2\pi}$ at the sample points are given in Table 5.3.

Point	δ	$\zeta_{2\pi}$
1	1.57	$3.120 \cdot +j3.470 \cdot 10^{-3}$
2	1.25	$3.122 + j3.297 \cdot 10^{-3}$
3	1.00	$3.125 + j2.960 \cdot 10^{-3}$
4	0.75	$3.131 + j2.532 \cdot 10^{-3}$
5	0.50	$3.136 + j2.105 \cdot 10^{-3}$
6	0.25	$3.140 + j1.562 \cdot 10^{-3}$
7	0.05	$3.141 + j4.092 \cdot 10^{-4}$
8	0.01	$3.142 + j8.401 \cdot 10^{-5}$

Table 5.3: Corresponding values of δ and $\zeta_{2\pi}$ for the points indicated in Figures 5.11.

5.2.6 Properties of branched solution

As for the 2π case, we have now succeeded in constructing a branched solution to (5.2a) having the form

$$w(\alpha,u) = \exp\int_{\alpha_0}^{\alpha}\Big\{v_0(\alpha',u) + v_1(\alpha',u) + \kappa_{2\pi}v^1_{2\pi}(\alpha',u) + \sigma_{2\pi}v^3_{2\pi}(\alpha',u) + \kappa_{4\pi}v^1_{4\pi}(\alpha',u) + \sigma_{4\pi}v^3_{4\pi}(\alpha',u)\Big\}\mathrm{d}\alpha' \quad (5.56)$$

with

$$v_0(\alpha,u) = -\frac{\alpha}{2\pi}\frac{u(\theta)}{u(\alpha)}\frac{\sin\alpha\cos\alpha}{\cos^2\alpha-\cos^2\theta}, \quad (5.57)$$

$$v_1(\alpha,u) = \frac{u(\theta)}{u(\alpha)}\frac{\cos(\alpha)}{\cos(\theta)}\frac{\varphi_1+\varphi_2\cos\frac{\alpha}{2}+\varphi_3\cos\alpha+\varphi_4\cos\frac{3\alpha}{2}}{\cos^2\alpha-\cos^2\theta}, \quad (5.58)$$

$$v^1_{2\pi}(\alpha,u) = \frac{1}{u(\alpha)}, \quad (5.59)$$

$$v^3_{2\pi}(\alpha,u) = \frac{u(\zeta_{2\pi})}{u(\alpha)}\frac{\cos\alpha}{\cos\zeta_{2\pi}}\frac{\sin\zeta_{2\pi}}{\cos\alpha-\cos\zeta_{2\pi}}, \quad (5.60)$$

$$v^1_{4\pi}(\alpha,u) = \frac{\cos\frac{\alpha}{2}}{u(\alpha)}, \quad (5.61)$$

$$v^3_{4\pi}(\alpha,u) = \frac{u(\zeta_{4\pi})}{u(\alpha)}\frac{\cos\alpha}{\cos\zeta_{4\pi}}\frac{1}{2}\frac{\sin\frac{\zeta_{4\pi}}{2}}{\cos\frac{\alpha}{2}-\cos\frac{\zeta_{4\pi}}{2}}, \quad (5.62)$$

where the coefficients φ_n are given in (5.8), the quantities $\zeta_{4\pi}$ and $\kappa_{4\pi}$ by (5.51) and (5.29), and the quantities $\zeta_{2\pi}$ and $\kappa_{2\pi}$ by (5.55) and (5.31). The lower limit α_0 may be any branch point in the strip $S_{4\pi}$, *i.e.* $\alpha_0 \in \{\pm\delta, \pm(\pi-\delta), \pm(\pi+\delta), \pm(2\pi-\delta)\}$. Furthermore, since the integrand has an even parity, $\alpha_0 = 0$ is also an allowed lower limit. The solution is multi-valued in the sense of the branch $u(\alpha)$ and each of the branches corresponds to a solution of one of the equations in (5.2). Despite the fact that the integrand has branch points commensurate with those of $u(\alpha)$, the path integral in (5.56) is well defined following the process of polar and cyclic period elimination carried out in the previous section. Aside for the poles and zeros introduced by this procedure, $w(\alpha,u)$ is otherwise free of poles and zeros in $S_{4\pi}$. Indeed, from (5.20) $v^3_{2\pi}(\alpha, u)$ has residues[1], on the positive branch of $u(\alpha)$,

$$\text{Res}\,\frac{u(\zeta_{2\pi})}{u(\alpha)}\,\frac{\cos\alpha}{\cos\zeta_{2\pi}}\,\frac{\sin\zeta_{2\pi}}{\cos\alpha-\cos\zeta_{2\pi}}\bigg|_{\alpha=\pm\zeta_{2\pi},\pm(\zeta_{2\pi}-\pi)} = \pm 1 \tag{5.63}$$

and $v^3_{4\pi}(\alpha, u)$ residues

$$\text{Res}\,\frac{u(\zeta_{4\pi})}{u(\alpha)}\,\frac{\cos\alpha}{\cos\zeta_{4\pi}}\,\frac{1}{2}\,\frac{\sin\frac{\zeta_{4\pi}}{2}}{\cos\frac{\alpha}{2}-\cos\frac{\zeta_{4\pi}}{2}}\bigg|_{\alpha=\pm\zeta_{4\pi}} = \pm 1. \tag{5.64}$$

Therefore $w(\alpha, u)$ has, in the strip $S_{4\pi}$, the poles and zeros

$$w(\alpha, u) \sim \frac{\alpha+\zeta_{2\pi}}{\alpha-\zeta_{2\pi}}\,\frac{\alpha+(\pi-\zeta_{2\pi})}{\alpha-(\pi-\zeta_{2\pi})}\,\frac{\alpha+\zeta_{4\pi}}{\alpha-\zeta_{4\pi}}, \tag{5.65a}$$

and

$$w(\alpha, -u) \sim \frac{\alpha-\zeta_{2\pi}}{\alpha+\zeta_{2\pi}}\,\frac{\alpha-(\pi-\zeta_{2\pi})}{\alpha+(\pi-\zeta_{2\pi})}\,\frac{\alpha-\zeta_{4\pi}}{\alpha+\zeta_{4\pi}}. \tag{5.65b}$$

These are of course also periodically located outside the strip and they are of order unity throughout. This differs from the θ poles and zeros — due to $v_0(\alpha, u)$ — which have been eliminated from the strip but occur periodically outside the strip. In their case, however, their order increases the further they are from $S_{4\pi}$ due to the α coefficient found in $v_0(\alpha, u)$, a behavior reminiscent of the Maliuzhinets's functions. Like its predecessor in Chapter 4, $w(\alpha, u)$ goes to its reciprocal when $\alpha \to -a$ or when $u(\alpha) \to -u(\alpha)$ and hence $w(\alpha, u) = 1/w(-a, u) = 1/w(\alpha, -u)$. Since the integrand vanishes as $|\text{Im}\,\alpha| \to \infty$ as required, then

$$w(\alpha, u) \sim O(1), \quad |\text{Im}\,\alpha| \to \infty. \tag{5.66}$$

5.2.6.1 The limit $\delta \to \pi/2$

As $\delta \to \pi/2$, $w(\alpha, u)$ recovers the known solution $\Psi_4(\alpha)$ given in (5.5) with an added multiplicative contribution from the integrals of the third kind associated with $v^3_{2\pi,4\pi}(\alpha, u)$. The limiting value can be exactly defined by determining the limiting values of $\kappa_{2\pi,4\pi}$ as well as $\zeta_{2\pi,4\pi}$. This can be carried out when $\delta \to \pi/2$ by considering the equation system

[1] It is now assumed for simplicity that $\sigma_{2\pi,4\pi} = 1$. The reader should bear in mind that expressions involving $v^3_{2\pi}(\alpha, u)$ and $v^3_{4\pi}(\alpha, u)$ have to be changed accordingly if instead $\sigma_{2\pi,4\pi} = -1$.

(5.28). For example, (5.28b) becomes

$$\lim_{\varepsilon\to 0}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon}\left\{\frac{u(\theta)}{u(\alpha)}\frac{\cos\alpha}{\cos\theta}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_4(\alpha)+\kappa_{2\pi}\frac{1}{u(\alpha)}+\sigma_{2\pi}\frac{u(\zeta_{2\pi})}{u(\alpha)}\frac{\cos\alpha}{\cos\zeta_{2\pi}}\frac{\sin\zeta_{2\pi}}{\cos\alpha-\cos\zeta_{2\pi}}\right.$$
$$\left.+\kappa_{4\pi}\frac{\cos\frac{\alpha}{2}}{u(\alpha)}+\sigma_{4\pi}\frac{u(\zeta_{4\pi})}{u(\alpha)}\frac{\cos\alpha}{\cos\zeta_{4\pi}}\frac{1}{2}\frac{\sin\frac{\zeta_{4\pi}}{2}}{\cos\frac{\alpha}{2}-\cos\frac{\zeta_{4\pi}}{2}}\right\}\mathrm{d}\alpha. \quad (5.67)$$

In the limit the integral is taken over a vanishing interval about $\alpha = \pi/2$ and the only contributing terms are those with singularities at $\alpha = \pi/2$: the second member $\kappa_{2\pi}v^1_{2\pi}(\alpha, u)$ and the fourth member $\kappa_{4\pi}v^1_{4\pi}(\alpha, u)$. This leads directly to

$$\kappa_{2\pi}+\kappa_{4\pi}\cos\frac{\pi}{4}=0. \quad (5.68)$$

The same also applies to (5.28d) which involves a vanishing interval about $\alpha = 3\pi/2$ and once again $v^1_{2\pi}(\alpha, u)$ and $v^1_{4\pi}(\alpha, u)$ are the only contributing terms. In this case, taking the limit yields

$$\kappa_{2\pi}-\kappa_{4\pi}\cos\frac{\pi}{4}=0. \quad (5.69)$$

The above implies

$$\kappa_{2\pi}\xrightarrow{\delta\to\pi/2}0,\qquad \kappa_{4\pi}\xrightarrow{\delta\to\pi/2}0, \quad (5.70)$$

in agreement with the results plotted in Figure 5.8. Consequently, the contributions from the integrals of the first kind vanish in that limit.

The values for the poles $\zeta_{2\pi,4\pi}$ follow from (5.28a) and (5.28c). Since, from (5.18),

$$v_0(\alpha,u)+v_1(\alpha,u)\xrightarrow{\delta\to\pi/2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_4(\alpha), \quad (5.71)$$

and, noting the two anti-derivatives

$$\int^{\alpha}\frac{\sin\zeta}{\cos\xi-\cos\zeta}\mathrm{d}\xi=\ln\frac{\tan\frac{\zeta}{2}+\tan\frac{\alpha}{2}}{\tan\frac{\zeta}{2}-\tan\frac{\alpha}{2}} \quad (5.72)$$

and

$$\int^{\alpha}\frac{1}{2}\frac{\sin\frac{\zeta}{2}}{\cos\frac{\xi}{2}-\cos\frac{\zeta}{2}}\mathrm{d}\xi=\ln\frac{\tan\frac{\zeta}{4}+\tan\frac{\alpha}{4}}{\tan\frac{\zeta}{4}-\tan\frac{\alpha}{4}}, \quad (5.73)$$

equation (5.28a) and (5.28c) then respectively become, as $\delta \to \pi/2$,

$$\Psi_4\left(\frac{\pi}{2}\right)\frac{\tan\frac{\zeta_{2\pi}}{2}+1}{\tan\frac{\zeta_{2\pi}}{2}-1}\frac{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\pi}{8}}{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\pi}{8}} = 1 \tag{5.74}$$

and

$$\frac{\Psi_4\left(\frac{3\pi}{2}\right)}{\Psi_4\left(\frac{\pi}{2}\right)}\left(\frac{\tan\frac{\zeta_{2\pi}}{2}-1}{\tan\frac{\zeta_{2\pi}}{2}+1}\right)^2\frac{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{3\pi}{8}}{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{3\pi}{8}}\frac{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\pi}{8}}{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\pi}{8}} = 1. \tag{5.75}$$

Solving for $\zeta_{2\pi}$ and $\zeta_{4\pi}$ in the above equation pair yields the quadratic in $\tan\zeta_{4\pi}/4$

$$\tan^2\frac{\zeta_{4\pi}}{4}+\left(\tan\frac{\pi}{8}+\tan\frac{3\pi}{8}\right)\frac{\Psi_4\left(\frac{\pi}{2}\right)\Psi_4\left(\frac{3\pi}{2}\right)+1}{\Psi_4\left(\frac{\pi}{2}\right)\Psi_4\left(\frac{3\pi}{2}\right)-1}\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\pi}{8}\tan\frac{3\pi}{8} = 0, \tag{5.76}$$

and, for $\zeta_{2\pi}$,

$$\zeta_{2\pi} = 2\arctan\frac{1+\Psi_4\left(\frac{\pi}{2}\right)\dfrac{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\pi}{8}}{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\pi}{8}}}{1-\Psi_4\left(\frac{\pi}{2}\right)\dfrac{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\pi}{8}}{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\pi}{8}}}. \tag{5.77}$$

When $\delta = \pi/2$, equation (5.76) gives a value of $\zeta_{4\pi} = 5.796 + j0.1015\cdot 10^{-1}$ and (5.77) a value of $\zeta_{2\pi} = 3.120 + j3.470\cdot 10^{-3}$, in agreement to the accuracy provided with the values in Tables 5.2 and 5.3 obtained by carrying out the entire inversion procedure when $\delta = 1.57$. Since the κ vanish when $\delta = \pi/2$, the solution $w(\alpha, u)$ in (5.56) assumes the simple form

$$w(\alpha, u) = \frac{\tan\frac{\zeta_{2\pi}}{2}+\tan\frac{\alpha}{2}}{\tan\frac{\zeta_{2\pi}}{2}-\tan\frac{\alpha}{2}}\frac{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\alpha}{4}}{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\alpha}{4}}\Psi_4(\alpha), \tag{5.78}$$

and, similarly,

$$w(\alpha, -u) = \frac{\tan\frac{\zeta_{2\pi}}{2}-\tan\frac{\alpha}{2}}{\tan\frac{\zeta_{2\pi}}{2}+\tan\frac{\alpha}{2}}\frac{\tan\frac{\zeta_{4\pi}}{4}-\tan\frac{\alpha}{4}}{\tan\frac{\zeta_{4\pi}}{4}+\tan\frac{\alpha}{4}}\frac{1}{\Psi_4(\alpha)}, \tag{5.79}$$

where the rational terms are the contributions from $v^3_{2\pi,4\pi}(\alpha, u)$. The recovery of the known solution $\Psi_4(\alpha)$ is achieved by properly constructing the branch-free solutions, the topic of the next section.

5.2.6.2 The limit $\delta \to 0$

As noted in Section 4.2.5, the other limit of interest $\delta \to 0$ is considerably more complicated and is not currently well understood. The best we could do right now is derive the limiting values of the κ and express them in terms of the poles ζ. In essence, we could obtain a form equivalent to (4.90) but now with unknowns $\zeta_{2\pi}$ and $\zeta_{4\pi}$. Moreover, apart from the behaviors indicated in Figures 5.9 and 5.11, the limiting values of the ζ poles are not well understood. We postpone further study of this limit, since it is more appropriate to pursue it once the simpler case in Section 4.2.5.2 is completed.

5.3 Branch-free solutions

We now have in hand well-defined solutions to the first order equations (5.2) which are also solutions, albeit unacceptable individually, to the second order difference equation (5.1). The reason for this is that $w(\alpha, u)$ still has branch points as well as poles and zeros in the strip $S_{4\pi}$. Meromorphic solutions to the second order difference equation can be obtained through the use of expressions (3.3). We recall however that solutions free of poles and zeros in the strip $S_{4\pi}$ are required, and we therefore seek to use specific linear combinations of (3.3) to finalize the construction of the solutions. Knowledge of the poles and zeros of $w(\alpha, u)$ is required to successfully complete this endeavor and this is provided in (5.65).[2] With this information the poles of any linear combinations involving (3.3) are easily determined. Zeros are by nature more elusive and we rely on knowledge of the limiting functions in order to determine their number as well as general location. The cancellation of the poles and zeros is also complicated by the order requirement on the solutions which must be $O(1)$ as $|\text{Im}\,\alpha| \to \infty$. We first present an entirely analytic approach from which two independent solutions are obtained, and its only drawback is that the proposed solutions vanish as $\delta \to 0$. Though knowledge of (5.10) circumvents this difficulty, we also provide two additional approaches where the limiting functions do not vanish in that limit and this is achieved at the cost of numerically locating zeros. In the following, we use the primed functions $t'_n(\alpha)$ to denote intermediate branch-free solutions of the second order difference equation (5.1) which still have undesired poles and zeros in the strip $S_{4\pi}$ whereas the unprimed functions $t_n(\alpha)$ denote the appropriate branch-free and pole/zero-free solutions.

[2] We still assume $\sigma_{2\pi} = \sigma_{4\pi} = 1$ for simplicity in what follows.

5.3.1 Analytical solution

We proceed by constructing two meromorphic solutions of the second order equation (5.1) $t_1'(\alpha)$ and $t_2'(\alpha)$ sharing a common pole at $\zeta_{4\pi}$ but having distinct unknown zeros α_1 and α_2 and then use a linear combination to obtain an expression with a known pole/zero pair. Proceeding in a manner similar to the technique presented in Section 4.3, we write

$$t_1'(\alpha) = \frac{Q_{2\pi}(\alpha)}{2}\left\{\left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\}, \tag{5.80}$$

which is a linear combination of the branch-free forms (3.3). The functions $r_1(\alpha)$ and $Q_{2\pi}(\alpha)$ are 4π periodic; $r_1(\alpha)$ is a trigonometric polynomial used to introduce zeros at appropriate locations in the α plane while the external multiplicative function $Q_{2\pi}(\alpha)$ is a rational trigonometric function used to annul poles and zeros. Once again, we introduce double zeros coincident with the poles of either $w(\alpha, u)$ or $w(\alpha, -u)$ to produce a simple zero in the term in curly braces above. Specifically, if we require

$$1 - \frac{r_1(\alpha)}{u(\alpha)} \sim (\alpha + \zeta_{2\pi})^2(\alpha + (\zeta_{2\pi} - 2\pi))^2(\alpha + \zeta_{4\pi}) \tag{5.81}$$

then the second term in braces in (5.80) has simple zeros coinciding with those of $w(\alpha, u)$ at $\alpha = -\zeta_{2\pi}, 2\pi - \zeta_{2\pi}$ and is finite at $\alpha = -\zeta_{4\pi}$. This implies

$$\left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, -u) \sim \frac{\alpha + \zeta_{2\pi}}{\alpha - \zeta_{2\pi}}\frac{\alpha + (\zeta_{2\pi} - 2\pi)}{\alpha - (\zeta_{2\pi} - 2\pi)}\frac{\alpha - \alpha_1}{\alpha - \zeta_{4\pi}} \tag{5.82}$$

where α_1 is the unknown location of a zero in the $S_{4\pi}$ strip. While the exact location of α_1 is not easily determined, its general location is known when δ is in the neighborhood of $\pi/2$. Indeed, as $\delta \to \pi/2$ we have $r_1(\alpha)/u(\alpha) \to 1$ and

$$\frac{1}{2}\left\{\left(1 + \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r_1(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\} \xrightarrow{\delta \to \pi/2} w(\alpha, u), \tag{5.83}$$

and we conclude with the help of (5.78) that when δ is in the neighborhood of $\pi/2$, α_1 is in the neighborhood of $-\zeta_{4\pi}$. Choosing $Q_{2\pi}(\alpha)$ to eliminate the poles and zeros associated with $\zeta_{2\pi}$ gives

$$Q_{2\pi}(\alpha) = \frac{\tan\frac{\zeta_{2\pi}}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_{2\pi}}{2} + \tan\frac{\alpha}{2}} \tag{5.84}$$

so that

$$t_1'(\alpha) \sim \frac{\alpha - \alpha_1}{\alpha - \zeta_{4\pi}}. \tag{5.85}$$

The 4π periodic function $r_1(\alpha)$ is obtained by letting

$$r_1(\alpha) = \rho_1 + \rho_2 \cos\alpha + \rho_3 \sin\alpha + \rho_4 \cos\frac{\alpha}{2} + \rho_5 \sin\frac{\alpha}{2} \tag{5.86}$$

and enforcement of (5.81) produces

$$\rho_1 = \frac{1}{1 - \cos(\zeta_{2\pi} - \zeta_{4\pi})}\left\{u(\zeta_{2\pi}) - \frac{1}{u(\zeta_{4\pi})}(\sin^2\delta\cos\zeta_{2\pi}\cos\zeta_{4\pi} - \cos^2\delta\sin\zeta_{2\pi}\sin\zeta_{4\pi})\right\}, \tag{5.87a}$$

$$\rho_2 = \cos\zeta_{2\pi}\left(-\zeta_{2\pi} + \frac{\sin^2\delta}{u(\zeta_{2\pi})}\right), \tag{5.87b}$$

$$\rho_3 = \sin\zeta_{2\pi}\left(\zeta_{2\pi} + \frac{\cos^2\delta}{u(\zeta_{2\pi})}\right), \tag{5.87c}$$

with $\rho_4 = \rho_5 = 0$. We note that, in agreement with the analyticity requirements, the function $Q_{2\pi}(\alpha)$ and the ratio $r_1(\alpha)/u(\alpha)$ are $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. A related meromorphic solution to the second order difference equation sharing the same pole but having a different zero is

$$t_2'(\alpha) = \frac{Q_{2\pi}(\alpha)}{2}\left\{\left(1 - \frac{r_2(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 + \frac{r_2(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\}, \tag{5.88}$$

where, following the same kind of procedure as for $r_1(\alpha)$, $r_2(\alpha)$ is now chosen such that

$$1 - \frac{r_2(\alpha)}{u(\alpha)} \sim (\alpha - \zeta_{2\pi})^2(\alpha - (\zeta_{2\pi} - 2\pi))^2, \tag{5.89}$$

$$1 + \frac{r_2(\alpha)}{u(\alpha)} \sim (\alpha + \zeta_{4\pi}), \tag{5.90}$$

and hence

$$t_2'(\alpha) \sim \frac{\alpha - \alpha_2}{\alpha - \zeta_{4\pi}}. \tag{5.91}$$

By the same reasoning as above, the zero α_2 is also in the neighborhood of $\alpha = -\zeta_{4\pi}$ when δ in in the neighborhood of $\pi/2$. The meromorphic solutions $t_1'(\alpha)$ and $t_2'(\alpha)$ then share the same pole and a linear combination can be used to introduce a zero at $\alpha = -\zeta_{4\pi}$ such that

$$t_1'(\alpha) + \chi t_2'(\alpha) \sim \frac{\alpha + \zeta_{4\pi}}{\alpha - \zeta_{4\pi}} \tag{5.92}$$

and this requires

$$\chi = -\frac{t_1'(-\zeta_{4\pi})}{t_2'(-\zeta_{4\pi})} = -Q_{2\pi}^2(-\zeta_{4\pi})\frac{\sin\zeta_{4\pi}\cos\zeta_{4\pi} - u(\zeta_{4\pi})(\rho_2\sin\zeta_{4\pi} + \rho_3\cos\zeta_{4\pi})}{\sin\zeta_{4\pi}\cos\zeta_{4\pi} + u(\zeta_{4\pi})(\rho_2\sin\zeta_{4\pi} + \rho_3\cos\zeta_{4\pi})}. \tag{5.93}$$

An acceptable solution of the second order difference equation (5.1), free of poles and zeros in $S_{4\pi}$ and $O(1)$ as $|\text{Im}\,\alpha| \to \infty$, is then

$$t_1(\alpha) = \frac{\tan\frac{\zeta_{4\pi}}{4} - \tan\frac{\alpha}{4}}{\tan\frac{\zeta_{4\pi}}{4} + \tan\frac{\alpha}{4}}\left\{t_1'(\alpha) + \chi t_2'(\alpha)\right\}, \tag{5.94}$$

and

$$t_1(\alpha) \xrightarrow{\delta\to\pi/2} \Psi_4(\alpha), \qquad t_1(\alpha) \xrightarrow{\delta\to 0} 0. \tag{5.95}$$

The first limit stems from the known behavior for $w(\alpha, u)$ given in (5.78) together with the fact that as $\delta \to \pi/2$, $1 - r_1(\alpha)/u(\alpha) \to 0$ and $\chi \to 0$. The second limit follows since $t_1' \to t_2'$ and $\chi \to -1$ as $\delta \to 0$. Though undesirable, this is not a serious drawback as a pair of linearly independent solutions are provided by (5.10) and its reciprocal.

Following the same prescription as above, a second independent solution $t_2(\alpha)$ can be derived by seeking instead a common pole at $-\zeta_{4\pi}$ and it can be shown that

$$t_2(\alpha) = t_1(-\alpha), \tag{5.96}$$

but we now have, using the same arguments as above,

$$t_2(\alpha) \xrightarrow{\delta\to\pi/2} \frac{1}{\Psi_4(\alpha)}, \qquad t_2(\alpha) \xrightarrow{\delta\to 0} 0. \tag{5.97}$$

Both $t_1(\alpha)$ and $t_2(\alpha)$ recover the same limit as $\delta \to 0$; this is also the case for the numerical approaches described below and it will generally be necessary to use one (or both, in this particular case) of the known solutions (5.10) to obtain a pair of independent solutions.

Sample curves for $|t_1(\alpha)|$ are provided in Figure 5.12 for various values of δ when $\theta = 0.25(1 + \mathrm{j})$. We observe that $t_1(\alpha) \to \Psi_4(\alpha)$ as $\delta \to \pi/2$, and $t_1(\alpha) \to 0$ as $\delta \to 0$. Due to their vanishing nature as $\delta \to 0$, the solutions $t_1(\alpha)$ and $t_2(\alpha)$ fall short of the more ideal behavior obtained in Section 4.3 where the solution varies smoothly as a function of

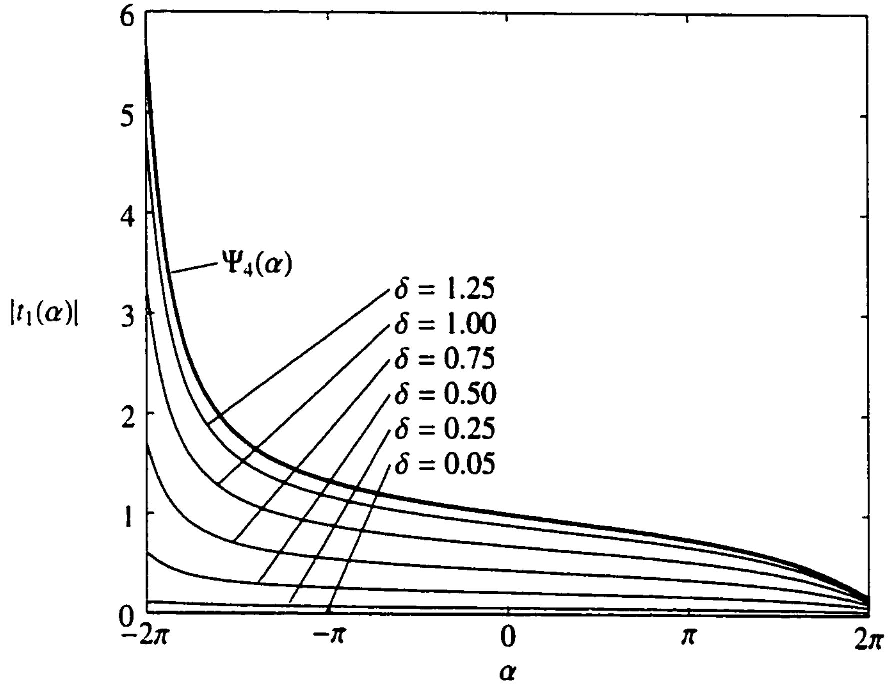


Figure 5.12: Magnitude of the branch-free solution $t_1(\alpha)$ given in (5.94) when $\theta = 0.25(1 + j)$ for various values of δ. The thicker line corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$. The case for $\delta = 1.57$ is indistinguishable from $\Psi_4(\alpha)$.

δ between the two known solutions when $\delta = \pi/2$ and $\delta = 0$. We now examine alternatives to this purely analytical approach to partially overcome this shortcoming.

5.3.2 Numerical solutions

By foregoing an entirely analytical approach, it is possible to construct solutions which, unlike $t_{1,2}(\alpha)$, do not vanish as $\delta \to 0$ and still recover the known solution $\Psi_4(\alpha)$ as $\delta \to \pi/2$. In such cases we must resort to the numerical identification of zeros and we consider two such variations. However, these approaches have so far been only partially successful as the procedure fails when $\delta \simeq 0$.

In the first instance we proceed by writing

$$t_3'(\alpha) = \frac{Q_{2\pi}(\alpha)}{2}\left\{\left(1 + \frac{r_3(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r_3(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\}, \tag{5.98}$$

and, as in Section 4.3, we choose $r_3(\alpha)$ such that — compare with (5.81) —

$$1 - \frac{r_3(\alpha)}{u(\alpha)} \sim (\alpha + \zeta_{2\pi})^2(\alpha + (\zeta_{2\pi} - 2\pi))^2, \tag{5.99}$$

requiring

$$r_3(\alpha) = \frac{\cos\zeta_{2\pi}\sin^2\delta\cos\alpha + \sin\zeta_{2\pi}\cos^2\delta\sin\alpha}{u(\zeta_{2\pi})}, \tag{5.100}$$

and (5.65) now implies that

$$t_3'(\alpha) \sim \frac{(\alpha-\alpha_1)(\alpha-\alpha_2)}{(\alpha-\zeta_{4\pi})(\alpha+\zeta_{4\pi})}. \tag{5.101}$$

Knowledge on the location and behavior of the zeros α_1 and α_2 is again obtained by examining the limiting behavior

$$t_3'(\alpha) \xrightarrow{\delta\to\pi/2} \frac{\tan\frac{\zeta_{4\pi}}{4} + \tan\frac{\alpha}{4}}{\tan\frac{\zeta_{4\pi}}{4} - \tan\frac{\alpha}{4}}\Psi_4(\alpha), \tag{5.102}$$

which suggests that, as $\delta \to \pi/2$, the two zeros $\alpha_{1,2}$ will coalesce with the pole at $\alpha = -\zeta_{4\pi}$ to produce a simple zero. We therefore expect the zeros $\alpha_{1,2}$ to gravitate near the pole at $-\zeta_{4\pi}$ when δ is in the neighborhood of $\pi/2$ and computations show this is indeed the case. In order to obtain a solution free of poles and zeros, the final solution will now have the form

$$t_3(\alpha) = \frac{\sin\frac{\alpha_1}{4}\sin\frac{\alpha_2}{4}}{\sin^2\frac{\zeta_{4\pi}}{4}}\frac{\sin\frac{1}{4}(\zeta_{4\pi}-\alpha)\sin\frac{1}{4}(\zeta_{4\pi}+\alpha)}{\sin\frac{1}{4}(\alpha_1-\alpha)\sin\frac{1}{4}(\alpha_2-\alpha)}t_3'(\alpha) \tag{5.103}$$

where it is necessary to locate the zeros α_1 and α_2 of $t_3'(\alpha)$ numerically. The solution $t_3(\alpha)$ is such that

$$t_3(\alpha) \xrightarrow{\delta\to\pi/2} \Psi_4(\alpha), \tag{5.104}$$

as desired, and it also tends to a known non-zero function as $\delta \to 0$. In that limit $\zeta_{2\pi} \to \pi$, $\zeta_{4\pi} \to 2\pi$, $f(\alpha)/u(\alpha) \to 0$, and $\alpha_1 \to -2\pi$ leading to

$$t_3(\alpha) \xrightarrow{\delta\to 0} \frac{1}{2}\frac{\sin\frac{\alpha_2}{4}\cos\frac{\alpha}{4}}{\sin\frac{1}{4}(\alpha_2-\alpha)}\left(\Psi_5(\alpha) + \frac{1}{\Psi_5(\alpha)}\right) = \Upsilon(\alpha), \tag{5.105}$$

where α_2 corresponds to a (specific) zero of $\Psi_5(\alpha) + 1/\Psi_5(\alpha)$. Note that the zeros of $\cos\alpha/4$ at $\pm 2\pi$ coincide with the poles of $1/\Psi_5(\alpha)$. The limiting function $\Upsilon(\alpha)$ is however flawed as the aforementioned zero α_2 strays outside of the strip $S_{4\pi}$ when $\delta \simeq 0$, thereby jeopardizing the analyticity of the solution as the zero cancelling term in α_2 then gives rise to a pole within the strip. The solution $t_3(\alpha)$ is therefore valid only when δ is not in the

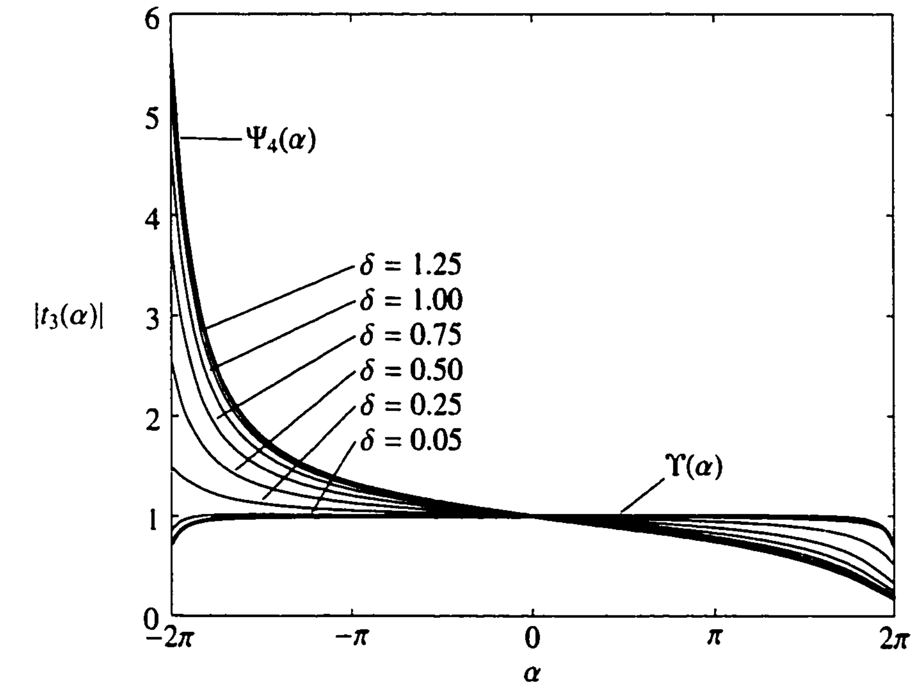


Figure 5.13: Magnitude of the branch-free solution $t_3(\alpha)$ given in (5.103) when $\theta = 0.25(1 + \mathrm{j})$ for various values of δ. The thicker lines corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$ and $\Upsilon(\alpha)$, per (5.105), for $\delta = 0$.

neighborhood of zero. A second solution is provided by $t_3(-\alpha)$ which recovers $1/\Psi_4(\alpha)$ when $\delta \to \pi/2$ and, as for $t_3(\alpha)$, $\Upsilon(\alpha)$ when $\delta \to 0$. Sample curves for $|t_3(\alpha)|$ are provided in Figure 5.13 for various values of δ when $\theta = 0.25(1+\mathrm{j})$ along with the limiting functions $\Psi_4(\alpha)$ and $\Upsilon(\alpha)$.

Interestingly, it is possible to reproduce the more desirable behavior obtained in Section 4.3 where the 2π periodic equation was solved. There, the solutions obtained vary smoothly between two known limiting functions $\Psi_1(\alpha)$ as $\delta \to \pi/2$ and $\Psi_2(\alpha)$ as $\delta \to 0$. A similar result can be achieved here but at the cost of having to numerically locate four zeros. Turning once again to a familiar form, we write

$$t_4'(\alpha) = \frac{Q_{4\pi}(\alpha)}{2}\left\{\left(1 + \frac{r_4(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r_4(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\}, \tag{5.106}$$

and examination of the behavior of the solution in Section 4.3 suggests using

$$r_4(\alpha) = \frac{\cos\zeta_{4\pi}\sin^2\delta\cos\alpha + \sin\zeta_{4\pi}\cos^2\delta\sin\alpha}{u(\zeta_{4\pi})}, \tag{5.107}$$

so that

$$1 - \frac{r_4(\alpha)}{u(\alpha)} \sim (\alpha + \zeta_{4\pi})^2(\alpha + (\zeta_{4\pi} - 2\pi))^2. \tag{5.108}$$

Selection of (5.107) comes about by considering the limiting values assumed by δ and $\zeta_{4\pi}$ and picking $r_4(\alpha)$ so that, in the limits $\delta \to 0, \pi/2$, the expressions $1 \pm r_4(\alpha)/u(\alpha)$ assume values of 0 or 2, recovering the behavior obtained when dealing with the equations with period 2π. The term in braces in (5.106) is then such that

$$\{\} \sim \frac{\alpha + \zeta_{4_\pi}}{\alpha - \zeta_{4_\pi}} \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4)}{(\alpha + \zeta_{2\pi})(\alpha - \zeta_{2\pi})(\alpha + (\zeta_{2\pi} - 2\pi))(\alpha + (\zeta_{2\pi} + 2\pi))} \tag{5.109}$$

and the pole and zero associated with $\zeta_{4\pi}$ are eliminated by choosing

$$Q_{4\pi}(\alpha) = \frac{\tan\frac{\zeta_{4\pi}}{4} - \tan\frac{\alpha}{4}}{\tan\frac{\zeta_{4\pi}}{4} + \tan\frac{\alpha}{4}}. \tag{5.110}$$

In the limit as $\delta \to \pi/2$ we have

$$t_4'(\alpha) \sim \frac{\alpha + \zeta_{2\pi}}{a - \zeta_{2\pi}} \frac{\alpha + (\zeta_{2\pi} - 2\pi)}{\alpha - (\zeta_{2\pi} - 2\pi)} \tag{5.111}$$

which implies that, when δ is in the neighborhood of $\pi/2$, the poles at $\alpha = -\zeta_{2\pi}$ and $\alpha = -\zeta_{2\pi} + 2\pi$ will each have a closely located pair of zeros α_n. Once the location of these zeros has been obtained numerically, the desired solution may be written as

$$t_4(\alpha) = \frac{\cos\alpha - \cos\zeta_{2\pi}}{1 - \cos\zeta_{2\pi}} \left(\prod_{n=1}^{4} \frac{\sin\frac{\alpha_n}{4}}{\sin\frac{1}{4}(\alpha_n - \alpha)} \right) t_4'(\alpha) \tag{5.112}$$

so that $t_4(\alpha)$ is free of poles and zeros in $S_{4\pi}$. It is easily shown that

$$t_4(\alpha) \xrightarrow{\delta \to \pi/2} \Psi_4(\alpha), \tag{5.113}$$

and since $r_4(\alpha)/u(\alpha) \to \pm 1$ as $\delta \to 0$, albeit in a branched fashion, it can also be shown (numerically) that

$$t_4(\alpha) \xrightarrow{\delta \to 0} \Psi_5(\alpha). \tag{5.114}$$

Figure 5.14 provides sample curves for $|t_4(\alpha)|$ for various values of δ when $\theta = 0.25(1 + j)$. The behavior obtained should be compared with that of Figure 4.11 on page 93 as the solution now varies smoothly between the two limiting functions $\Psi_{4,5}(\alpha)$ as a function of δ.

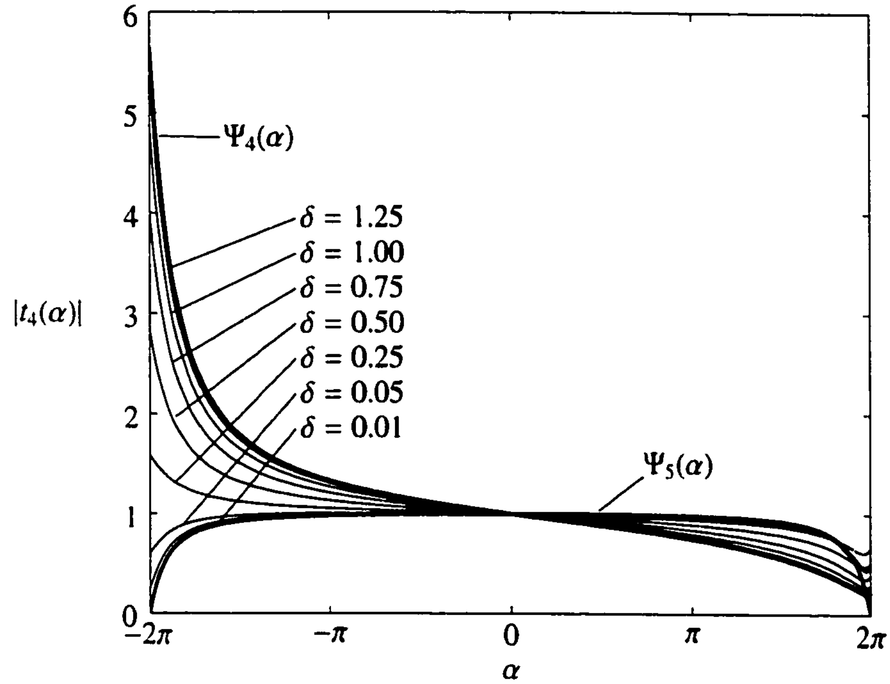


Figure 5.14: Magnitude of the branch-free solution $t_4(\alpha)$ given in (5.112) when θ = 0.25(1 + j) for various values of δ. The thicker lines corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$ and $\Psi_5(\alpha)$, per (5.10), for $\delta = 0$.

Once again, a second solution is provided by $t_4(-\alpha)$ and this recovers $1/\Psi_4(\alpha)$ as $\delta \to \pi/2$ and $\Psi_5(\alpha)$ as $\delta \to 0$. This approach also displays, however, a problem similar to the previous case since one of the numerical zeros strays slightly outside $S_{4\pi}$ when $\delta \simeq 0.001$ which, strictly speaking, means that the solution also fails in that limit. It is unclear at this time if this is due to numerical inaccuracies or a fundamental limitation of the approach. It is intriguing that this almost succeeds in recovering $\Psi_5(\alpha)$ but ultimately falls short.

5.4 Summary

The proposed technique was successfully applied to a generalization of the equation for the penetrable wedge solved in Chapter 4 where the period of the difference equation is doubled from 2π to 4π. This doubles the width of the strip of analyticity and hence doubles the number of poles and branch points found therein. The analysis parallels closely the one given in the previous chapter with two important distinctions. The first is that the elimination of the cyclic periods now required analysis on a Riemann surface of genus three. This was substantially more complicated that the one done previously but was nevertheless carried out fully analytically. Secondly, the construction of meromorphic solutions free of poles and zeros in the strip of width 4π presented a significant challenge due to a paucity of degrees of freedom. Solutions could be constructed analytically but only just. Furthermore, despite recovering the desired solution when $\delta \to \pi/2$, the solutions vanish as $\delta \to 0$. Two attempts at circumventing this were made by constructing meromorphic solutions whose zeros must be located numerically. In the first instance it is required to locate two zeros numerically whereas the second one requires the identification of four zeros. Given the added complication of numerically locating zeros, their behavior is at best marginally superior to the analytical solution: both are almost successful in recovering known limits when $\delta \to 0$ but ultimately fail when δ is in that neighborhood.

CHAPTER 6

THE METHOD OF PERIOD REDUCTION

COMPARISON of the results in Chapters 4 and 5 shows that a larger number of branch points in the strip of analyticity generally calls for analyses on Riemann surfaces of higher genus and this significantly complicates the derivation of the bilinear relation of Riemann. Looking ahead to the problem of the anisotropic half-plane, which has sixteen branch points in the strip of analyticity, this provides an incentive to develop a variation of the technique presented in Chapter 3 where the analysis required to eliminate the cyclic periods is simplified. This is achieved by deriving first order equations whose period is reduced by half, thereby halving the width of the strip of analyticity and this in turn halves the numbers of singularities that must be dealt with. The resulting reduced equation, obtained in a purely algebraic approach, has symmetry properties which differ fundamentally from those of the original equation and this has interesting ramifications on the properties of its solution. Unfortunately, the simplification comes at the price of making the construction of the branch-free solutions much more difficult. Two variants of the technique, termed the method of global elimination and the method of split elimination, are discussed.

6.1 Benefits of a reduced period

Two closely related second order difference equations were solved in the proceeding two chapters. In Chapter 4 an equation of period 2π associated with the problem of the penetrable composite wedge, discussed in Section 2.4, was solved. This was followed in Chapter 5 by a solution to an equation of the same form but with the period doubled to 4π. A comparison of the analyses required, particularly the part pertaining to the elimination of cyclic periods, reveals that an increase in the number of singularities in the strip of analyticity entails a more complicated analysis. In this light, there is cause for concern if one contemplates the problem of the anisotropic impedance half-plane, introduced in

Section 2.2, whose strip of analyticity is populated by no less than 16 branch points. This is better appreciated by comparing the strips of analyticity of the equation of period 2π, period 4π and that of the anisotropic half-plane which are collected for convenience in Figure 6.1. Simply put, the number of branch point singularities in the strip of analyticity for the anisotropic half-plane is twice that of the equation with period 4π tackled in Chapter 5, a solution which required analysis on a Riemann surface of genus three. Indications are therefore that direct attempts to solve the problem of the anisotropic half-plane will at best be very difficult and this is the subject of more discussion in Chapter 7. A reduction of the period of analyticity of the first order difference equation would reduce the problem to one with a strip of analyticity of width 2π, corresponding to the inner strip in Figure 6.1(c). This is the same as for the first generalization discussed in Section 4.4.1 for which a solution was sketched out using only two degrees of freedom. The prospect of such a simplification warrants a close examination of the derivation and solution of equations with reduced period. To garner insight, we first apply it to the case of the second order difference equation of period 4π solved in the previous chapter whose strip of analyticity is illustrated in Figure 6.1(b).

6.2 Preliminary Analysis

We consider again the second order difference equation

$$t(\alpha+5\pi) - 2\left\{1 - 2\frac{\cos^2\delta - \cos^2\theta}{\cos^2\alpha - \cos^2\theta}\right\} t(\alpha+\pi) + t(\alpha-3\pi) = 0 \tag{6.1}$$

which was solved in Chapter 5. As shown there, (6.1) can be recast as the pair of first order difference equations of period 4π

$$\frac{w(\alpha+2\pi, \pm u)}{w(\alpha-2\pi, \pm u)} = g(\alpha, \pm u) = \left(\frac{u(\alpha) - u(\theta)}{u(\alpha) + u(\theta)}\right)^{\pm 1} \tag{6.2}$$

where

$$u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta}. \tag{6.3}$$

Solutions to (6.1) are sought that are meromorphic, $O(1)$ as $|\text{Im}\,\alpha| \to \infty$ and free of poles and zeros in the strip $S_{4\pi}$. This, as carried out in Chapter 5, can be achieved by first obtaining well-defined branched solutions to (6.2) and taking proper branch-free linear combinations. In this chapter the overall approach remains mostly unchanged save for the

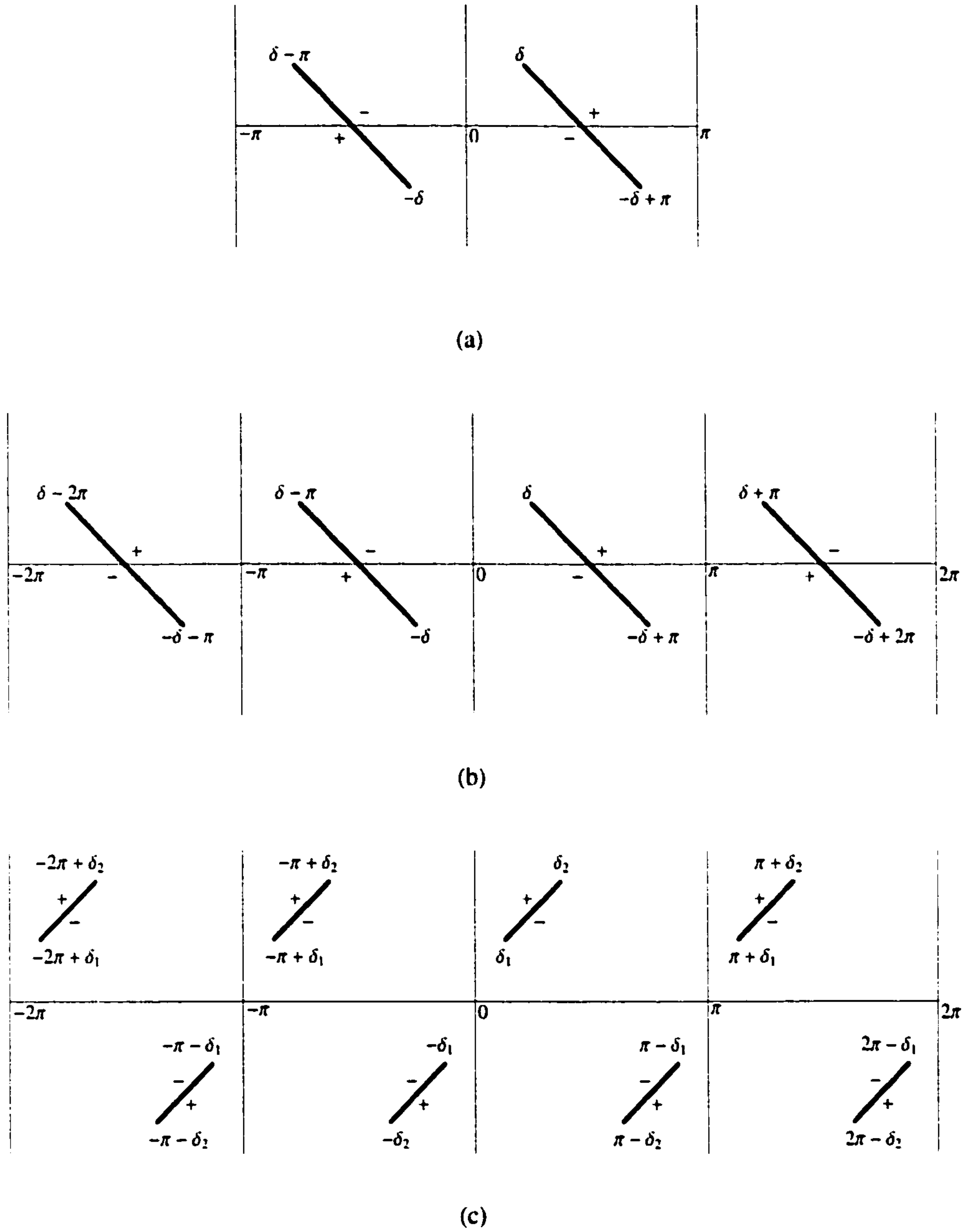


Figure 6.1: A comparison of the strips of analyticity for the (a) equation of period 2π solved in Chapter 4, (b) equation of period 4π solved in Chapter 5 and (c) the anisotropic half-plane discussed in Section 2.2.

introduction of an intermediate step where a first order difference equation of period 2π is derived from the above first order equations of period 4π. The key lies in rewriting the right-hand side of (6.2) as

$$\frac{u(\alpha)-u(\theta)}{u(\alpha)+u(\theta)} = \frac{\cos(\alpha)-\cos(\theta)}{\cos(\alpha)+\cos(\theta)}\left(\frac{\cos(\alpha)+\cos(\theta)}{u(\alpha)+u(\theta)}\right)^2. \tag{6.4}$$

The first member on the right-hand side is familiar: it corresponds to $g(\alpha,u)$ when $\delta=\pi/2$ and, writing

$$\frac{w_1(\alpha+2\pi)}{w_1(\alpha-2\pi)} = \frac{\cos(\alpha)-\cos(\theta)}{\cos(\alpha)+\cos(\theta)}, \tag{6.5}$$

it follows immediately from Section 5.1.1.1 that

$$w_1(\alpha) = \Psi_4(\alpha) = \frac{\psi_\pi\left(\alpha+\frac{\pi}{2}-\theta\right)}{\psi_\pi\left(\alpha-\frac{\pi}{2}+\theta\right)}. \tag{6.6}$$

Recall that $\Psi_4(\alpha)$ is free of poles and zeros in $\mathcal{S}_{4\pi}$ and $\mathcal{O}(1)$ as $|\text{Im}\,\alpha|\to\infty$. The second member on the right-hand side of (6.4) is significantly more interesting — and more complicated. The possibility of reducing the period is opened up by the presence of the square. Indeed, note that if

$$\frac{w_2(\alpha+\pi,\pm u)}{w_2(\alpha-\pi,\pm u)} = \frac{\cos\alpha-\cos\theta}{\pm u(\alpha)-u(\theta)} = g_2(\alpha,\pm u) \tag{6.7}$$

then

$$\frac{w_2(\alpha+2\pi,u)}{w_2(\alpha-2\pi,u)} = \frac{w_2(\alpha+2\pi,u)}{w_2(\alpha,u)}\frac{w_2(\alpha,u)}{w_2(\alpha-2\pi,u)} = \left(\frac{\cos(\alpha)+\cos(\theta)}{u(\alpha)+u(\theta)}\right)^2 \tag{6.8}$$

and the solution to (6.2) can be written as

$$w(\alpha,u) = C_{4\pi}(\alpha)w_1(\alpha)w_2(\alpha,u) \tag{6.9}$$

where $C_{4\pi}(\alpha)$ is a 4π periodic function. We now seek a solution $w_2(\alpha,u)$ to (6.7) which is well defined in the strip $\mathcal{S}_{2\pi}$ shown in Figure 6.1(a) as opposed to $\mathcal{S}_{4\pi}$ shown in Figure 6.1(b). The cancellation of cyclic periods should introduce a minimum number of poles and zeros and while it is also desirable for $w_2(\alpha,u)$ to be $\mathcal{O}(1)$ as $|\text{Im}\,\alpha|\to\infty$, we shall soon see that this is not possible. This is hinted at by the form of $g_2(\alpha,u)$ in (6.7) which differs from its previously encountered counterparts where $g(\alpha,u)=1/g(\alpha,-u)$. The effect of this loss of symmetry will be made clear when the construction of a solution is carried out. It must also be mentioned that, like its predecessors, the first order difference equations (6.2) also have a parent second order difference equation given, in this particular case, by

$$(\cos\alpha - \cos\theta)t(\alpha + 4\pi) + 2u(\theta)t(\alpha + 2\pi) - (\cos\alpha + \cos\theta)t(\alpha) = 0. \tag{6.10}$$

Properly constructed solutions of equation with reduced period (6.7) will therefore also satisfy the second order equation (6.10). Furthermore, branch-free solutions to (6.10) are also possible if the branch-free forms (3.3) are used.

6.2.1 Construction of the branched solution

A branched solution to the difference equation with reduced period (6.7) is sought. Taking the logarithmic derivative of (6.7), we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}w_2(\alpha+\pi,u) - \frac{\mathrm{d}}{\mathrm{d}\alpha}w_2(\alpha-\pi,u) = -\left(1 - \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta}\right)\frac{\cos\theta\sin\alpha}{\cos^2\alpha - \cos^2\theta}, \tag{6.11}$$

which leads to a preliminary solution

$$w_2(\alpha,u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi,u) + v_{2\pi}(\xi,u)\right\}\mathrm{d}\xi \tag{6.12}$$

where, from (6.11), it follows that

$$v_0(\alpha,u) = \frac{\alpha}{2\pi}\left(1 - \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta}\right)\frac{\cos\theta\sin\alpha}{\cos^2\alpha - \cos^2\theta}. \tag{6.13}$$

This is somewhat similar to its counterparts (4.26) for the penetrable wedge and (5.13) for the extension with period 4π but, as suggested by $g_2(\alpha,u)$ above, the symmetry $v_0(\alpha,u) = -v_0(\alpha,-u)$ has now been lost. Consequently, as opposed to having poles on one Riemann sheet and corresponding poles with negative residues on the other sheet, $v_0(\alpha,u)$ has poles at $\alpha = (\pm\theta, -u(\theta)), (\pm(\pi-\theta), -u(\pi-\theta))$ but is free of poles on the sheet $+u(\alpha)$. However, it still vanishes as $|\mathrm{Im}\,\alpha| \to \infty$, like its predecessors. The obligatory 2π periodic functions in (6.12), which eliminate the polar and cyclic periods of the integrand, are required to obtain a well defined path integral. Once this condition is fulfilled the lower limit α_0 may be chosen to be any of the branch points in the strip of width 2π.

We now turn to the familiar exercise of polar period elimination. Exploiting the even parity of $v_0(\alpha,u)$, a total of two degrees of freedom are required to eliminate its poles. We introduce the 2π periodic function

$$v_1(\alpha, u) = \left(1 - \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta}\right)\frac{\varphi_1 + \varphi_2\cos\alpha}{\cos^2\alpha - \cos^2\theta} \tag{6.14}$$

which has the same symmetry properties as $v_0(\alpha, u)$: it vanishes as $|\mathrm{Im}\,\alpha| \to \infty$, is even, and has poles only on the $-u(\alpha)$ Riemann sheet. It should also be noted that (6.14) is the only such expression available if the order requirement is observed. Enforcement of vanishing residues leads to

$$\varphi_1 = -\frac{1}{4}\sin\theta\cos\theta, \quad \text{and} \quad \varphi_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{\theta}{\pi}\right)\sin\theta. \tag{6.15}$$

Summing up $v_0(\alpha, u)$ and $v_1(\alpha, u)$, we obtain

$$v_0(\alpha, u) + v_1(\alpha, u) = \frac{1}{2}\left(1 - \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta}\right) \cdot \frac{\frac{1}{\pi}(\alpha\cos\theta\sin\alpha - \theta\sin\theta\cos\alpha) + \frac{1}{2}\sin\theta(\cos\alpha - \cos\theta)}{\cos^2\alpha - \cos^2\theta} \tag{6.16}$$

which, in view of (4.14), becomes

$$v_0(\alpha, u) + v_1(\alpha, u) = \frac{1}{2}\left(1 - \frac{u(\theta)\cos\alpha}{u(\alpha)\cos\theta}\right)\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_1(\alpha). \tag{6.17}$$

Intriguingly, this implies

$$v_0(\alpha, u) + v_1(\alpha, u) \xrightarrow{\delta\to\pi/2} \begin{cases} 0 & u(\alpha) = +u(\alpha) \\ \dfrac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_1(\alpha) & u(\alpha) = -u(\alpha), \end{cases} \tag{6.18}$$

where the distinct behavior of the solution on the different branches of $u(\alpha)$ is made clear. With the elimination of the polar periods, the solution (6.12) is now

$$w_2(\alpha, u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi, u) + v_1(\xi, u) + v_{2\pi}(\xi, u)\right\}\mathrm{d}\xi \tag{6.19}$$

where, if we were to proceed as in Chapters 4 and 5, we would select the remaining 2π periodic functions $v_{2\pi}(\alpha, u)$, which eliminate the cyclic periods, so that they vanish as $|\mathrm{Im}\,\alpha| \to \infty$. However, if we do so the resulting expression *fails* to satisfy the difference equation (6.7). To see this we resort to the technique introduced in Section 3.2.1 and inspect the first order difference equation when $|\mathrm{Im}\,\alpha| \to \infty$. Assuming $v_{2\pi}(\alpha, u)$ is such that the cyclic periods have been annulled, the left-hand side of (6.7) becomes

$$\frac{w_2(\alpha+\pi,u)}{w_2(\alpha-\pi,u)} = \exp\int_{\alpha-\pi}^{\alpha+\pi}\left\{v_0+v_1+v_{2\pi}\right\}d\xi \xrightarrow{|\text{Im }\alpha|\to\infty} 1 \tag{6.20}$$

since by assumption all the terms in the integrand vanish in that limit. On the other hand, the right-hand side of (6.7) becomes

$$g_2(\alpha,\pm u) = \frac{\cos\alpha-\cos\theta}{\pm u(\alpha)-u(\theta)} \xrightarrow{|\text{Im }\alpha|\to\infty} \pm 1. \tag{6.21}$$

We are led to conclude that under the current assumptions $w_2(\alpha,u)$ satisfies the first order equation (6.7) when $u(\alpha) = +u(\alpha)$ but fails to do so when $u(\alpha) = -u(\alpha)$. More precisely, $w_2(\alpha,u)$ actually satisfies

$$\frac{w_2(\alpha+\pi,\pm u)}{w_2(\alpha-\pi,\pm u)} = \frac{\cos\alpha-\cos\theta}{u(\alpha)\mp u(\theta)} = \pm g_2(\alpha,\pm u). \tag{6.22}$$

Despite this, it is easily verified by inspection that $w_2(\alpha,u)$ still satisfies the original first order equation (6.8) and this may prompt us to admit $w_2(\alpha,u)$ as an acceptable solution regardless of the fact that it only partially satisfies (6.7). Close scrutiny of its analytic continuation, however, reveals that while branch-free solutions can be constructed from (3.3) in the srip $S_{2\pi}$, their continuation outside the strip fails to be meromorphic. To see this, suppose that $\alpha \in \overline{S_{2\pi}} \cap S_{4\pi}$. Then the continuation of $w_2(\alpha,u)$ to the outer strip by means of (6.22) produces

$$w_2(\alpha,u)+w_2(\alpha,-u) = \frac{\cos\alpha+\cos\theta}{\cos^2\alpha-\cos^2\theta} \cdot\Big[u(\alpha)\big\{w(\alpha-2\pi,u)+w(\alpha-2\pi,-u)\big\} - u(\theta)\big\{w(\alpha-2\pi,u)-w(\alpha-2\pi,-u)\big\}\Big] \tag{6.23}$$

which, letting $u(\alpha) \to -u(\alpha)$, is manifestly branch dependent. This may lead one to forego (6.22) and rely instead on (6.7) to carry out the continuation but such an approach is inadmissible since it leads to discontinuities at $\alpha = \pm\pi$, a hardly surprising result given that $w_2(\alpha,u)$ does not satisfy (6.7). Any notion that $w_2(\alpha,u)$, as proposed above, could serve as an acceptable solution despite the discrepancy noted in (6.22) is therefore effectively quenched. The sign error in (6.22) must somehow be corrected and this can apparently be achieved only by introducing terms that are non-vanishing as $|\text{Im }\alpha| \to \infty$.

6.2.2 The higher order term

It is recognized that if a function $w_\infty(\alpha, u)$ can be constructed which satisfies

$$\frac{w_\infty(\alpha+\pi, \pm u)}{w_\infty(\alpha-\pi, \pm u)} = \pm 1, \tag{6.24}$$

then the product $w_2(\alpha, u)w_\infty(\alpha, u)$ is effectively a solution to the difference equation (6.7). The only acceptable expression found so far which makes this possible and allows a graceful recovery of the limiting case $\delta \to \pi/2$ is provided by

$$w_\infty(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_\infty(\xi, u) + v_{2\pi}(\xi, u) \right\} \mathrm{d}\xi \tag{6.25}$$

where

$$v_\infty(\alpha, u) = \frac{\mathrm{j}}{4}\left(1 - \frac{\cos\alpha}{u(\alpha)}\right). \tag{6.26}$$

This has a number of interesting properties. We first confirm that $w_\infty(\alpha, u)$ is indeed a solution to (6.24). It is assumed, as we must always do, that all the cyclic periods have been eliminated with a suitable 2π periodic function $v_{2\pi}(\alpha, u)$, itself vanishing as $|\mathrm{Im}\,\alpha| \to \infty$. Equation (6.24) is then verified by direct substitution of $w_\infty(\alpha, u)$. Closing the path of integration out at $\mathrm{Im}\,\alpha \to \infty$ (see Figure 3.1) produces, since $v_{2\pi}(\alpha, u) \to 0$ and $\pm\cos\alpha/u(\alpha) \to \pm 1$ as $|\mathrm{Im}\,\alpha| \to \infty$,

$$\begin{aligned} \frac{w_\infty(\alpha+\pi, \pm u)}{w_\infty(\alpha-\pi, \pm u)} &= \exp \int_{\alpha-\pi}^{\alpha+\pi} \left\{ v_\infty(\xi, \pm u) + v_{2\pi}(\xi, \pm u) \right\} \mathrm{d}\xi \\ &= \exp \int_{\alpha-\pi+\mathrm{j}\infty}^{\alpha+\pi+\mathrm{j}\infty} \left\{ v_\infty(\xi, \pm u) + v_{2\pi}(\xi, \pm u) \right\} \mathrm{d}\xi \\ &= \exp \int_{\alpha-\pi}^{\alpha+\pi} \frac{\mathrm{j}}{4}(1 \mp 1)\,\mathrm{d}\xi \\ &= \pm 1, \end{aligned} \tag{6.27}$$

fulfilling (6.24) explicitly. Any contribution from residues is ignored above as they are legitimately assumed to have integer values. We also note that

$$v_\infty(\alpha, \pm u) \xrightarrow{\delta\to\pi/2} \begin{cases} 0 & u(\alpha) = +u(\alpha) \\ \dfrac{\mathrm{j}}{2} & u(\alpha) = -u(\alpha) \end{cases} \tag{6.28}$$

and, hence,

$$\exp \int^{\alpha} v_\infty(\xi, u) \mathrm{d}\xi = \begin{cases} 1 & u(\alpha) = +u(\alpha) \\ \mathrm{e}^{\mathrm{j}\alpha/2} & u(\alpha) = -u(\alpha). \end{cases} \tag{6.29}$$

The solution to the first order equation with reduced period (6.7) then has the form

$$w_2(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_1(\xi, u) + v_\infty(\xi, u) + v_{2\pi}(\xi, u) \right\} \mathrm{d}\xi \tag{6.30}$$

where the (as yet) unspecified 2π periodic function $v_{2\pi}(\alpha, u)$ are chosen to eliminate the cyclic periods. In the previous two chapters, a change of the branch of $u(\alpha)$ resulted in a change in sign of the integrand of $w(\alpha, u)$ so that $w(\alpha, u) = 1/w(\alpha, -u)$. Instead of this neat relationship, inspection of (6.30) reveals the $w_2(\alpha, u)$ and $w_2(\alpha, -u)$ satisfy

$$w_2(\alpha, -u) = \frac{\mathrm{e}^{\mathrm{j}\alpha/2} \Psi_1(\alpha)}{w_2(\alpha, u)}. \tag{6.31}$$

It is also noted that $w_2(\alpha, u)$ can be slightly generalized. Direct substitution in (6.27) shows that $-v_\infty(\alpha, u)$ is also a suitable expression. This is accounted for by introducing the notation

$$w_2^{\pm}(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_1(\xi, u) \pm v_\infty(\xi, u) + v_{2\pi}(\xi, u) \right\} \mathrm{d}\xi. \tag{6.32}$$

Thus, the reduction in period, while alleviating difficulties with the elimination of the cyclic periods, introduces its own share of complications. A reduction of the strip of analyticity is a double-edged sword: the number of singularities captured is halved but the solution must nevertheless be made free of singularities in the wider strip. Since the elimination of the polar periods by means of $v_1(\alpha, u)$ is carried out in the reduced strip, θ poles remain in outer strip $\pi \leq |\mathrm{Re}\, \alpha| \leq 2\pi$. It would seem that the stage at which these are best dealt with is during the construction of the final meromorphic solutions, a difficult process to begin with. Needless to say, these "extra" θ poles only further complicate matters.

Another added complication is provided by the non-vanishing term $v_\infty(\alpha, u)$, and its presence marks a significant departure from the solutions constructed in Chapter 4 and 5. Since the desired solutions to the second order equation (6.1) must be $O(1)$ as $|\mathrm{Im}\, \alpha| \to \infty$, a way must be found to correct the order of the solution. From the expressions for $w_2^{\pm}(\alpha, u)$

and $v_\infty(\alpha, u)$, it can be shown that as $|\text{Im}\,\alpha| \to \infty$

$$w_2^\pm(\alpha, +u) \sim O(1), \tag{6.33}$$

$$w_2^\pm(\alpha, -u) \sim O\left(\exp \mp \frac{1}{2} \text{Im}\,\alpha\right), \tag{6.34}$$

and the order of $w_2^\pm(\alpha, -u)$ differs depending on whether $\text{Im}\,\alpha$ tends to $+\infty$ or $-\infty$. The branch-free form (3.3a) then behaves as

$$w_2^\pm(\alpha, u) + w_2^\pm(\alpha, -u) \sim \begin{cases} O(1) & \text{Im}\,\alpha \to \pm\infty \\ O\left(\exp \frac{1}{2} |\text{Im}\,\alpha|\right) & \text{Im}\,\alpha \to \mp\infty. \end{cases} \tag{6.35}$$

Since

$$\frac{e^{\mp j\alpha/4}}{\cos \alpha/4} \sim \begin{cases} O(1) & \text{Im}\,\alpha \to \pm\infty \\ O\left(\exp -\frac{1}{2} |\text{Im}\,\alpha|\right) & \text{Im}\,\alpha \to \mp\infty, \end{cases} \tag{6.36}$$

this provides, under the guise of a 4π periodic, a function which exactly cancels the order in (6.35) though the cosine in the denominator will likely introduce poles at the edges of the strip. Bearing this in mind, the form (6.36) is used to correct the order when constructing branch-free expressions.

Once the cyclic periods are eliminated, solutions to the second order equation (6.1) are sought by means of branch-free combinations (3.3) of $w_1(\alpha, u) w_2(\alpha, u)$ and particular attention has to be paid to the elimination of the poles and zeros in the outer strip as well as to the order correction. The technique presented does allow us to also solve another class of difference equation, namely the one given by (6.10), whose associated first order difference equations are (6.7). These equations, which are perhaps less firmly embedded in the realm of physics, are interesting in their own right due to their peculiar symmetry and their associated solutions $w_2(\alpha, \pm u)$ (6.30) whose order and singularities are branch dependent. Furthermore, since (6.10) is itself meromorphic, meromorphic solutions are allowable using the branch-free expressions (3.3).

6.2.3 The cyclic periods

The analysis required to eliminate the cyclic periods parallels very closely the one developed in Chapter 4. This comes as no surprise: the expressions being dealt with have the same even parity and the strip of analyticity has the exact same configuration. Only two

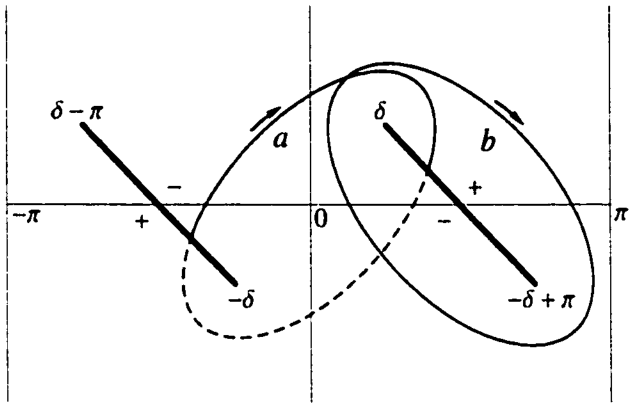


Figure 6.2: The clockwise cyclic paths a and b used to define the cyclic periods occurring in the strip $S_{2\pi}$. Note that cycle a crosses from the upper Riemann sheet (solid line) to the lower Riemann sheet (dashed line) whereas b is confined to the upper sheet.

cyclic periods, associated with the cycles a and b shown in Figure 6.2.3, must therefore be eliminated. Contrast this to the direct approach carried out in Chapter 5, where four cyclic periods had to be annulled using a much more intricate analysis on a surface of genus three. There is an important dissimilarity between the expressions $v_n(\alpha, u)$ found in this section and those employed in previous analyses: they have both unbranched and branched parts as opposed to being strictly branched. In the process of eliminating the cyclic periods, only the branched portion of the expressions is considered.

Two different approaches to the problem of cyclic period elimination are considered. In both instances, the analysis required to eliminate the cyclic periods parallels closely the one carried out in Section 4.2.3: only two degrees of freedom are required and the derivation of the required bilinear relationship is carried out on the dissected torus. As mentioned above, the construction of branch-free solutions is made difficult in both approaches due to the large number of poles to be annulled and the required correction to the order. In the first approach, termed the method of global elimination, the cyclic periods of the branched portion of v_0, v_1 and v_∞ are simultaneously annulled. In the second approach, termed the method of split elimination, two pairs of 2π periodic functions are introduced: one to eliminate the cyclic periods of v_0 and v_1 only, the other to eliminate those of $v_\infty(\alpha, u)$. Compared to the global approach, this leads to interesting symmetries in the solution but it is achieved at the price of introducing an extra pair of poles during the process of cyclic period elimination. The extra set of poles makes the construction of meromorphic solutions with the desired properties quite difficult. Due to the distinctively different nature of the solutions obtained with the two methods, the process of constructing branch-free solutions is examined separately for each of the two methods. We first consider the more intuitive global elimination.

6.3 The method of global elimination

The most obvious way to complete the construction of the branched solution $w_w^{\pm}(\alpha, u)$ (6.32) is to use $v_{2\pi}(\alpha, u)$ to simultaneously cancel the cyclic periods of all the terms in the integrand, including the higher order term $v_\infty(\alpha, u)$. The benefit of this approach is that it introduces only one set of ζ poles. Despite its apparent simplicity, the requirement for a solution in $S_{4\pi}$ makes the construction of a branch-free solution difficult. We present a few of the possibilities investigated, all of which involve some numerical work to locate zeros.

6.3.1 Elimination of the cyclic periods

This leads to the same equation system, with a few minor changes, as the one obtained in Section 4.2.3. As just mentioned above, we require only two degrees of freedom and the final solution is[1]

$$w_2^+(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_1(\xi, u) + v_\infty(\alpha, u) + \kappa_1 v_{2\pi}^1(\xi, u) + \sigma_1 v_{2\pi}^3(\xi, u; \zeta_1) \right\} d\xi, \quad (6.37)$$

with the 2π periodic $v_{2\pi}^{1,3}(\alpha, u)$, both of which give rise to elliptic integrals, defined in (4.44):

$$v_{2\pi}^1(\alpha, u) = \frac{1}{u(\alpha)}, \qquad v_{2\pi}^3(\alpha, u; \zeta_1) = \frac{u(\zeta_1)}{u(\alpha)} \frac{\cos\alpha}{\cos\zeta_1} \frac{\sin\zeta_1}{\cos\alpha - \cos\zeta_1}. \quad (6.38)$$

Note that the pole associated with $v_{2\pi}^3(\alpha, u; \zeta)$ now appears explicitly in the function definition to simplify the forthcoming analysis. We also choose to indicate this explicitly in the associated cyclic periods by writing $A_{2\pi}^3(\zeta_1)$ and $A_{2\pi}^3(\zeta_1)$. The two degrees of freedom κ_1 and ζ_1, are readily obtained from the analysis in Section 4.2.3. Accounting for $v_\infty(\alpha, u)$, we now define

$$A'_{0+1+\infty} = \int_a \left\{ v'_0(\alpha, u) + v'_1(\alpha, u) + v'_\infty(\alpha, u) \right\} d\alpha, \quad (6.39a)$$

$$B'_{0+1+\infty} = \int_b \left\{ v'_0(\alpha, u) + v'_1(\alpha, u) + v'_\infty(\alpha, u) \right\} d\alpha. \quad (6.39b)$$

[1]To avoid clutter, the analysis that follows applies to $w_2^+(\alpha, u)$. It is exactly the same for $w_2^-(\alpha, u)$ though different values for κ_1 and ζ_1 are then obtained.

where the primes indicate the branched portions so that

$$v_0'(\alpha, u) = -\frac{\alpha}{2\pi} \frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{\cos\theta \sin\alpha}{\cos^2\alpha - \cos^2\theta}, \tag{6.40a}$$

$$v_1'(\alpha, u) = -\frac{u(\theta)}{u(\alpha)} \frac{\cos\alpha}{\cos\theta} \frac{\varphi_1 + \varphi_2 \cos\alpha}{\cos^2\alpha - \cos^2\theta}, \tag{6.40b}$$

$$v_\infty'(\alpha, u) = -\frac{\mathrm{j}}{4} \frac{\cos\alpha}{u(\alpha)}. \tag{6.40c}$$

The constants φ_1 and φ_2 are given in (6.15). We remark that the branched portion of $v_\infty(\alpha, u)$ given above has an exact anti-derivative

$$\int^\alpha \frac{\cos\xi}{u(\xi)} \mathrm{d}\xi = \mathrm{j} \operatorname{arccosh} \frac{\sin\alpha}{\sin\delta}, \tag{6.41}$$

and has cyclic periods $A_\infty' = -2\pi$ and $B_\infty' = 0$. Enforcement of vanishing cyclic periods on the cycles a and b gives

$$A_{0+1+\infty}' + \kappa_1 A_{2\pi}^1 + A_{2\pi}^3(\zeta_1) = 0, \tag{6.42a}$$

$$B_{0+1+\infty}' + \kappa_1 B_{2\pi}^1 + B_{2\pi}^3(\zeta_1) = 0, \tag{6.42b}$$

with the periods $A_{2\pi}^{1,3}$ and $B_{2\pi}^{1,3}$ defined per (4.45). The analysis provided in Section 4.2.3 now applies verbatim if we let $A_{0+1} \to A_{0+1+\infty}'$ and $B_{0+1} \to B_{0+1+\infty}'$. Hence,

$$\zeta_1 = \arccos\left[k \operatorname{sn}(\mathrm{j}K' + 3K + \sigma_1 \Lambda_1, k)\right], \qquad \Lambda_1 \in \mathcal{P}^{\sigma_1}, \tag{6.43}$$

where

$$\Lambda_1 = \frac{1}{8\pi} \left(A_{2\pi}^1 B_{0+1+\infty}' - B_{2\pi}^1 A_{0+1+\infty}' \right), \tag{6.44}$$

and the period parallelograms $\mathcal{P}$ are identified in Figure 5.10. In (6.43), the parameter $k = \cos\delta/2$ with K and K' defined in the standard fashion (see Section 3.5.1). The constant κ_1 follows immediately from either equation in (6.42), completing the construction of the branched solution to equation (6.37).

6.3.2 Properties of the solution

The solution just constructed solves the first order difference equation (6.7) and, together with the branch-free combinations (3.3), can be used to obtain meromorphic solutions to the associated second order difference equation given in (6.10). We note that this procedure can be readily carried out using the branch-free solution given in Section 4.3 in a fully analytic manner if we are willing to admit solutions which are not $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. Our goal however is to find solutions to equation (6.1) which, from Chapter 5, does admit a solution $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$ that is free of poles and zeros in the wider $S_{4\pi}$. The solution $w^{+}(\alpha, u)$ to (6.2) is, from (6.9),

$$
\begin{aligned}
w^{+}(\alpha,u) &= w_1(\alpha) w_2^{+}(\alpha,u) \\
&= \Psi_4(\alpha) \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi,u) + v_1(\xi,u) + v_\infty(\alpha,u) + \kappa_1 v_{2\pi}^{1}(\xi,u) + \sigma_1 v_{2\pi}^{3}(\xi,u;\zeta_1) \right\} \mathrm{d}\xi .
\end{aligned} \tag{6.45}
$$

This shares many properties with the solution $w(\alpha, u)$ described in Section 5.2.6 under the direct approach yet it is provides a significantly greater challenge when constructing proper meromorphic solutions. There are two main reasons for this. The first is the need to construct a pole and zero free solution in the wider strip $S_{4\pi}$ as opposed to $S_{2\pi}$. The second is the needed correction in the order to offset the growth (or decay) due to $v_\infty(\alpha, u)$.

The wider strip is populated by θ poles, from $v_0(\alpha, u)$, as well as ζ_1 poles, from $v_{2\pi}^{3}(\alpha, u; \zeta_1)$. The poles and zeros of $w(\alpha, u)$ associated with the poles of $v_{2\pi}^{3}(\alpha, u; \zeta_1)$ follow from (5.65). The θ poles are due to $v_0(\alpha, u) + v_1(\alpha, u)$ and this has residues, on the $-u(\alpha)$ Riemann sheet *only*,

$$
\begin{aligned}
&\mathrm{Res} \left\{ v_0(\alpha,u) + v_1(\alpha,u) \right\} \Big|_{\alpha=\pm(2\pi-\theta),\pm(\pi-\theta)} \\
&\quad = \left. \frac{\frac{1}{\pi}(\alpha\cos\theta\sin\alpha - \theta\sin\theta\cos\alpha) + \frac{1}{2}\sin\theta(\cos\alpha - \cos\theta)}{-2\sin\alpha\cos 2\alpha} \right|_{\alpha=\pm(2\pi-\theta),\pm(\pi-\theta)}
\end{aligned} \tag{6.46}
$$

which produces

$$
\mathrm{Res} \left\{ v_0(\alpha,u) + v_1(\alpha,u) \right\} = \pm 1, \qquad \alpha = \pm(\pi+\theta,-u), \pm(-2\pi+\theta,-u). \tag{6.47}
$$

Interestingly, the elimination of the polar periods in the narrow strip is sufficient to produce integer residues outside the strip; if it were not so the integral would still be ill-defined out of the strip $S_{2\pi}$. The end result is that, in addition to the ζ_1 poles found on both Riemann sheets, we must also deal with the θ poles which solely reside on the $-u(\alpha)$ sheet. Noting that $\Psi_4(\alpha)$, given in (5.5), is free of poles and zeros in $S_{4\pi}$, we therefore have, assuming

$\sigma_1 = 1$,

$$w^{\pm}(\alpha, u) \sim \frac{\alpha + \zeta_1}{\alpha - \zeta_1} \frac{\alpha + (2\pi - \zeta_1)}{\alpha - (2\pi - \zeta_1)}, \tag{6.48a}$$

and

$$w^{\pm}(\alpha, -u) \sim \frac{\alpha - \zeta_1}{\alpha + \zeta_1} \frac{\alpha - (2\pi - \zeta_1)}{\alpha + (2\pi - \zeta_1)} \frac{\alpha - (\pi + \theta)}{\alpha + (\pi + \theta)} \frac{\alpha - (-2\pi + \theta)}{\alpha + (-2\pi + \theta)}, \tag{6.48b}$$

where ζ_1 is used instead of $\zeta_1^{\pm}$ to reduce clutter.

The limiting behaviors of $w^{\pm}(\alpha, u)$ and $w^{\pm}(\alpha, -u)$ are easily obtained. Following the same approach as in Sections 4.2.5 or 5.2.6, it is a simple task to show that κ_1 vanishes as $\delta \to \pi/2$. It immediately follows, making use of prior analysis, that

$$w^{\pm}(\alpha, u) \xrightarrow{\delta \to \pi/2} \frac{\tan\frac{\zeta_1}{2} + \tan\frac{\alpha}{2}}{\tan\frac{\zeta_1}{2} - \tan\frac{\alpha}{2}} \Psi_4(\alpha) \tag{6.49}$$

and

$$w^{\pm}(\alpha, -u) \xrightarrow{\delta \to \pi/2} \frac{\tan\frac{\zeta_1}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_1}{2} + \tan\frac{\alpha}{2}} \Psi_4(\alpha) e^{\pm j\alpha/2} \Psi_1(\alpha), \tag{6.50}$$

which is in agreement with (6.31). Moreover, using the properties of Maliuzhinets's functions, it can be shown that

$$\Psi_1(\alpha) = \frac{\cos\frac{1}{4}(\alpha - \theta + \pi)\cos\frac{1}{4}(\alpha - \theta)}{\cos\frac{1}{4}(\alpha + \theta - \pi)\cos\frac{1}{4}(\alpha + \theta)} \frac{1}{\Psi_4^2(\alpha)} \tag{6.51}$$

and (6.50) becomes

$$w^{\pm}(\alpha, -u) \xrightarrow{\delta \to \pi/2} \frac{\tan\frac{\zeta_1}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_1}{2} + \tan\frac{\alpha}{2}} \frac{\cos\frac{1}{4}(\alpha - \theta + \pi)\cos\frac{1}{4}(\alpha - \theta)}{\cos\frac{1}{4}(\alpha + \theta - \pi)\cos\frac{1}{4}(\alpha + \theta)} \frac{e^{\pm j\frac{1}{2}\alpha}}{\Psi_4(\alpha)}, \tag{6.52}$$

providing a glimpse of how $w^{\pm}(\alpha, u)$ recovers expressions related to the desired limiting functions $\Psi_4(\alpha)$ and its reciprocal. Note also that both (6.49) and (6.52) are in agreement with (6.48).

6.3.3 Branch-free solutions

It is not obvious how a fully analytical approach can be used to obtain branch-free solutions to the second order equation (6.1) based on the branched solution (6.45). Experience suggests the use of the approaches, or variations thereof, employed in Chapter 5 where the very same equation was solved fully analytically. Both the elimination of the θ and ζ_1 poles as well as the required correction to the order must be borne in mind. The approach followed to correct the order of $w(\alpha, u)$ so as to remove the exponential growth (or decay) associated with $v_\infty(\alpha, u)$ was previously hinted at in Section 6.2.2: we make use expression (6.36) in the construction of the branch-free solutions in order to obtain the desired order. The most general approach is a liner combination of (3.3a) and (3.3b) which leads us to contemplate once again the forms

$$t_1^{\pm}(\alpha) = \frac{Q(\alpha)}{2}\left\{\left(1 + \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, u) + \left(1 - \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, -u)\right\}, \tag{6.53}$$

$$t_2^{\pm}(\alpha) = \frac{1}{2Q(\alpha)}\left\{\left(1 - \frac{r_2^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, u) + \left(1 + \frac{r_2^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, -u)\right\}, \tag{6.54}$$

where[2] $r_1(\alpha)$ is the familiar 4π periodic polynomial with five degrees of freedom ρ_n:

$$r_1(\alpha) = \rho_1 + \rho_2 \cos\alpha + \rho_3 \sin\alpha + \rho_4 \cos\frac{\alpha}{2} + \rho_5 \sin\frac{\alpha}{2}. \tag{6.55}$$

Function $r_2(\alpha)$ has the exact same form save that it uses ϱ_n instead of ρ_n. Since the functions $r_n(\alpha)$ are, based on preceding approaches, used to introduce a double zeros, we can effectively eliminate two poles while simultaneously creating two zero in the term in curly braces above. Apparently, the best approach here is to use this freedom to eliminate the ζ_1 poles. This uses up four degrees of freedom and duplicates the functions $r_{1,2}(\alpha)$ used back in (4.98)

$$r_1(\alpha) = \frac{\rho_1(1 - \cos(\zeta_1 + \alpha)) + \cos\zeta_1 \sin^2\delta\cos\alpha + \sin\zeta_1\cos^2\delta\sin\alpha}{u(\zeta_1)}, \tag{6.56}$$

and

$$r_2(\alpha) = \frac{\varrho_1(1 - \cos(\zeta_1 - \alpha)) + \cos\zeta_1 \sin^2\delta\cos\alpha - \sin\zeta_1\cos^2\delta\sin\alpha}{u(\zeta_1)}. \tag{6.57}$$

[2]We drop the $\pm$ superscripts henceforth but it is implied that $r_n(\alpha)$ is distinct depending on the choice of the superscript.

This leaves a single degree of freedom, the arbitrary constants ρ_1 and ϱ_1 which were set to zero in most of the prior analyses. This will enable us to discretely eliminate a single pole of our choice. The form of $r_n(\alpha)$ chosen above leads to symmetric poles and zeros in the term in curly braces and their elimination is easily carried out with $Q(\alpha)$, which is then identical to $Q_{2\pi}(\alpha)$ in (5.84). In short, the procedure followed to eliminate the ζ_1 poles is identical to the one followed in Chapter 4 (ζ poles) or in Chapter 5 ($\zeta_{2\pi}$ poles). This completes the elimination of the ζ_1 poles and leaves only the θ poles to be dealt with as well as the required correction in the order of the solution. The best way to proceed from here on is unclear; a myriad of possibilities are available and many were examined. A small sample of those which were seriously entertained are presented below. Since our goal is to proceed as fully analytically as possible, we first present an almost fully analytical solution and then go one to examine solutions which involve locating a progressively larger number of zeros numerically.

6.3.3.1 Approach I: Fully analytical θ pole elimination

We seek a solution where the θ poles are eliminated fully analytically. This can be achieved by choosing ρ_1 in (6.55) such that the coefficient $1 - r_1(\alpha)/u(\alpha)$ eliminates the pole of $w(\alpha, -u)$ at $\alpha = 2\pi - \theta$. A short analysis produces

$$\rho_1 = \frac{-u(\zeta_1)u(\theta) - \cos\theta\cos\zeta_1 + \cos^2\delta\cos(\zeta_1 + \theta)}{u(\zeta_1)(1 - \cos(\zeta_1 - \theta))}. \tag{6.58}$$

In the same fashion, we define $r_2(\alpha)$ such that $1 + r_2(\alpha)/u(\alpha)$ also eliminates the pole of $w(\alpha, -u)$ at $\alpha = 2\pi - \theta$. The constant ϱ_1 in this case is given by

$$\varrho_1 = \frac{u(\zeta_1)u(\theta) - \cos\theta\cos\zeta_1 + \cos^2\delta\cos(\zeta_1 - \theta)}{u(\zeta_1)(1 - \cos(\zeta_1 + \theta))}. \tag{6.59}$$

Both $t_1(\alpha)$ and $t_2(\alpha)$ now share a common pole at $\alpha = -\pi - \theta$ which can be eliminated with a linear combination. Define

$$t_3^{\pm}(\alpha) = t_1^{\pm}(\alpha) + \chi^{\pm} t_2^{\pm}(\alpha) \tag{6.60}$$

where χ is chosen to eliminate the common pole of $t_1(\alpha)$ and $t_2(\alpha)$. Examination of the residues gives

$$\chi^{\pm} = -Q^2(\alpha)\frac{u(\alpha) - r_1^{\pm}(\alpha)}{u(\alpha) + r_2^{\pm}(\alpha)}\bigg|_{\alpha=-\pi-\theta}. \tag{6.61}$$

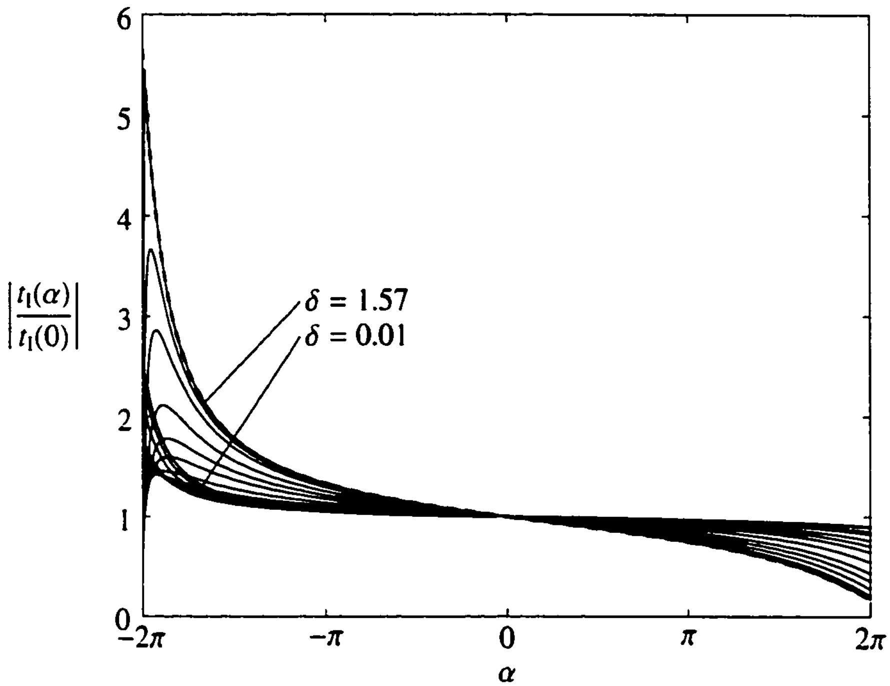


Figure 6.3: Magnitude (solid lines) of the normalized branch-free solution $t_l(\alpha)$ given in (6.62) when $\theta = 0.25(1 + j)$ for various values of δ. The thicker dashed line corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$. The change in curvature of $t_l(\alpha)$ near -2π occurs roughly when $\delta \sim 0.80$.

We now make use of our knowledge of the desired order correcting function (6.36) and write, finally,

$$t_l(\alpha) = \frac{1}{2} \frac{\sin \frac{1}{4}\alpha_l}{\sin \frac{1}{4}(\alpha_l - \alpha)} \left\{ e^{-j\alpha/4} t_3^+(\alpha) + e^{j\alpha/4} t_3^-(\alpha) \right\} ; \tag{6.62}$$

it can be showed that as $\delta \to \pi/2$, $t_3^\pm(\alpha) \to \Psi_4(\alpha)$ and the (numerically located) zero $\alpha_l \to 2\pi$. Hence, in that limit, the known solution $\Psi_4(\alpha)$ (5.5) is recovered. The solution $t_l(\alpha)$ has a somewhat disenchanting behavior near -2π, the result of an interacting pole-zero pair in the neighborhood of $\alpha = -2\pi$ just outside the strip of analyticity. As $\delta \to \pi/2$, the pair is closely located and its effect highly localized. The behavior of $t_l(\alpha)$ is illustrated in Figure 6.3 for a large number of values of δ. It is seen that whereas the behavior is as expected in the limit as $\delta \to \pi/2$, the limiting behavior as $\delta \to 0$ is not easily related to known solution in that limit such as $\Psi_5(\alpha)$ or its inverse. Mathematically, $t_l(\alpha)$ is a legitimate solution to the second order equation (6.1) but it is nowhere as well behaved as the ones obtained in the previous chapter. In an attempt to recover the more ideal behavior obtained in Chapter 5, variations of the procedure just presented are examined.

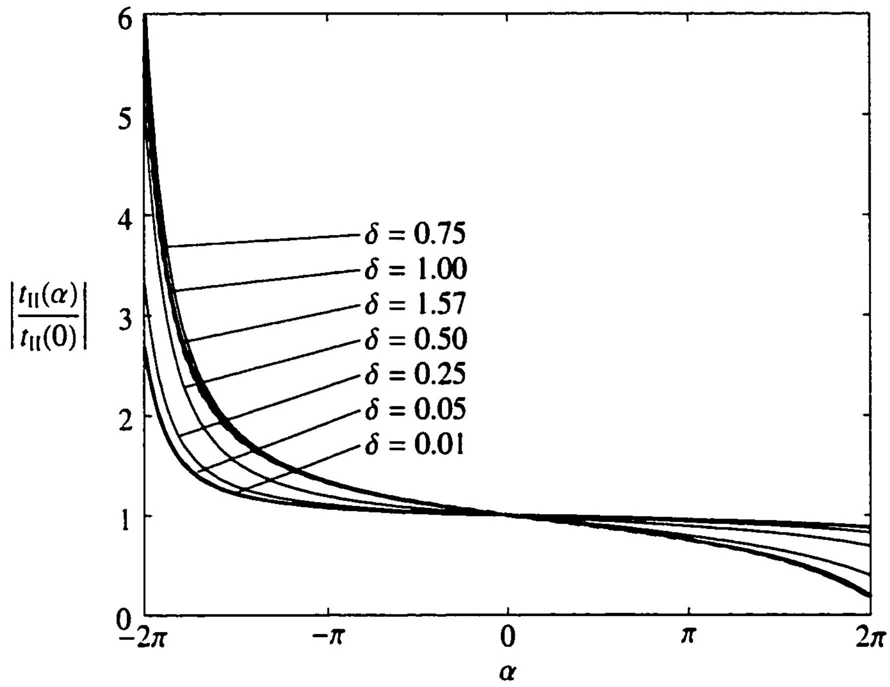


Figure 6.4: Magnitude (solid lines) of the normalized branch-free solution $t_{II}(\alpha)$ given in (6.63) when $\theta = 0.25(1+\mathrm{j})$ for various values of δ. The thicker dashed line corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$.

6.3.3.2 Approach II: Fully analytical θ pole elimination

This is almost identical to solution I, save that we now consider

$$t_{II}(\alpha) = \frac{e^{-j\alpha/4}}{\sin\frac{1}{4}(\alpha_1 - \alpha)} t_3^+(\alpha) \tag{6.63}$$

and omit the linear combination of $t_3^+(\alpha)$ and $t_3^-(\alpha)$ used in (6.62). The zero α_1 is once again located numerically and, since $\operatorname{Im}\alpha_1 \to \infty$ as $\delta \to \pi/2$, the normalized solution $t_{II}(\alpha)/t_{II}(0)$ recovers the known solution in that limit. As illustrated in Figure 6.4, this produces a less uniform convergence as $\delta \to \pi/2$ and tends to an unknown solution as $\delta \to 0$. Removal of the linear combination does partially eliminate the undesirable behavior near $\alpha = -2\pi$ but indications are that this is achieved at the cost of a non-uniform convergence to $\Psi_4(\alpha)$ as $\delta \to \pi/2$.

6.3.3.3 Approach III: Numerical elimination of $-\pi - \theta$ pole

In an effort to recover a better behaved branch-free solution, we consider using $t_1^{\pm}(\alpha)$ directly, dispensing with the linear combination used to cancel the pole at $\alpha = -\pi - \theta$. This

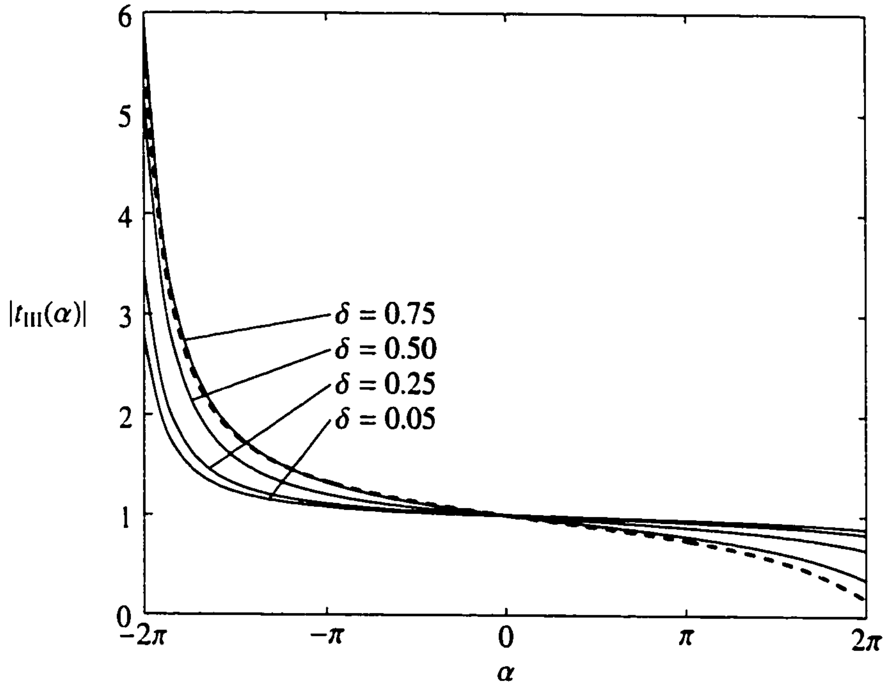


Figure 6.5: Magnitude (solid lines) of the branch-free solution $t_{\rm III}(\alpha)$ given in (6.64) when $\theta = 0.25(1 + {\rm j})$ for various values of δ. The thicker dashed line corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$.

leads to the expression

$$t_{\rm III}(\alpha) = \frac{\sin\frac{1}{4}(\alpha_\theta - \alpha)}{\sin\frac{1}{4}(\alpha_1 - \alpha)} \frac{\sin\frac{1}{4}\alpha_2}{\sin\frac{1}{4}(\alpha_2 - \alpha)} \frac{e^{-{\rm j}\alpha/4}}{2} t_1^+(\alpha) \tag{6.64}$$

where $\alpha_\theta = -\pi - \theta$ and the zeros α_1, associated with the pole at $\alpha = -\pi - \theta$, and α_2, associated with the order correcting term, must be located numerically. As $\delta \to \pi/2$, the known solution $\Psi_4(\alpha)$ is recovered since $\alpha_1 \to -\pi - \theta$ and ${\rm Im}\,\alpha_2 \to \infty$. The behavior of the magnitude of $t_{\rm III}(\alpha)$ is given in Figure 6.5. Once again, the solution overshoots the limiting function $\Psi_4(\alpha)$, a rather sharp departure from the results obtained in Chapters 4 and 5. For values as small as $\delta = 0.75$, the solution $t_{\rm III}(\alpha)$ has already partly overshot the desired limiting function $\Psi_4(\alpha)$. The behavior when $\delta \to 0$ is similar to the one in Figure 6.4. The increase in values in the neighborhood of -2π is due to a pole which sits just outside the strip of analyticity. All of the solutions presented so far are afflicted by this pole, solution I to lesser extent since there is a closely situated zero in that case. This pole is in fact the 4π periodic counterpart of the one eliminated analytically using ρ_1 and ϱ_1.

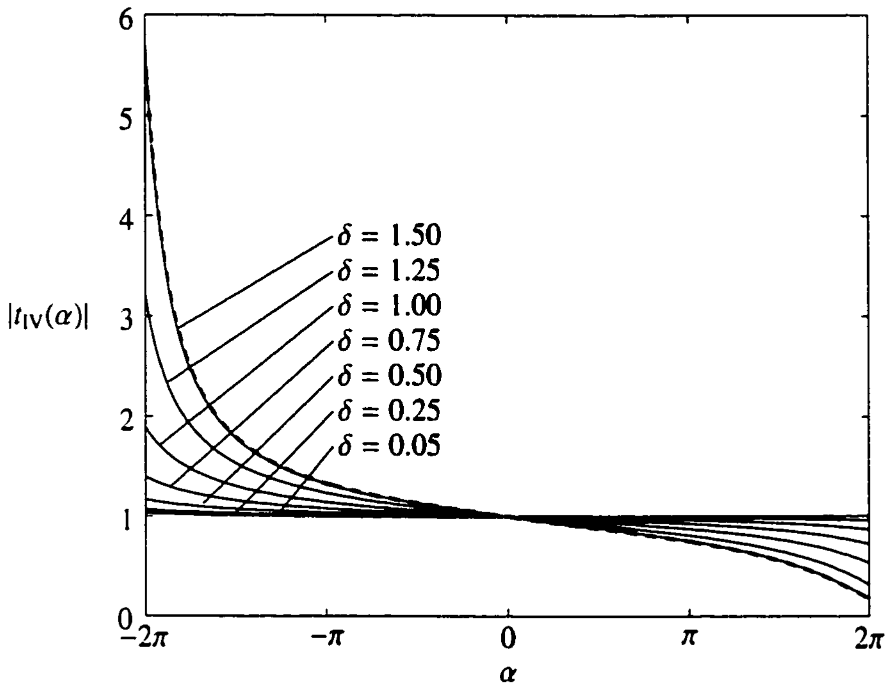


Figure 6.6: Magnitude (solid lines) of the branch-free solution $t_{IV}(\alpha)$ given in (6.65) when $\theta = 0.25(1 + j)$ for various values of δ. The thicker dashed line corresponds to the known limiting function $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$.

6.3.3.4 Approach IV: Numerical elimination of $2\pi - \theta$ pole

In the previous approach, the pole at $2\pi - \theta$ was eliminated analytically whereas the one at $-\pi - \theta$ was eliminated numerically. We now follow the same procedure but choose $r_1(\alpha)$ so that $1 + r_1(\alpha)/u(\alpha)$ eliminates the pole of $w(\alpha, -u)$ at $\alpha = -\pi - \theta$ and then proceed to eliminate the remaining pole at $\alpha = 2\pi - \theta$ numerically. To distinguish the resulting function $t_1(\alpha)$ from the closely related (6.53), it is denoted as $t_1'(\alpha)$. Hence, $t_1'(\alpha)$ is free of ζ_1 poles and has a pole at $\alpha = -\pi - \theta$ whereas $t_1(\alpha)$, also free of ζ_1 poles, has a pole at $\alpha = 2\pi - \theta$. Eliminating the remaining θ pole and correcting the order, we have

$$t_{IV}(\alpha) = \frac{\sin\frac{1}{4}(\alpha_\theta - \alpha)}{\sin\frac{1}{4}(\alpha_1 - \alpha)} \frac{\sin\frac{1}{4}\alpha_2}{\sin\frac{1}{4}(\alpha_2 - \alpha)} \frac{e^{-j\alpha/4}}{2} t_1'^{+}(\alpha) \tag{6.65}$$

where $\alpha_\theta = 2\pi - \theta$ and the zeros α_1, associated with the pole at $\alpha = 2\pi - \theta$, and α_2, associated with the order correcting term, must be located numerically. As shown in Figure 6.6, this seemingly minor change has a dramatic effect on the behavior of $t_{IV}(\alpha)$. Indeed, we now recover a function that converges smoothly towards the limiting function $\Psi_4(\alpha)$. Furthermore, the increase in the neighborhood of $\alpha = -2\pi$ has been eliminated altogether.

6.3.4 Summary of global elimination

Apart from those presented above, numerous other combinations were examined and so far none has successfully recovered the desired behavior obtained in the earlier chapters. The upshot is that these efforts have made apparent a number of features. For instance, the method chosen to eliminate poles, whether by means of ρ_1 and ϱ_1 or by employing linear combinations appears to have far reaching consequences on the behavior of the branch-free solutions. On the one hand, linear combinations of $t_3^{\pm}(\alpha)$ when the $2\pi - \theta$ pole is eliminated with ρ_1 and ϱ_1 are typically plagued with an undesirable behavior near $\alpha = -2\pi$ and this is exemplified in Figure 6.3. On the other hand, those where the linear combination is used to eliminate the $-\pi - \theta$ pole, as seen in Figure 6.6, display a much more acceptable behavior. A variation of solutions III and IV was also examined where the θ poles were eliminated solely by means of numerically locating zeros when $\rho_1 = 0$ and $\varrho_1 = 0$. This failed rather miserably as some of the numerical zeros migrate outside the strip of analyticity for values of δ as large as 0.75. This has the unfortunate consequence that the 4π periodic zero cancelling terms reintroduce poles in the strip of analyticity, further emphasizing how critically dependent the solution is on the method chosen to eliminate the poles. On a related note, it is also apparent that using the constants ρ_1 and ϱ_1 to eliminate poles generally seems to jeopardize the recovery of at least one limit and should therefore be avoided if possible. A comparison of solutions III shown in Figure 6.5 and solution IV in Figure 6.6 provides a quick snapshot of the capricious nature of the process. At first glance, they are both equivalent: in the first the $2\pi - \theta$ pole is eliminated using ρ_1, ϱ_1 and the $-\pi - \theta$ pole is dealt with numerically; in the second, this is reversed and the $2\pi - \theta$ pole is eliminated numerically. Yet, their behaviors are quite dissimilar in both limits of δ that are of interest. This may be an indication that the procedures lead to poles having different orders and/or residues outside the strip of analyticity. Clearly, more consideration is required to shed light on these issues.

6.4 The method of split elimination

An alternative to the global approach which is not quite as intuitive and harder to justify at this point is to take a two-step approach to the problem of cyclic period elimination. In this method, two separate pairs of elliptic integrands are introduced to annul the cyclic periods, thereby annulling individually the cyclic periods of $v_0(\alpha, u) + v_1(\alpha, u)$ and of $v_\infty(\alpha, u)$. This yields expressions which are more symmetric than for the case of the method of global elimination, albeit at the substantial cost of introducing an extra set of ζ poles. This is admittedly a hard-sell at this juncture given our current inability to recover the solution from

Chapter 5 using the simpler global elimination just discussed. However, it turns out that to reduce the period of the difference equation for the problem of the anisotropic half-plane this is the only known way to proceed since, for reasons explained in the next chapter, the method of global elimination fails to apply in that context. The construction of a branch-free form in this case will be conducted using a fully numerical approach and we do succeed in recovering the known limits as $\delta \to \pi/2$ and $\delta \to 0$.

6.4.1 Elimination of the cyclic periods

This is relatively simple given our current experience. Much like the case for the global elimination, we recover relationships which differ only slightly from those obtained in Section 4.2.3. Following the prescription outlined above, we write

$$\begin{aligned} w_2(\alpha, u) &= w_{2a}(\alpha, u) w_{2b}^{\pm}(\alpha, u) \\ &= \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + \kappa_a v_{2\pi}^1(\xi, u) + \sigma_a v_{2\pi}^3(\xi, u, \zeta_a) \Big\} \mathrm{d}\xi \\ &\quad \cdot \exp \int_{\alpha_0}^{\alpha} \Big\{ \pm v_\infty(\alpha, u) + \kappa_b v_{2\pi}^1(\xi, u) + \sigma_b v_{2\pi}^3(\xi, u, \zeta_b) \Big\} \mathrm{d}\xi \end{aligned} \tag{6.66}$$

where $w_{2a}(\alpha, u)$ is a solution to (6.22) while $w_{2b}(\alpha, u)$ is a solution to (6.24). The $\pm$ superscript is omitted below to avoid making the notation too cumbersome. Since it indicates a change in the sign associated with $v_\infty(\alpha, u)$, only the quantities κ_b and ζ_b are affected by its choice; it follows from symmetry that $\kappa_b^+ = -\kappa_b^-$, $\zeta_b^+ = \zeta_b^-$ and $\sigma_b^+ = -\sigma_b^-$. We define

$$A'_{0+1} = \int_a \Big\{ v'_0(\alpha, u) + v'_1(\alpha, u) \Big\} \mathrm{d}\alpha, \tag{6.67a}$$

$$B'_{0+1} = \int_b \Big\{ v'_0(\alpha, u) + v'_1(\alpha, u) \Big\} \mathrm{d}\alpha, \tag{6.67b}$$

where the primed integrands refer to the branched portion of the respective unprimed function as given in (6.40). Enforcing vanishing cyclic periods over the cycles a and b produces

$$A'_{0+1} + \kappa_a A_{2\pi}^1 + A_{2\pi}^3(\zeta_a) = 0, \tag{6.68a}$$

$$B'_{0+1} + \kappa_a B_{2\pi}^1 + B_{2\pi}^3(\zeta_a) = 0, \tag{6.68b}$$

with the periods $A_{2\pi}^{1,3}$ and $B_{2\pi}^{1,3}$ defined in (4.45). The dependence on ζ_a is explicitly indicated to avoid confusion with the forthcoming analysis for $w_{2b}(\alpha, u)$. Following once again the analysis in Section 4.2.3, we obtain

$$\zeta_a = \arccos\left[k\,\mathrm{sn}\left(\mathrm{j}K' + 3K + \sigma_a\Lambda_a, k\right)\right], \qquad \Lambda_a \in \mathcal{P}^{\sigma_a}, \tag{6.69}$$

where

$$\Lambda_a = \frac{1}{8\pi}\left(A^1_{2\pi}B'_{0+1} - B^1_{2\pi}A'_{0+1}\right), \tag{6.70}$$

and the period parallelograms are identified in Figure 5.10. In (6.69), the parameter $k = \cos\delta/2$ with K and K' defined in the standard fashion (see Section 3.5.1.) The constant κ_a follows immediately from either equation in (6.68).

Repeating the analysis for $w_{2b}(\alpha, u)$ we define

$$A'_\infty = \int_a v'_\infty(\alpha, u)\mathrm{d}\alpha, \qquad B'_\infty = \int_b v'_\infty(\alpha, u)\mathrm{d}\alpha, \tag{6.71}$$

where the primed integrand refers to the branched portion of $v_\infty(\alpha, u)$ given in (6.40c). As noted on page 152, $v'_\infty(\alpha, u)$ has an exact anti-derivative and it is easily shown that $A'_\infty = \mathrm{j}\pi/2$ and $B'_\infty = 0$. Unfortunately, since A'_∞ is not an integer multiple of $2\pi\mathrm{j}$, vanishing cyclic periods must be enforced. Doing so over the cycles a and b produces

$$A'_\infty + \kappa_b A^1_{2\pi} + A^3_{2\pi}(\zeta_b) = 0, \tag{6.72a}$$

$$B'_\infty + \kappa_b B^1_{2\pi} + B^3_{2\pi}(\zeta_b) = 0, \tag{6.72b}$$

with the periods $A^{1,3}_{2\pi}$ and $B^{1,3}_{2\pi}$ defined in (4.45). Paralleling the analysis in Section 4.2.3,

$$\zeta_b = \arccos\left[k\,\mathrm{sn}\left(\mathrm{j}K' + 3K + \sigma_b\Lambda_b, k\right)\right], \qquad \Lambda_b \in \mathcal{P}^{\sigma_b}, \tag{6.73}$$

where

$$\Lambda_b = \frac{1}{8\pi}\left(A^1_{2\pi}B'_\infty - B^1_{2\pi}A'_\infty\right). \tag{6.74}$$

The period parallelograms $\mathcal{P}$ are identified in Figure 5.10 and the parameter $k = \cos\delta/2$ with K and K' defined as above for the case of ζ_a. Either equation in (6.72) can now be used to obtain κ_b. The determination of $\kappa_{a,b}$ and $\zeta_{a,b}$ completes the construction of the branched solution $w^\pm(\alpha, u)$ and we now turn our attention to obtaining branch-free solutions.

6.4.2 Properties of the solution

This alternative solution $w_2(\alpha, u)$ to (6.37), like its predecessor obtained by global elimination, can be used to obtain meromorphic solutions to the associated second order difference equation given in (6.10). Once again, this can be achieved in a fully analytic manner, though one must now resort to using the constant term in $r(\alpha)$ as well as linear combinations of $t_1(\alpha)$ and $t_2(\alpha)$ given in Section 4.3. This procedure however precludes the recovery of the known solution even as $\delta \to \pi/2$ and the expression recovered is not $O(1)$ as $|\text{Im}\,\alpha| \to \infty$. It can thus be appreciated that even at this stage, the method of split elimination yields results which are of a higher complexity than the method of global elimination. We turn to constructing a solution to equation (6.1) from the branched function

$$w^{\pm}(\alpha, u) = w_1(\alpha)w_2^{\pm}(\alpha, u) = \Psi_4(\alpha)w_{2a}(a, u)w_{2b}^{\pm}(a, u). \tag{6.75}$$

This shares many properties of the solution constructed by global elimination. Again, the removal of the poles in the strip $S_{4\pi}$ and the correction of the order make obtaining the desired branch-free solution a difficult task. In fact, it is even more so in this case as two sets of poles ζ_a and ζ_b (versus one when using global elimination) must now be contended with. The functions $w_{2a}(\alpha, u)$ and $w_{2b}^{+}(\alpha, u)$ have[3] poles and zeros such that

$$w_{2a}(\alpha, u) \sim \frac{\alpha + \zeta_a}{\alpha - \zeta_a}\,\frac{\alpha + (2\pi - \zeta_a)}{\alpha - (2\pi - \zeta_a)}, \tag{6.76a}$$

$$w_{2a}(\alpha, -u) \sim \frac{\alpha - \zeta_a}{\alpha + \zeta_a}\,\frac{\alpha - (2\pi - \zeta_a)}{\alpha + (2\pi - \zeta_a)}\,\frac{\alpha - (\pi + \theta)}{\alpha + (\pi + \theta)}\,\frac{\alpha - (-2\pi + \theta)}{\alpha + (-2\pi + \theta)}, \tag{6.76b}$$

and

$$w_{2b}^{+}(\alpha, u) \sim \frac{\alpha + \zeta_b}{\alpha - \zeta_b}\,\frac{\alpha + (2\pi - \zeta_b)}{\alpha - (2\pi - \zeta_b)}, \tag{6.77a}$$

$$w_{2b}^{+}(\alpha, -u) \sim \frac{\alpha - \zeta_b}{\alpha + \zeta_b}\,\frac{\alpha - (2\pi - \zeta_b)}{\alpha + (2\pi - \zeta_b)}. \tag{6.77b}$$

The corresponding expressions for $w_{2b}^{-}(\alpha, u)$ are simply the reciprocal of those for $w_{2b}^{+}(\alpha, u)$. Equations (6.76) and (6.77) when taken together identify the poles and zeros of $w^{\pm}(\alpha, u)$ in the strip $S_{4\pi}$. Needless to say, the complete elimination of these singularities is a daunting

[3]It is assumed that $\sigma_a = \sigma_b = 1$.

task. In the limit as $\delta \to \pi/2$, we have

$$w^{+}(\alpha, u) \xrightarrow{\delta\to\pi/2} \frac{\tan\frac{\zeta_a}{2} + \tan\frac{\alpha}{2}}{\tan\frac{\zeta_a}{2} - \tan\frac{\alpha}{2}} \frac{\tan\frac{\zeta_b}{2} + \tan\frac{\alpha}{2}}{\tan\frac{\zeta_b}{2} - \tan\frac{\alpha}{2}} \Psi_4(\alpha) \tag{6.78}$$

and

$$w^{+}(\alpha, -u) \xrightarrow{\delta\to\pi/2} \frac{\tan\frac{\zeta_a}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_a}{2} + \tan\frac{\alpha}{2}} \frac{\tan\frac{\zeta_b}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_b}{2} + \tan\frac{\alpha}{2}} \frac{\cos\frac{1}{4}(\alpha - \theta + \pi)\cos\frac{1}{4}(\alpha - \theta)}{\cos\frac{1}{4}(\alpha + \theta - \pi)\cos\frac{1}{4}(\alpha + \theta)} \frac{e^{\pm j\frac{1}{2}\alpha}}{\Psi_4(\alpha)}, \tag{6.79}$$

where use has been made of (6.51). These results are a simple extension of their counterparts encountered for the global elimination. They are, of course, in agreement with (6.76) and (6.77).

6.4.3 Branch-free solutions

Results obtained with the method of global elimination during the investigation of a large number of branch-free forms have revealed that in many instances, barring numerical zeros straying out of $S_{4\pi}$, the more highly numerical approaches (requiring the numerical identification of larger numbers of numerical zeros) generally produce better behaved solutions. Recognizing this, we attempt to construct branch-free solutions by attacking the problem without resorting to the use of constants or linear combinations to eliminate poles. Our starting point are the branch-free forms

$$t_1^{\pm}(\alpha) = \frac{Q(\alpha)}{2}\left\{\left(1 + \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right) w^{\pm}(\alpha, u) + \left(1 - \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right) w^{\pm}(\alpha, -u)\right\}, \tag{6.80}$$

$$t_2^{\pm}(\alpha) = \frac{1}{2Q(\alpha)}\left\{\left(1 - \frac{r_2^{\pm}(\alpha)}{u(\alpha)}\right) w^{\pm}(\alpha, u) + \left(1 + \frac{r_2^{\pm}(\alpha)}{u(\alpha)}\right) w^{\pm}(\alpha, -u)\right\}, \tag{6.81}$$

where $r(\alpha)$ is the familiar 4π periodic polynomial (see (6.82) and (6.83)) having five degrees of freedom. Proceeding as in Section 4.3, we eliminate the ζ_a poles by using $r(\alpha)$ to introduce double zeros at the poles of $w^{\pm}(\alpha, -u)$ which results in

$$r_1(\alpha) = \frac{\cos\zeta_a \sin^2\delta\cos\alpha + \sin\zeta_a\cos^2\delta\sin\alpha}{u(\zeta_a)}. \tag{6.82}$$

and

$$r_2(\alpha) = \frac{\cos\zeta_a \sin^2\delta\cos\alpha - \sin\zeta_a\cos^2\delta\sin\alpha}{u(\zeta_a)}. \tag{6.83}$$

Since we have resigned ourselves to using a more numerical approach we have set the constants ρ_1 and ϱ_1 to zero. The hope is that such a maneuver will facilitate the recovery of known limiting functions. Setting

$$Q(\alpha) = \frac{\tan\frac{\zeta_a}{2} - \tan\frac{\alpha}{2}}{\tan\frac{\zeta_a}{2} + \tan\frac{\alpha}{2}}, \tag{6.84}$$

the branch-free solutions (6.80) and (6.81) are now free of ζ_a poles. This leaves us with the θ related poles, the ζ_b poles, as well as the required correction to the order. The core of the solution is the combination

$$\frac{e^{j\alpha/4}t_1^+(\alpha) + e^{-j\alpha/4}t_1^-(\alpha)}{\cos\frac{\alpha}{4}} \tag{6.85}$$

which is $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. More importantly perhaps, complex analysis in the α plane shows that the numerator is always zero at $\alpha = \pm 2\pi$, making possible a correction to the order without introducing additional poles. This is one of the advantages that the method of split elimination has over the method of global elimination: the latter has much more asymmetric expressions and it is unclear how a similar feat could be accomplished. All that is required now is to eliminate the θ and ζ_b poles. Write

$$\begin{aligned} t_V(\alpha) = {}& \frac{1}{2}\frac{\sin\frac{1}{4}(2\pi-\theta-\alpha)}{\sin\frac{1}{4}(2\pi-\theta)}\frac{\sin\frac{1}{4}(\pi+\theta+\alpha)}{\sin\frac{1}{4}(\pi+\theta)} \\ & \cdot\frac{\sin\frac{1}{4}(\zeta_b-\alpha)}{\sin\frac{1}{4}\zeta_b}\frac{\sin\frac{1}{4}(\zeta_b+\alpha)}{\sin\frac{1}{4}\zeta_b}\frac{\sin\frac{1}{4}(2\pi-\zeta_b-\alpha)}{\sin\frac{1}{4}(2\pi-\zeta_b)}\frac{\sin\frac{1}{4}(-2\pi+\zeta_b-\alpha)}{\sin\frac{1}{4}(-2\pi+\zeta_b)} \\ & \cdot\prod_{n=1}^{6}\frac{\sin\frac{1}{4}\alpha_n}{\sin\frac{1}{4}(\alpha_n-\alpha)}\frac{e^{j\alpha/4}t_1^+(\alpha)+e^{-j\alpha/4}t_1^-(\alpha)}{\cos\frac{1}{4}\alpha} \end{aligned} \tag{6.86}$$

which can be slightly simplified using trigonometric identities but is more instructive in the provided factored representation. The first six normalized factors annul the θ poles of $w_{2u}(\alpha,-u)$ in the outer strips, see (6.76b), and the ζ_b poles of $w_{2b}^{\pm}(\alpha,-u)$, see (6.77b). The product of factors that follows annuls the zeros α_n, which must be located numerically, and offset the order of the six preceding terms. The results are shown in Figure 6.7 which gives the magnitude of $t_V(\alpha)$ for a variety of δ and known solutions are seemingly recovered in both limits. There is however a problem when $\delta \to 0$ since one of the numerical zeros strays outside the strip $S_{4\pi}$ when $\delta \sim 0.01$. It is unclear whether this is a fundamental flaw in the process or if it is due to numerical inaccuracies. Regardless, solution $t_V(\alpha)$ is invalid when $\delta \sim 0$ since it is then not pole-free in $S_{4\pi}$.

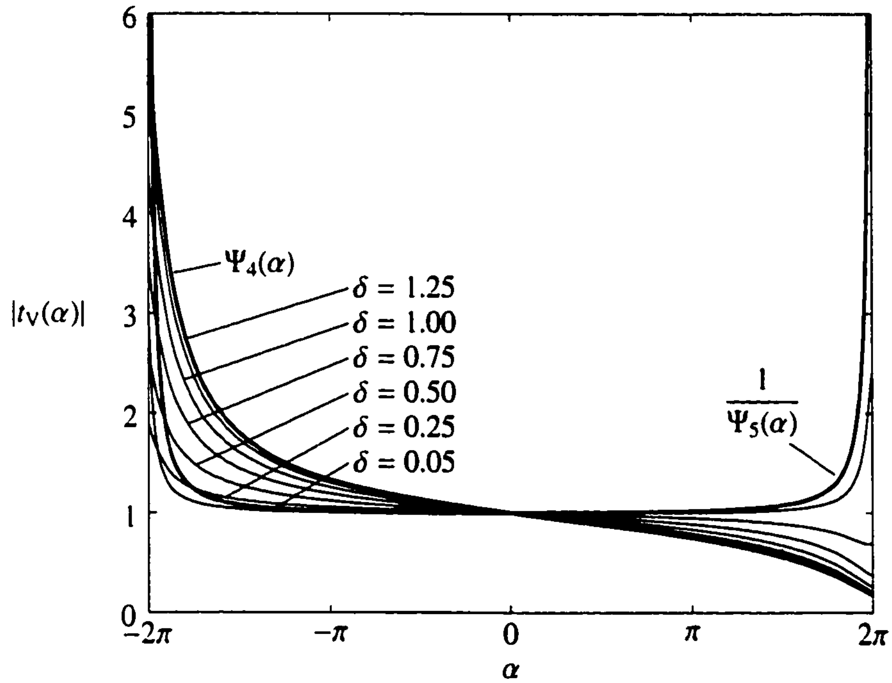


Figure 6.7: Magnitude of the branch-free solution $t_V(\alpha)$ given in (6.86) when $\theta = 0.25(1+j)$ for various values of δ. The thicker lines corresponds to the known limiting functions $\Psi_4(\alpha)$, per (5.5), for $\delta = \pi/2$ and $1/\Psi_5(\alpha)$, per (5.10), for $\delta = 0$. The $\delta = 1.57$ curve is juxtaposed with $\Psi_4(\alpha)$. The small glitch between π and 2π is attributed to insufficient accuracy in the value of one of the zeros α_n.

6.5 Possible alternative

In both the global and split elimination approaches, the key problems arise when constructing the branch-free solutions. The large number of poles coupled with their asymmetry as well as the required correction to the order make this process difficult at best and puts purely analytical approaches out of reach. A variation to both approaches that shows promise is to employ

$$v_3(\alpha, u) = \frac{1}{2}\left(1 - \frac{\cos(\alpha)}{\cos(\zeta)}\frac{u(\zeta)}{u(\alpha)}\right)\frac{\sin\zeta}{\cos\alpha - \cos\zeta} \tag{6.87}$$

as a substitute to the currently used

$$v_{2\pi}^3(\alpha, u) = \frac{u(\zeta)}{u(\alpha)}\frac{\cos(\alpha)}{\cos(\zeta)}\frac{\sin\zeta}{\cos\alpha - \cos\zeta} \tag{6.88}$$

when eliminating the cyclic periods. The apparently more complex (6.87) actually has a number of advantages over $v_{2\pi}^3(\alpha, u)$, the most important of which is that it has poles only on the $-u(\alpha)$ Riemann sheet, a characteristic shared with $v_0(\alpha, u)$ and $v_1(\alpha, u)$. The upshot is that the number of poles introduced in the elimination of cyclic periods is reduced by half as all of the poles in the integrands are now confined solely to the $-u(\alpha)$ Riemann sheet. Preliminary indications are encouraging. In the case of global elimination, this means that if we consider

$$t_1^{\pm}(\alpha) = \left\{\left(1 + \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, u) + \left(1 - \frac{r_1^{\pm}(\alpha)}{u(\alpha)}\right)w^{\pm}(\alpha, -u)\right\}, \tag{6.89}$$

the term $1+r_1^{\pm}(\alpha)/u(\alpha)$ can be used to introduce four zeros coinciding with those of $w(\alpha, -u)$ or, alternatively, $1 - r_1^{\pm}(\alpha)/u(\alpha)$ can be used to directly eliminate the poles of $w(\alpha, -u)$. This can be achieved without resorting to constant terms in $r_1(\alpha)$ or linear combinations, a source of problems as we have seen. This approach is also applicable to the method of split elimination though it is unclear at this point what its ramifications are on the symmetry of the solution. The key worry here is that it may jeopardize the ability to correct the order without introducing poles. However, at the very least, it reduces by half the number of ζ poles in both instances. It is perhaps a more reasonable way to proceed, since it takes advantage, after all, of symmetries ingrained in the expressions involved.

6.6 Summary

A modified version of the technique was presented where, in an effort to simplify the elimination of the cyclic periods, first order difference equations with reduced period were derived. The technique was applied to the second order equation solved in Chapter 5 and analysis was now required on a surface of genus one instead of genus three. The price paid was the need to introduce terms that are non-vanishing as $|\mathrm{Im}\ \alpha| \to \infty$ and the need to eliminate additional poles. Consequently, the construction of the desired meromorphic solutions is significantly more difficult and cannot be achieved fully analytically: zeros had to be numerically identified to complete the solutions. A large number of attempts were made in the context of the method of global elimination but the preferred behavior of Chapter 4 was not achieved. These efforts did show however that constants and linear combinations tend to disrupt the recovery of limiting functions and should preferably be avoided when eliminating poles during the construction of meromorphic solutions. The all out numerical approach taken in the method of split elimination almost succeeded in recovering known limits but the solution fails in the neighborhood of $\delta \sim 0$. The best hope to alleviate these difficulties is the alternate integrand of the third kind $v_3(\alpha, u)$ discussed in the previous section which halves the number of poles introduced during the elimination of the cyclic periods.

CHAPTER 7

The Anisotropic Impedance Half-Plane

THE problem of the anisotropic half-plane with common face impedances illuminated at oblique incidence is now examined. Owing to the large number of singularities present in the strip of analyticity, the associated difference equations are considerably more complex than those solved in Chapters 4, 5 and 6. These difficulties can be side-stepped by making an approximation to the square root term that appears and this is used to construct an approximate solution. The difference equations then all involve ratios of trigonometric polynomials and their solutions are expressible in terms of Maliuzhinets functions. A discussion of possible exact treatments based on the material from the preceding chapters follows and two alternatives are discussed. In the first, a direct approach is taken as in Chapters 4 and 5. As suspected, the main problem with such an approach is the elimination of the cyclic periods. The second approach is based on the method of period reduction of Chapter 6, which simplifies the elimination of the periods at the cost of introducing more poles in the strip of analyticity. In both cases, shortages in the number of degrees of freedom preclude the completion of fully analytical solutions.

7.1 The difference equations

Consider the anisotropic half-plane shown in Figure 7.1 illuminated by an obliquely incident plane wave characterized by an azimuthal angle ϕ_0 and a skew angle β — see (2.8) and Figure 2.1(a). The difference equations derived in Section 2.2 were obtained by specializing the generalized formulation of the anisotropic impedance wedge. We avoid the general but heavier notation employed therein and revert instead to the one employed by Senior and Legault [1998]. Recall that the quantities $t_e(\alpha)$ and $t_h(\alpha)$, related to the unknown spectral functions $s_e(\alpha)$ and $s_h(\alpha)$, were defined in Section 2.1.1 when diagonal-

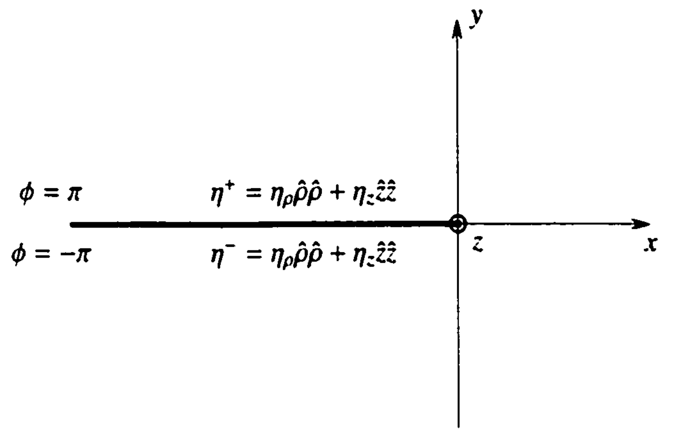


Figure 7.1: The anisotropic half-plane with $\Phi = \pi$.

izing the matrix expressions for the spectral functions $s_{e,h}(\alpha)$ in (2.15). The second order difference equation satisfied by either $t_e(\alpha)$ or $t_h(\alpha)$, which must both be even by virtue of (2.19), may be written as[1]

$$(\Gamma^{++})^2\, t(\alpha + 6\pi) - \left\{2\Gamma^{++}\Gamma^{--} + (\Gamma^{+-} - \Gamma^{-+})^2\right\} t(\alpha + 2\pi) + (\Gamma^{--})^2\, t(\alpha - 2\pi) = 0, \qquad (7.1)$$

which is (2.53) under a different guise. We have defined

$$\Gamma^{\sigma_1\sigma_2} = \cos^2\alpha\cos^2\beta + \left(\sin\alpha + \sigma_1 \frac{1}{\eta_\rho}\sin\beta\right)(\sin\alpha + \sigma_2\eta_z\sin\beta) \qquad (7.2)$$

where $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$ so that, for example,

$$\Gamma^{+-} = \cos^2\alpha\cos^2\beta + \left(\sin\alpha + \frac{1}{\eta_\rho}\sin\beta\right)(\sin\alpha - \eta_z\sin\beta). \qquad (7.3)$$

The quantity β is the skew angle of the illuminating wave (see Figure 2.1(a)) where $\beta = \pi/2$ corresponds to normal incidence. In that particular case, the second order equation reduces

[1]Equations (2.17) in this work, where $t_{e,h}(\alpha)$ are defined in terms of $s_{e,h}(\alpha + \pi)$, differ from those used in [Senior and Legault, 1998] which instead express $t_{1,2}(\alpha+\pi)$ in terms of $s_{e,h}(\alpha+\pi)$ so that $t_{1,2}(\alpha+\pi) = t_{e,h}(\alpha)$. This also results in a change of $\alpha \to \alpha - \pi$ in the coefficients of the difference equations.

to the first order difference equations

$$\frac{t_e(\alpha+2\pi)}{t_e(\alpha-2\pi)}=\left(\frac{\sin\alpha-1/\eta_\rho}{\sin\alpha+1/\eta_\rho}\right)^2, \tag{7.4}$$

$$\frac{t_h(\alpha+2\pi)}{t_h(\alpha-2\pi)}=\left(\frac{\sin\alpha-\eta_z}{\sin\alpha+\eta_z}\right)^2, \tag{7.5}$$

which are also respectively satisfied by $s_e(\alpha-\pi)$ and $s_h(\alpha-\pi)$. Letting $\sin\theta_1 = 1/\eta_\rho$ and $\sin\theta_2 = \eta_z$, the solutions to (7.4) and (7.5) are then written

$$t_e(\alpha) = \Psi_6^2(\alpha,\theta_1), \tag{7.6}$$

$$t_h(\alpha) = \Psi_6^2(\alpha,\theta_2), \tag{7.7}$$

where, in terms of the Maliuzhinets half-plane function $\psi_\pi(\alpha)$,

$$\begin{aligned}\Psi_6(\alpha,\theta) &= \frac{\psi_\pi\left(\alpha+\theta-\frac{\pi}{2}\right)\psi_\pi\left(\alpha-\theta+\frac{\pi}{2}\right)}{\psi_\pi^2\left(\theta-\frac{\pi}{2}\right)} \\ &= \exp\int_0^\alpha \frac{\frac{\xi}{2\pi}\sin\theta\cos\xi+\varphi_1\sin\frac{\xi}{2}+\varphi_2\cos\xi+\varphi_3\cos\frac{3\xi}{2}+\varphi_4\sin 2\xi}{\sin^2\xi-\sin^2\theta}\mathrm{d}\xi\end{aligned} \tag{7.8}$$

and

$$\varphi_1 = -\frac{1}{4}\left(\cos\frac{3\theta}{2}-\sin\frac{3\theta}{2}\right), \tag{7.9a}$$

$$\varphi_2 = \left(\frac{1}{4}-\frac{\theta}{2\pi}\right), \tag{7.9b}$$

$$\varphi_3 = -\frac{1}{4}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right), \tag{7.9c}$$

$$\varphi_4 = \frac{1}{8}. \tag{7.9d}$$

Note that $t_{e,h}(\alpha) \sim \exp\frac{1}{4}|\mathrm{Im}\,\alpha|$ as $|\mathrm{Im}\,\alpha| \to \infty$ which is consistent with the order requirement for this particular problem. The function $\Psi_6(\alpha,\theta)$ actually satisfies

$$\frac{\Psi_6(\alpha+2\pi,\theta)}{\Psi_6(\alpha-2\pi,\theta)}=-\frac{\sin\alpha-\sin\theta}{\sin\alpha+\sin\theta} \tag{7.10}$$

but the minus sign is of no consequence since the function $\Psi_6(\alpha,\theta)$ is squared in the solution $t_{e,h}(\alpha)$. This should be compared to $\Psi_5(\alpha,\theta)$ defined in (5.10).

The second order difference equation (7.1) can be recast as a pair of first order difference equations by using either technique given in Section 2.1.2. Adhering to our convention of denoting solutions to first order difference equations by $w(\alpha, u)$, the resulting expressions are

$$\frac{w(\alpha+2\pi, \pm u)}{w(\alpha-2\pi, \pm u)} = \frac{\Gamma^{--}}{\Gamma^{++}} \frac{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} - \gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} + \gamma\sin\alpha} \tag{7.11}$$

and, letting $w(\alpha, u) = w_1(\alpha)w_2(\alpha, u)$, we have

$$\frac{w_1(\alpha+2\pi)}{w_1(\alpha-2\pi)} = \frac{\Gamma^{--}}{\Gamma^{++}} = \frac{(\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)} \tag{7.12}$$

and

$$\frac{w_2(\alpha+2\pi, \pm u)}{w_2(\alpha-2\pi, \pm u)} = g(\alpha, \pm u) = \frac{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} - \gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} + \gamma\sin\alpha}, \tag{7.13}$$

with

$$\sin\theta_{1,2} = \frac{1}{2\sin\beta}\left\{\eta_z + \frac{1}{\eta_\rho} \mp \sqrt{\left(\eta_z - \frac{1}{\eta_\rho}\right)^2 + 4\left(\frac{\eta_\rho}{\eta_z} - 1\right)\cos^2\beta}\right\}, \tag{7.14}$$

$$\sin\delta_{1,2} = \frac{1}{\sin\beta}\left\{\sqrt{\frac{\eta_z}{\eta_\rho}} \pm \cos\beta\sqrt{\frac{\eta_z}{\eta_\rho} - 1}\right\}, \tag{7.15}$$

and

$$\gamma = \left(\eta_z - \frac{1}{\eta_\rho}\right)\frac{1}{\sin\beta}. \tag{7.16}$$

Note that when $\beta = \pi/2$, $\delta_1 = \delta_2$, $\sin\delta_1 = \sin\delta_2 = \sqrt{\eta_\rho/\eta_z}$, $\sin\theta_1 = 1/\eta_\rho$ and $\sin\theta_2 = \eta_z$. The square root $u(\alpha)$ is, from above,

$$u(\alpha) = \sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} \tag{7.17}$$

and its cuts are chosen such that $u(\alpha) = u(-\alpha) = u(\alpha+\pi)$. The associated dissections in the strip of analyticity are illustrated in Figure 7.2. The root $u(\alpha)$ is also such that on the positive branch (top Riemann sheet) $u(\alpha) = \sin\delta_1 \sin\delta_2$ when $\sin\alpha = 0$. This implies that

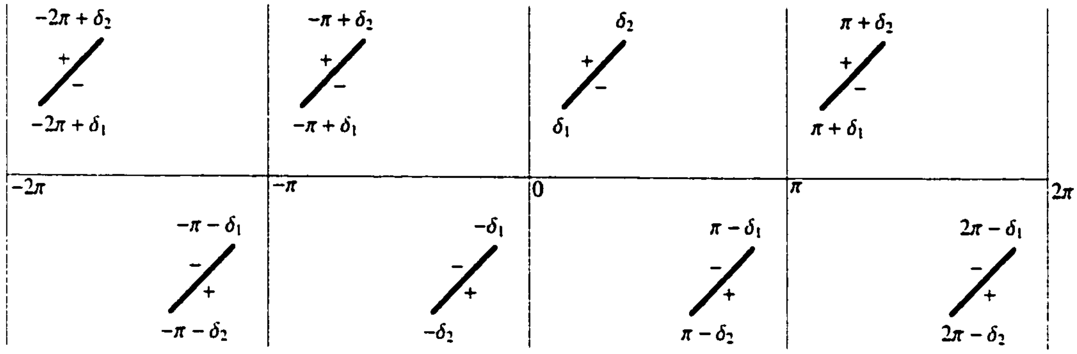


Figure 7.2: The 4π strip of analyticity with the branch points and associated cuts for the square root (7.17) appearing in (7.13).

when $\beta = \pi/2$, $u(\alpha) = \sin^2\delta_1 - \sin^2\alpha = \sin^2\delta_2 - \sin^2\alpha$.

The two required solutions $t_e(\alpha)$ and $t_h(\alpha)$ are expressed as linear combinations of $w_1(\alpha)w_2(\alpha, u)$ using the branch-free expressions (3.3). They must satisfy (7.1), be $O\{\exp[(1-\varepsilon)|\mathrm{Im}\,\alpha|]\}$ as $|\mathrm{Im}\,\alpha| \to \infty$ (here ε is positive real), and be free of singularities in the strip $S_{4\pi}$ apart from a simple (optics) pole at $\alpha = \phi_0$. A solution to equation (7.12) that is free of poles and zeros in $S_{4\pi}$ is, since $\Psi_6(\alpha,\theta)$ satisfies (7.10),

$$w_1(\alpha) = \Psi_6(\alpha,\theta_1)\Psi_6(\alpha,\theta_2) \tag{7.18}$$

and as $|\mathrm{Im}\,\alpha| \to \infty$,

$$w_1(\alpha) \sim O\left\{\exp\left(\frac{1}{2}|\mathrm{Im}\,\alpha|\right)\right\}. \tag{7.19}$$

The crux of the problem lies in finding the solution $w_2(\alpha, u)$ to (7.13). As always, it is useful to examine the solution when $\beta = \pi/2$ and the branch points vanish. Equation (7.13) then becomes

$$\frac{w_2(\alpha+2\pi,\pm u)}{w_2(\alpha-2\pi,\pm u)} = g(\alpha,\pm u) = \left\{\frac{\sin\alpha-\sin\theta_1}{\sin\alpha+\sin\theta_1}\,\frac{\sin\alpha+\sin\theta_2}{\sin\alpha-\sin\theta_2}\right\}^{\pm 1} \tag{7.20}$$

and using once again (7.10), the solution $w_2(\alpha,\pm u)$ in that limit is

$$w_2(\alpha,\pm u) = w_2(\alpha)^{\pm 1} = \left(\frac{\Psi_6(\alpha,\theta_1)}{\Psi_6(\alpha,\theta_2)}\right)^{\pm 1} \tag{7.21}$$

which is $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. Since $w(\alpha,u) = w_1(\alpha)w_2(\alpha,u)$ it can been appreciated, using (7.18) and (7.21), that the solutions (7.6) and (7.7) are recovered. We now examine an approximate solution to (7.13) that recovers the known expressions when $\beta = \pi/2$ (normal incidence) and when $\eta_\rho = \eta_z$ (isotropic impedance.)

7.2 Approximate solution

The complications due to the branch points of $u(\alpha)$ can be avoided by approximating the square root. The difference equations then all involve rational trigonometric functions and are amenable to solution using Maliuzhinets functions. Consider those $\sin\alpha$ for which $|\sin\alpha| \ll |\sin\delta_{1,2}|$. This is not an unreasonable approximation if $|\sin\delta_{1,2}| > 1$ and we recall that, under a steepest descent approximation, the integrals in (2.12) are dominated by the behavior of the integrands at the saddle points on the real axis of the α plane where $|\sin\alpha| \leq 1$. We then have

$$\sqrt{(\sin^2\alpha - \sin^2\delta_1)(\sin^2\alpha - \sin^2\delta_2)} \simeq \sin\delta_1 \sin\delta_2 \left\{ 1 - \frac{1}{2}\left(\frac{1}{\sin^2\delta_1} + \frac{1}{\sin^2\delta_2}\right)\sin^2\alpha \right\} \tag{7.22}$$

consistent with the definition of the square root in (7.17), thereby eliminating the branch points of $g(\alpha, u)$. This is an exact representation if $\sin\delta_1 = \sin\delta_2$ corresponding to an anisotropic half–plane at normal incidence ($\beta = \pi/2$) and an isotropic ($\eta_\rho = \eta_z$) half–plane for arbitrary β. With this simplification the expression for $g(\alpha, u)$ becomes

$$g(\alpha) = \frac{(\sin\alpha - \sin\bar{\theta}_1)(\sin\alpha + \sin\bar{\theta}_2)}{(\sin\alpha + \sin\bar{\theta}_1)(\sin\alpha - \sin\bar{\theta}_2)} \tag{7.23}$$

where

$$\sin\bar{\theta}_{1,2} = \frac{1}{\sin\beta}\frac{\sin\delta_1\sin\delta_2}{\sin^2\delta_1 + \sin^2\delta_2}\left\{\mp\left(\eta_z - \frac{1}{\eta_\rho}\right) + \sqrt{\left(\eta_z - \frac{1}{\eta_\rho}\right)^2 + 2(\sin^2\delta_1 + \sin^2\delta_2)\sin^2\beta}\right\}, \tag{7.24}$$

and from the expressions (7.15) for $\sin\delta_{1,2}$ we obtain

$$\sin\bar{\theta}_{1,2} = \frac{1}{2\sin\beta}\frac{\frac{\eta_z}{\eta_\rho} - \cos^2\beta\left(\frac{\eta_z}{\eta_\rho} - 1\right)}{\frac{\eta_z}{\eta_\rho} + \cos^2\beta\left(\frac{\eta_z}{\eta_\rho} - 1\right)}\left\{\mp\left(\eta_z - \frac{1}{\eta_\rho}\right) + \sqrt{\left(\eta_z + \frac{1}{\eta_\rho}\right)^2 + 4\cos^2\beta\left(\frac{\eta_z}{\eta_\rho} - 1\right)}\right\}. \tag{7.25}$$

It is assumed that $0 \leq \mathrm{Re}\,\bar{\theta}_1, \mathrm{Re}\,\bar{\theta}_2 \leq \pi/2$, and the square root is such that when $\beta = \pi/2$

$$\sin\bar{\theta}_1 = \frac{1}{2}\left\{-\left(\eta_z - \frac{1}{\eta_\rho}\right) + \left(\eta_z + \frac{1}{\eta_\rho}\right)\right\} = \frac{1}{\eta_\rho} = \sin\theta_1, \tag{7.26}$$

$$\sin\bar{\theta}_2 = \frac{1}{2}\left\{\left(\eta_z - \frac{1}{\eta_\rho}\right) + \left(\eta_z + \frac{1}{\eta_\rho}\right)\right\} = \eta_z = \sin\theta_2. \tag{7.27}$$

We are now able to express $w_2(\alpha, u)$, like its counterpart $w_1(\alpha)$, in terms of the Maliuzhinets half-plane function. Since (7.13) can now be approximated as

$$\frac{w_2(\alpha + 2\pi, \pm u)}{w_2(\alpha - 2\pi, \pm u)} = \left\{\frac{(\sin\alpha - \sin\bar{\theta}_1)(\sin\alpha + \sin\bar{\theta}_2)}{(\sin\alpha + \sin\bar{\theta}_1)(\sin\alpha - \sin\bar{\theta}_2)}\right\}^{\pm 1}, \tag{7.28}$$

it follows that

$$w_2(\alpha, \pm u) = w_2(\alpha)^{\pm 1} = \left\{\frac{\Psi_6(\alpha, \bar{\theta}_1)}{\Psi_6(\alpha, \bar{\theta}_2)}\right\}^{\pm 1}, \tag{7.29}$$

and $w_2(\alpha) \sim O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. The combinations (3.3) are not required since $w_2(\alpha)$ is meromorphic; the solutions to the second order difference equation (7.1) are given by

$$t_e(\alpha) = C_e(\alpha) w_1(\alpha) w_2(\alpha), \tag{7.30}$$

$$t_h(\alpha) = C_h(\alpha)\frac{w_1(\alpha)}{w_2(\alpha)}, \tag{7.31}$$

where $C_e(\alpha)$ and $C_h(\alpha)$ are 4π periodic functions. Functionally, this is the same scenario as the one dealt with in the previous chapters when the limit $\delta \to \pi/2$ and the branch points vanish.

In view of the order restriction on $s_e(\alpha)$ and $s_h(\alpha)$, the even parity requirement on $t_{e,h}(\alpha)$, and the requirement for a simple pole at $\alpha = \phi_0$, we now write

$$C_e(\alpha) = A_e q(\alpha) + a_e + b_e \cos\frac{\alpha}{2} \tag{7.32}$$

$$C_h(\alpha) = A_h q(\alpha) + a_h + b_h \cos\frac{\alpha}{2} \tag{7.33}$$

where

$$q(\alpha) = \frac{1}{2}\frac{\cos\frac{\phi_0}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\phi_0}{2}} \tag{7.34}$$

and A_e, A_h, a_e, a_h, b_e and b_h are constants to be determined. From (2.17)

$$t_e(\alpha - \pi) = \left(-\sin\alpha + \frac{1}{\eta_\rho}\sin\beta\right)s_e(\alpha) + \cos\alpha\cos\beta s_h(\alpha), \tag{7.35a}$$

$$t_h(\alpha - \pi) = (-\sin\alpha + \eta_z\sin\beta)s_h(\alpha) - \cos\alpha\cos\beta s_e(\alpha), \tag{7.35b}$$

and, to reproduce the incident field when $\alpha = \phi_0$, we require

$$A_e w_1(\phi_0 - \pi)w_2(\phi_0 - \pi) = \left(-\sin\phi_0 + \frac{1}{\eta_\rho}\sin\beta\right)e_z + \cos\phi_0\cos\beta h_z, \tag{7.36a}$$

$$A_h\frac{w_1(\phi_0 - \pi)}{w_2(\phi_0 - \pi)} = (-\sin\phi_0 + \eta_z\sin\beta)h_z - \cos\phi_0\cos\beta e_z. \tag{7.36b}$$

Equation (2.18) and (7.2) imply

$$s_e(\alpha) = \frac{1}{\Gamma^{--}}\left\{(-\sin\alpha + \eta_z\sin\beta)t_e(\alpha - \pi) - \cos\alpha\cos\beta t_h(\alpha - \pi)\right\}, \tag{7.37a}$$

$$s_h(\alpha) = \frac{1}{\Gamma^{--}}\left\{\left(-\sin\alpha + \frac{1}{\eta_\rho}\sin\beta\right)t_h(\alpha - \pi) + \cos\alpha\cos\beta t_e(\alpha) - \pi\right\}, \tag{7.37b}$$

and the remaining four constants are needed to eliminate the poles of $1/\Gamma^{--}$ at $\alpha = \theta_1$, $\pi - \theta_1$, θ_2 and $\pi - \theta_2$. Based on the expression for $s_e(\alpha)$, the requirements are

$$(-\sin\theta_1 + \eta_z\sin\beta)t_e(\theta_1 - \pi) = \cos\theta_1\cos\beta t_h(\theta_1 - \pi), \tag{7.38a}$$

$$(-\sin\theta_1 + \eta_z\sin\beta)t_e(\theta_1) = -\cos\theta_1\cos\beta t_h(\theta_1), \tag{7.38b}$$

$$(-\sin\theta_2 + \eta_z\sin\beta)t_e(\theta_2 - \pi) = \cos\theta_2\cos\beta t_h(\theta_2 - \pi), \tag{7.38c}$$

$$(-\sin\theta_2 + \eta_z\sin\beta)t_e(\theta_2) = -\cos\theta_2\cos\beta t_h(\theta_2), \tag{7.38d}$$

which are four equations from which a_e, a_h, b_e and b_h can be determined. It is easily verified that these also eliminate the poles from the expression for $s_h(\alpha)$. The determination of $t_e(\alpha)$ and $t_h(\alpha)$ and, from (7.37), $s_e(\alpha)$ and $s_h(\alpha)$, is now complete and it is seen that $s_e(\alpha)$, $s_h(\alpha) \sim O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$. As shown in [Senior and Legault, 1998, Appendix A], the expressions for $s_e(\alpha)$ and $s_h(\alpha)$ reduce to the known solutions when $\beta = \pi/2$.

With $s_e(\alpha)$ and $s_h(\alpha)$ specified as above, the expressions for the total field components E_z and Z_0H_z are as given in (2.12). The contour can be closed using paths from $j\infty - \pi/2$

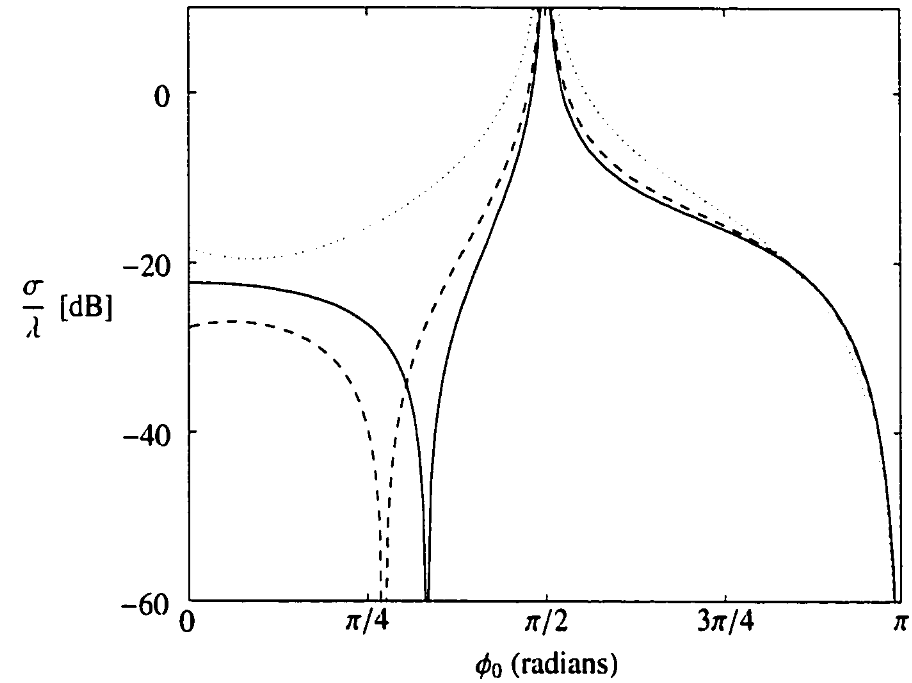


Figure 7.3: The normalized non-uniform backscatter echowidth when $e_z = 1$, $h_z = 0$ (E-polarization), $\eta_\rho = 2$ and $\eta_z = 4$: (——) $\beta = \pi/2$, (— —) $\beta = \pi/3$ and ($\cdots$) $\beta = \pi/6$. The angle $\phi_0 = 0$ corresponds to edge-on illumination and $\phi_0 = \pi/2$ to illumination from above.

to $-j\infty - 3\pi/2$ and $-j\infty + \pi/2$ to $j\infty + 3\pi/2$ shown in Figure 2.2, leading to the result

$$E_z(\rho,\phi) = \frac{e^{-jk_0 z\cos\beta}}{2\pi j}\int_{-\pi/2-j\infty}^{\pi/2+j\infty} e^{-jk_0\rho\sin\beta\cos\alpha}\Big\{s_e(\alpha+\phi-\pi) - s_e(\alpha+\phi+\pi)\Big\}d\alpha + 2\pi j\sum \text{Res} \quad (7.39)$$

with a similar expression for $Z_0 H_z$. The residues are those of the optics pole, giving rise to the incident wave if $\phi > \phi_0 - \pi$ and the reflected wave if $\phi > \pi - \phi_0$, as well as those of any surface wave poles of Ψ_6, but apart from contributions of these types, the diffracted field for $k_0\rho \gg 1$ is

$$E_z^d(\rho,\phi) = \frac{e^{-jk_0(z\cos\beta+\rho\sin\beta)-j\pi/4}}{\sqrt{2\pi k_0\rho\sin\beta}}\Big\{s_e(\phi-\pi) - s_e(\phi+\pi)\Big\}. \quad (7.40)$$

Since $s_e(\alpha)$ and $s_h(\alpha)$ have been expressed in terms of the Maliuzhinets half-plane function, the computation of the field is a simple task. In Figure 7.3 the backscattered ($\phi = \phi_0$) far field amplitude based on the non-uniform representation (7.40) is plotted as a function of ϕ_0, $0 \le \phi_0 \le \pi$, for $\eta_\rho = 2, \eta_z = 4$ and $\beta = \pi/2$ (normal incidence), $\pi/3$ and $\pi/6$. The quantity plotted is the normalized non-uniform echowidth defined as

$$\frac{\sigma}{\lambda} = \frac{1}{\lambda} \lim_{\rho \to \infty} 2\pi\rho \left| \frac{E_z^d}{E_z^i} \right|. \tag{7.41}$$

We now consider an exact treatment of the problem. From the previous section, it is clear that this entails obtaining an exact solution of the difference equation (7.13). Two approaches are considered: a direct approach and one based on a reduction of the period.

7.3 Exact solution: Direct approach

We now examine the solution of the branched first order equation (7.13) by means of a direct approach as carried out in Chapters 4 and 5. We first proceed to take its logarithmic derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha} \ln w_2(\alpha + 2\pi, u) - \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln w_2(\alpha - 2\pi, u) & \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln g(\alpha, u) \\ &= \frac{k(\alpha)}{u(\alpha)} \left(\frac{2\sin\theta_1 \cos\alpha}{\sin^2\alpha - \sin^2\theta_1} - \frac{2\sin\theta_2 \cos\alpha}{\sin^2\alpha - \sin^2\theta_2} \right)\end{aligned} \tag{7.42}$$

where

$$k(\alpha) = \gamma \frac{\sin\theta_1 \sin\theta_2 - \sin^2\alpha}{\sin\theta_2 - \sin\theta_1}, \tag{7.43}$$

and we note that

$$\operatorname{Res} \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln g(\alpha, u) = \begin{cases} \pm 1 & \sin\alpha = \pm \sin\theta_1 \\ \mp 1 & \sin\alpha = \pm \sin\theta_2. \end{cases} \tag{7.44}$$

Making use of the periodicity of $\mathrm{d} \ln g(\alpha, u)/\mathrm{d}\alpha$, it can be shown that if $\mathrm{d}w_2(\alpha, u)/\mathrm{d}\alpha = v_0(\alpha, u)$, then equation (7.42) is satisfied if

$$v_0(\alpha, u) = \frac{\alpha}{4\pi} \frac{k(\alpha)}{u(\alpha)} \left(\frac{2\sin\theta_1 \cos\alpha}{\sin^2\alpha - \sin^2\theta_1} - \frac{2\sin\theta_2 \cos\alpha}{\sin^2\alpha - \sin^2\theta_2} \right). \tag{7.45}$$

The solution therefore has the form

$$w_2(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \left\{ v_0(\xi, u) + v_{4\pi}(\xi, u) \right\} \mathrm{d}\xi, \tag{7.46}$$

where the 4π periodic $v_{4\pi}(\alpha, u)$ is needed to eliminate the polar and the cyclic periods of $v_0(\alpha, u)$ in order to obtain a well defined path integral. The function $v_0(\alpha, u)$ is odd, in contradistinction to all of the cases previously examined which had even parity, and it vanishes as $|\text{Im}\,\alpha| \to \infty$. Within the strip $\mathcal{S}_{4\pi}$, it has branch points associated with $1/u(\alpha)$ and poles at the zeros of $\sin^2\alpha - \sin^2\theta_{1,2}$ for a total of 16 branch points and 16 poles per sheet.

7.3.1 Elimination of polar periods

To eliminate the polar periods, introduce

$$v_1(\alpha, u) = \frac{k(\alpha)}{u(\alpha)}\left(\frac{f(\alpha, \theta_1)}{\sin^2\alpha - \sin^2\theta_1} - \frac{f(\alpha, \theta_2)}{\sin^2\alpha - \sin^2\theta_2}\right) \tag{7.47}$$

where, in order to match the parity of $v_0(\alpha, u)$,

$$f(\alpha, \theta) = \varphi_1 \sin\frac{\alpha}{2} + \varphi_2 \sin\alpha + \varphi_3 \sin\frac{3\alpha}{2} + \varphi_4 \sin 2\alpha. \tag{7.48}$$

Vanishing residues of $v_0(\alpha, u) + v_1(\alpha, u)$ are enforced and solving the resulting system of four equation in four unknowns φ_n recovers the values given in (7.9). It follows that

$$\begin{aligned} v_0(\alpha, \pm u) + v_1(\alpha, \pm u) &= \pm\frac{k(\alpha)}{u(\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_6(\alpha, \theta_1) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Psi_6(\alpha, \theta_2)\right) \\ &\xrightarrow{\beta\to\pi/2} \pm\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\frac{\Psi_6(\alpha, \theta_1)}{\Psi_6(\alpha, \theta_2)}. \end{aligned} \tag{7.49}$$

and (7.21) is recovered, as required, when $\beta = \pi/2$. It is worth pointing out that whereas the individual members appearing in the parentheses of (7.47) are $O(1)$ as $|\text{Im}\,\alpha| \to \infty$, their difference vanishes since the dominant terms are the same for both $f(\alpha, \theta_1)$ and $f(\alpha, \theta_2)$. In agreement with previous cases, $v_0(\alpha, u)$ and $v_1(\alpha, u)$ both vanish as $|\text{Im}\,\alpha| \to \infty$.

7.3.2 Elimination of cyclic periods

Barring the discovery of currently unknown favorable symmetries, the elimination of the cyclic periods requires a total of seven degrees of freedom. Indeed, exploiting the parity of $v_0(\alpha, u) + v_1(\alpha, u)$, we can limit our attention to the right-hand portion of $\mathcal{S}_{4\pi}$ as illustrated in Figure 7.4. The segments s_n are branch point to branch point half-cycles over which the integrals must vanish. We need not concern ourselves for now with path s_a because its contribution vanishes identically since $v_0(\alpha, u)$ and $v_1(\alpha, u)$ are odd. This leaves the seven segments s_b to s_h. Much effort has been devoted to finding symmetries of $v_0(\alpha, u)$ and

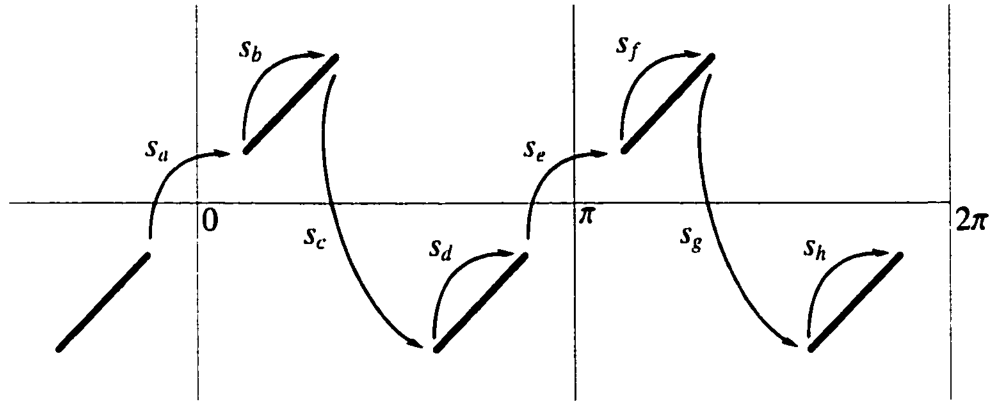


Figure 7.4: The branch point to branch point half-cycles over which the integrals must vanish.

$v_1(\alpha, u)$ to simplify the analysis but so far they remain elusive. Consequently, vanishing integrals must be individually enforced over the seven branch point to branch point paths s_b to s_h, which requires seven degrees of freedom. This is an intriguing result: in all of the previous cases, an even number of degrees of freedom was required. This fitted perfectly with the method proposed in this work since the degrees of freedom are introduced in pairs with one degree associated with an elliptic integral of the first kind and the other with an elliptic integral of the third kind. It is unclear how the proposed approach applies when an odd number of degrees is called for. The issue is further beclouded by the small number of acceptable odd expressions that are associated with integrals of the first kind. The three obvious choices are

$$\frac{\sin\frac{\alpha}{2}}{u(\alpha)}, \quad \frac{\sin\alpha}{u(\alpha)}, \quad \text{and} \quad \frac{\sin\frac{3\alpha}{2}}{u(\alpha)}. \tag{7.50}$$

All other options lead to non-vanishing expressions as $|\text{Im}\,\alpha| \to \infty$ and are therefore inadmissible. Together with the associated expressions leading to the integrals of the third kind, this yields a total of six degrees of freedom for a shortfall of one degree of freedom. Assuming we managed by some means to produce a seventh degree, a somewhat obscure proposition, the result would be a system of seven equations which would have to be decoupled into equation pairs as done in Chapter 5. No obvious symmetries indicate how this process can be carried through straightforwardly. The one redeeming feature is that, due to the vanishing period across the origin, the analysis would apparently be confined to surfaces of genus three. Besides the prohibitive analysis required there is another, though not so serious, source of puzzlement. Looking for example at the solutions in Chapter 4, the expressions obtained are naturally normalized to unity when $\alpha = 0$ and smoothly recover the known solution in the appropriate limits. This is not the case here due to the odd parity: the path integral from, say, δ_1 to the origin does not identically vanish. Our solution will therefore require normalization in order to recover the known limits. The only way to achieve this without resorting to normalization is to enforce a vanishing integral from 0 to δ_1, which would bring the path s_a back into play and increase the number of required degrees of freedom to eight, making it once again even. While this suggests that part of the answer probably lies in a better understanding of how to approach the problem when the integrand is odd, the shortfall in degrees of freedom and the complexity of the analysis suggest that a beneficial symmetry is being overlooked. If such a symmetry, as the one employed in Section 4.4.1, could be brought to bear, the number of required degrees of freedom could easily be reduced to five or perhaps four. The direct approach remains at an impasse until these questions can be addressed.

7.4 Exact solution: Method of period reduction

In the aftermath of the complications experienced in the direct approach, it is logical to turn to the method of period reduction of Chapter 6 which is tailor-made to deal with large numbers of branch points. It will be seen that the only variant which applies is the method of split elimination and an outline for carrying out the solution using this technique is provided. We proceed by first obtaining an equation with reduced period 2π related to the original equation for $w(\alpha, u)$ of period 4π (7.11)

$$\frac{w(\alpha + 2\pi, \pm u)}{w(\alpha - 2\pi, \pm u)} = \frac{\Gamma^{--}}{\Gamma^{++}} \frac{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} - \gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha - \sin^2\delta_1\right)\left(\sin^2\alpha - \sin^2\delta_2\right)} + \gamma\sin\alpha}. \tag{7.51}$$

7.4.1 The reduced equation

We seek $w_r(\alpha, u)$ — r stands for reduced — such that

$$\frac{w_r(\alpha + \pi, \pm u)}{w_r(\alpha - \pi, \pm u)} = g_r(\alpha, \pm u) \tag{7.52}$$

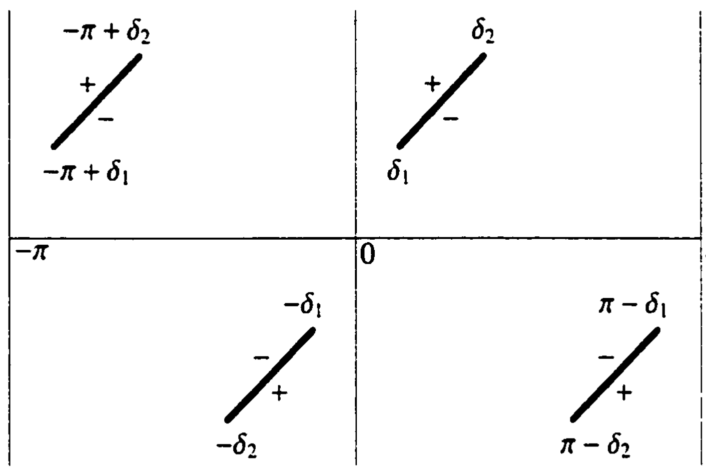


Figure 7.5: The configuration of the strip of analyticity of width 2π for the reduced equation (7.52). It corresponds to the inner part of the 4π wide strip illustrated in 7.2.

and

$$\begin{aligned}\frac{w_r(\alpha+2\pi,\pm u)}{w_r(\alpha-2\pi,\pm u)} &= g_r(\alpha+\pi,\pm u)g_r(\alpha-\pi,\pm u)\\ &= \frac{\Gamma^{--}}{\Gamma^{++}}\,\frac{\pm\sqrt{\left(\sin^2\alpha-\sin^2\delta_1\right)\left(\sin^2\alpha-\sin^2\delta_2\right)}-\gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha-\sin^2\delta_1\right)\left(\sin^2\alpha-\sin^2\delta_2\right)}+\gamma\sin\alpha}.\end{aligned} \tag{7.53}$$

A moderate amount of algebra shows that

$$\frac{\Gamma^{--}}{\Gamma^{++}}\,\frac{\pm\sqrt{\left(\sin^2\alpha-\sin^2\delta_1\right)\left(\sin^2\alpha-\sin^2\delta_2\right)}-\gamma\sin\alpha}{\pm\sqrt{\left(\sin^2\alpha-\sin^2\delta_1\right)\left(\sin^2\alpha-\sin^2\delta_2\right)}+\gamma\sin\alpha} = \left\{\frac{(\sin\alpha-\sin\theta_1)(\sin\alpha-\sin\theta_2)}{u+\gamma\sin\alpha}\right\}^2, \tag{7.54}$$

and the first order difference equation with reduced period to solve is

$$\frac{w_r(\alpha+\pi,\pm u)}{w_r(\alpha-\pi,\pm u)} = g_r(\alpha,\pm u) = \frac{(\sin\alpha+\sin\theta_1)(\sin\alpha+\sin\theta_2)}{u-\gamma\sin\alpha}. \tag{7.55}$$

Once $w_r(\alpha,u)$ has been obtained, the branched solution to (7.55) will have form $w(\alpha,u) = C_{4\pi}(\alpha)w_r(\alpha,u)$ where $C_{4\pi}(\alpha)$ is a unit (4π) periodic. It must be noted that, as was the case in Chapter 6, the behavior of $g_r(\alpha,u)$ is branch dependent when $|\text{Im}\,\alpha| \to \infty$ since $g_r(\alpha,\pm u) \to \pm 1$; it will be necessary to introduce terms that are non-vanishing as $|\text{Im}\,\alpha| \to \infty$ in order to compensate for this behavior. However, the solution can now be constructed in the strip $\mathcal{S}_{2\pi}$ which, for this particular problem, is illustrated in Figure 7.5. This duplicates the strip of analyticity examined in Section 4.4.1 where it was shown, provided the expressions have the right symmetry, that the complexity of the analysis was of the same order as the one required for the 2π periodic second order difference equation dealt with in Chapter 4. If the solution did not require higher order terms the process of elimination of cyclic periods would be fairly straightforward; it is unfortunately not the case.

7.4.1.1 The limit $\beta \to \pi/2$

First restricting ourselves to the case where $u(\alpha) = +u(\alpha)$, it can be shown that when $\beta = \pi/2$,

$$\frac{w_r(\alpha+\pi,+u)}{w_r(\alpha-\pi,+u)} = \frac{\sin\theta_1 + \sin\alpha}{\sin\theta_1 - \sin\alpha} \tag{7.56}$$

and, from the expression for $\Psi_6(\alpha,\theta)$ (7.8) and the equation (7.10) that it satisfies, it follows that we expect to recover $w_r(\alpha,+u) = \Psi_7(\alpha,\theta_1)$, where

$$\Psi_7(\alpha,\theta) = \frac{\psi_{\pi/2}\left(\alpha+\theta-\frac{\pi}{2}\right)\psi_{\pi/2}\left(\alpha-\theta+\frac{\pi}{2}\right)}{\psi^2_{\pi/2}\left(\theta-\frac{\pi}{2}\right)}. \tag{7.57}$$

The integral representation of $\Psi_7(\alpha,\theta)$ follows immediately from (4.18),

$$\Psi_7(\alpha) = \exp - \int_0^\alpha \frac{\frac{\xi}{\pi}\sin\theta\cos\xi - \left(\frac{\theta}{2}-\frac{1}{2}\right)\cos\theta\sin\xi - \frac{1}{4}\sin 2\xi}{\cos^2\xi - \cos^2\theta}\mathrm{d}\xi. \tag{7.58}$$

It is instructive to compare this with the expected expression for $t_e(\alpha)$ provided in (7.6). Indeed, for a given 4π periodic $C_{4\pi}(\alpha)$, we have $w(\alpha) = C_{4\pi}(\alpha)w_r(\alpha)$ implying that $\Psi_6^2(\alpha,\theta_1) = C_{4\pi}(\alpha,\theta_1)\Psi_7(\alpha,\theta_1)$. Interestingly, it can be showed, using the properties of Maliuzhinets's functions — see for example [Maliuzhinets, 1958*a*] or [Senior and Volakis, 1995] — that

$$\Psi_7(\alpha,\theta_1) = 4\frac{\psi_\pi^4(\pi)}{\psi_\pi^8\left(\frac{\pi}{2}\right)}\frac{1}{\left(\cos\frac{\alpha}{2}+\cos\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\theta_1}{2}\right)}\Psi_6^2(\alpha,\theta_1) \tag{7.59}$$

leading to

$$C_{4\pi}(\alpha,\theta_1) = \frac{1}{4}\frac{\psi_\pi^8\left(\frac{\pi}{2}\right)}{\psi_\pi^4(\pi)}\left(\cos\frac{\alpha}{2}+\cos\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\theta_1}{2}\right) \tag{7.60}$$

and this enables us to recover identically the desired solution $\Psi_6^2(\alpha,\theta_1)$ from the reduced equation. If we now turn to the case where we have the branch $-u(\alpha)$, algebraic manipulations yield

$$\frac{w_r(\alpha+\pi,-u)}{w_r(\alpha-\pi,-u)} = \frac{\sin\theta_2 + \sin\alpha}{\sin\theta_2 - \sin\alpha} \tag{7.61}$$

and, from (7.57), we now expect to recover $w_r(\alpha,-u) = \Psi_7(\alpha,\theta_2)$ with $\Psi_7(\alpha,\theta)$ defined in (7.57). This also recovers the desired solution (7.7) since

$$w(\alpha,-u) = \Psi_6^2(\alpha,\theta_2) = C_{4\pi}(\alpha,\theta_2)w_r(\alpha,-u) = C_{4\pi}(\alpha,\theta_2)\Psi_7(\alpha,\theta_2) \tag{7.62}$$

with $C_{4\pi}(\alpha,\theta)$ defined in (7.60).

7.4.2 The logarithmic derivative

A general solution that still applies when $\beta \neq \pi/2$ is now sought. Taking the logarithmic derivative of (7.55),

$$\begin{aligned} &v_0(\alpha+\pi,u) - v_0(\alpha-\pi,u) = \\ &\qquad \cos\alpha\left[\left(1+\frac{k(\alpha)}{u(\alpha)}\right)\frac{\sin\theta_1}{\cos^2\alpha-\cos^2\theta_1} + \left(1-\frac{k(\alpha)}{u(\alpha)}\right)\frac{\sin\theta_2}{\cos^2\alpha-\cos^2\theta_2}\right] \end{aligned} \tag{7.63}$$

where $v_0(\alpha,u) = \mathrm{d}/\mathrm{d}\alpha \ln w_r(\alpha,u)$. This implies

$$\begin{aligned} v_0(\alpha,u) &= -\frac{\alpha}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln g_r(\alpha,u) \\ &= -\frac{\alpha}{2\pi}\cos\alpha\left[\left(1+\frac{k(\alpha)}{u(\alpha)}\right)\frac{\sin\theta_1}{\cos^2\alpha-\cos^2\theta_1} + \left(1-\frac{k(\alpha)}{u(\alpha)}\right)\frac{\sin\theta_2}{\cos^2\alpha-\cos^2\theta_2}\right]. \end{aligned} \tag{7.64}$$

The function $v_0(\alpha,u)$ is odd, vanishes as $|\mathrm{Im}\,\alpha| \to \infty$ and has poles at the zeros of $(\cos^2\alpha - \cos^2\theta_1)$ on the $+u(\alpha)$ Riemann sheet and at the zeros of $(\cos^2\alpha - \cos^2\theta_2)$ on the $-u(\alpha)$ Riemann sheet. The solution to (7.52) is now

$$w_r(\alpha,u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi,u) + v_1(\xi,u) + v_{2\pi}(\xi,u)\right\}\mathrm{d}\xi \tag{7.65}$$

with the 2π periodic $v_1(\alpha,u)$ eliminating the polar periods and $v_{2\pi}(\alpha,u)$ the cyclic periods of the integrand. As shown below, $v_{2\pi}(\alpha,u)$ will also have to include a higher order term, a hardly surprising development based on the experience of Chapter 6.

7.4.3 Elimination of polar periods

Introduce

$$v_1(\alpha,u) = \left(1 + \frac{k(\alpha)}{u(\alpha)}\right)\frac{f(\alpha,\theta_1)}{\cos^2\alpha - cos^2\theta_1} + \left(1 - \frac{k(\alpha)}{u(\alpha)}\right)\frac{f(\alpha,\theta_2)}{\cos^2\alpha - cos^2\theta_2} \tag{7.66}$$

where $f(\alpha,\theta) = \varphi_1 \sin\alpha + \varphi_2 \sin 2\alpha$ so that $v_1(\alpha,u)$ is 2π periodic, odd and has the required degrees of freedom to eliminate the poles of $v_0(\alpha,u)$. Accordingly, $v_1(\alpha,u)$ shares with $v_0(\alpha,u)$ the same poles which, in $S_{2\pi}$, are at $\alpha = \pm\theta_{1,2}$ and $\alpha = \pm(\pi - \theta_{1,2})$. We expect the constant φ_2 to be parameter independent in order for the behavior of $v_1(\alpha,u)$ to be parameter independent as $|\mathrm{Im}\,\alpha| \to \infty$. It is indeed the case; enforcing vanishing residues leads to

$$\varphi_1 = -\left(\frac{1}{4} - \frac{\theta}{2\pi}\right)\cos\theta, \qquad \varphi_2 = \frac{1}{8}, \tag{7.67}$$

so that

$$f(\alpha,\theta) = -\left(\frac{1}{4} - \frac{\theta}{2\pi}\right)\cos\theta + \frac{1}{8}\sin 2\alpha \tag{7.68}$$

Bearing (7.68) in mind, a comparison of the sum of $v_0(\alpha,u)$ (7.64) and $v_1(\alpha,u)$ (7.66) with the integral form of $\Psi_7(\alpha,\theta)$ in (7.58) shows that

$$v_0(\alpha,u) + v_1(\alpha,u) = \frac{1}{2}\left(1 + \frac{k(\alpha)}{u(\alpha)}\right)\Psi_7(\alpha,\theta_1) + \frac{1}{2}\left(1 - \frac{k(\alpha)}{u(\alpha)}\right)\Psi_7(\alpha,\theta_2) \tag{7.69}$$

so that, as desired,

$$v_0(\alpha,u) + v_1(\alpha,u) \xrightarrow{\beta\to\pi/2} \begin{cases} \Psi_7(\alpha,\theta_1) & u = +u(\alpha) \\ \Psi_7(\alpha,\theta_2) & u = -u(\alpha). \end{cases} \tag{7.70}$$

Consistent with (7.57), $v_1(\alpha,u)$ is non-vanishing as $|\mathrm{Im}\,\alpha| \to \infty$. More precisely, since

$$\frac{\sin 2\alpha}{\cos^2\alpha} \xrightarrow{\mathrm{Im}\,\alpha\to\pm\infty} \pm \mathrm{j}2, \tag{7.71}$$

then $v_1(\alpha, u) \sim \pm j/2$ as $\text{Im}\,\alpha \to \pm\infty$. This in turn implies, letting $\text{Im}\,\alpha \to \infty$ and assuming the cyclic periods have been eliminated with vanishing 2π periodic functions, that

$$\begin{aligned}\frac{w_r(\alpha+\pi,u)}{w_r(\alpha-\pi,u)} &= \exp\int_{\alpha-\pi+j\infty}^{\alpha+\pi+j\infty}\left\{v_0(\xi,u)+v_1(\xi,u)+v_{2\pi}(\xi,u)\right\}d\xi \\ &= \exp\int_{\alpha-\pi}^{\alpha+\pi}\pm\frac{j}{2}d\xi \\ &= -1.\end{aligned} \tag{7.72}$$

The left-hand side of (7.55) in the same limit produces

$$g_r(\alpha,\pm u) = \frac{(\sin\alpha+\sin\theta_1)(\sin\alpha+\sin\theta_2)}{\pm u - \gamma\sin\alpha} \xrightarrow{\text{Im}\,\alpha\to\infty} \mp 1. \tag{7.73}$$

The first order equation (7.55) is thus satisfied only on the "positive" branch $+u(\alpha)$ and fails to hold on the "negative" branch $-u(\alpha)$ due to a discrepancy in sign. This situation was previously encountered back in Section 6.2.1 in our initial discussion of the method of period reduction. As noted then, its rectification requires the introduction of a higher order term.

7.4.4 The higher order term

As done in Section 6.2.2, we seek to introduce a non-vanishing term as $|\text{Im}\,\alpha| \to \infty$ so that (7.52) holds for both branches of $u(\alpha)$. We are led to contemplate the higher order *even* expression

$$v_\infty(\alpha,u) = \frac{j}{4}\left(1 - \frac{\sin\delta_1\sin\delta_2 - \sin^2\alpha}{u(\alpha)}\right) \tag{7.74}$$

and since

$$v_\infty(\alpha,u) \xrightarrow{|\text{Im}\,\alpha|\to\infty} \begin{cases} 0 & u = +u(\alpha) \\ \frac{j}{2} & u = -u(\alpha), \end{cases} \tag{7.75}$$

it is seen that the correct sign correction factor is introduced. To cast a bit more light on this issue, define

$$w_\infty(\alpha,u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_\infty(\xi,u)+v_{2\pi}(\xi,u)\right\}d\xi \tag{7.76}$$

where $v_{2\pi}(\alpha, u)$ is assumed to eliminate the cyclic periods of $v_\infty(\alpha, u)$ and vanishes as $|\mathrm{Im}\,\alpha| \to \infty$. This effectively reproduces the behavior of $v_\infty(\alpha, u)$ within the framework of the solution without the trappings of $v_{0,1}(\alpha, u)$. Indeed, closing the integration path at, say, j∞ (see Figure 3.1),

$$\frac{w_\infty(\alpha+\pi, u)}{w_\infty(\alpha-\pi, u)} = \exp\int_{\alpha-\pi+\mathrm{j}\infty}^{\alpha+\pi+\mathrm{j}\infty}\left\{v_\infty(\xi, u) + v_{2\pi}(\xi, u)\right\}\mathrm{d}\xi$$

$$= \begin{cases} 1 & u = +u(\alpha) \\ \exp\displaystyle\int_{\alpha-\pi}^{\alpha+\pi} \pm\frac{\mathrm{j}}{2}\mathrm{d}\xi = -1 & u = -u(\alpha), \end{cases} \tag{7.77}$$

thereby producing the appropriate correction in sign and recovering the same behavior as its counterpart (6.26). The similarities do not end there. As the branch points vanish,

$$v_\infty(\alpha, \pm u) \xrightarrow{\beta\to\pi/2} \begin{cases} 0 & u(\alpha) = +u(\alpha) \\ \dfrac{\mathrm{j}}{2} & u(\alpha) = -u(\alpha), \end{cases} \tag{7.78}$$

so that the exponential of the anti-derivative in that limit is

$$\exp\int^{\alpha} v_\infty(\xi, u)\mathrm{d}\xi \xrightarrow{\beta\to\pi/2} \begin{cases} 1 & u(\alpha) = +u(\alpha) \\ \mathrm{e}^{\mathrm{j}\alpha/2} & u(\alpha) = -u(\alpha), \end{cases} \tag{7.79}$$

reproducing (6.29). We now seek a solution to (7.52) of form

$$w_r(\alpha, u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi, u) + v_1(\xi, u) + v_\infty(\xi, u) + v_{2\pi}(\xi, u)\right\}d\alpha \tag{7.80}$$

where the as of yet unspecified 2π periodic function $v_{2\pi}(\alpha, u)$ are chosen to eliminate the cyclic periods. There is a lack of symmetry when $u(\alpha) \to -u(\alpha)$ and

$$w_r(\alpha, -u) = \frac{\mathrm{e}^{\mathrm{j}\alpha/2}\Psi_7(\alpha, \theta_1)\Psi_7(\alpha, \theta_2)}{w_r(\alpha, u)}, \tag{7.81}$$

where $\Psi_7(\alpha, \theta)$ is defined in (7.57). The undesired contribution to the behavior of $w_r(\alpha, u)$ due to $v_\infty(\alpha, u)$ as $|\mathrm{Im}\,\alpha| \to \infty$ can be mitigated by using a unit periodic such as (6.36). Note also that $-v_\infty(\alpha, u)$ can be used throughout without compromising the aforementioned properties.

7.4.5 Elimination of cyclic periods

It was seen in Chapter 6 that we could take (at least) two different approaches to the problem of cyclic period elimination. These two options were referred as the method of global elimination and the method of split elimination. It was also hinted on page 161 that the method of split elimination was the only one that applied to the reduced equation for the problem of the anisotropic impedance half-plane. Indeed, observe that in

$$w_r(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + v_\infty(\xi, u) + v_{2\pi}(\xi, u) \Big\} d\xi \tag{7.82}$$

where v_0, v_1 and v_∞ are respectively given in (7.64), (7.66) and (7.74), expressions v_0 and v_1 have an odd parity whereas v_∞ has an even parity. One of the keystones in our ability to eliminate cyclic periods in all of the cases examined up to now is the ability to use unit periodics having the same parity as $v_0(\alpha, u) + v_1(\alpha, u)$, thereby literally halving the number of degrees of freedom required. In this instance, attempting to use unit periodics of either even or odd parity in order to complete the global elimination of the cyclic periods would require seven degrees of freedom. This is the same as the amount required in the direct approach and the extra complication of the reduction then buys nothing. A much better way to proceed is to consider instead the method of split elimination which would allow for the elimination of the cyclic periods of $v_0 + v_1$ using odd unit periodics and those of v_∞ using even unit periodics. The solution is then the product of one function with odd integrands and one with even integrands,

$$w_r(\alpha, u) = w_o(\alpha, u) w_e(\alpha, u), \tag{7.83}$$

where

$$w_o(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + v_{o,2\pi}(\xi, u) \Big\} d\xi, \tag{7.84}$$

$$w_e(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_\infty(\xi, u) + v_{e,2\pi}(\xi, u) \Big\} d\xi. \tag{7.85}$$

$$\tag{7.86}$$

The terms $v_{o,2\pi}(\alpha, u)$ and $v_{e,2\pi}(\alpha, u)$ respectively denote odd and even 2π periodics used to annul the cyclic periods. The strip of analyticity and the required cycles are illustrated in Figure 7.6.

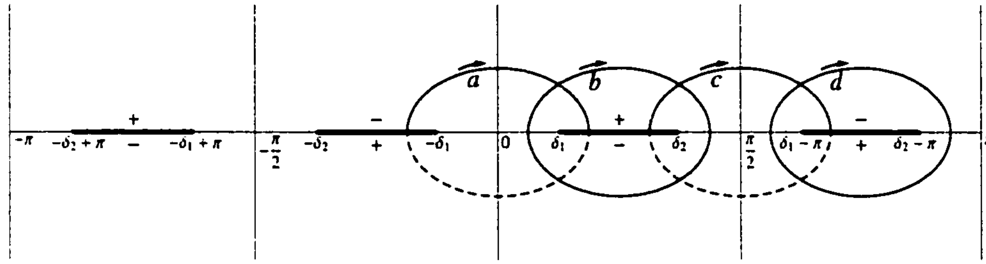


Figure 7.6: The four cycles of interest in the 2π wide strip of analyticity associated with $w_r(\alpha, u)$.

7.4.5.1 Analysis for odd parity

The elimination of the cyclic periods applies only to the branched portions of $v_0(\alpha, u)$ and $v_1(\alpha, u)$. Extending the notation introduced in Sections 6.3 and 6.4, we identify the branched portion of v_0 and v_1 as

$$v_0'(\alpha, u) = -\frac{\alpha}{2\pi}\frac{k(\alpha)}{u(\alpha)}\left(\frac{\sin\theta_1\cos\alpha}{\cos^2\alpha - cos^2\theta_1} - \frac{\sin\theta_2\cos\alpha}{\cos^2\alpha - cos^2\theta_2}\right) \tag{7.87}$$

$$v_1'(\alpha, u) = \frac{k(\alpha)}{u(\alpha)}\left(\frac{f(\alpha,\theta_1)}{\cos^2\alpha - cos^2\theta_1} - \frac{f(\alpha,\theta_2)}{\cos^2\alpha - cos^2\theta_2}\right) \tag{7.88}$$

with $f(\alpha, \theta)$ given in (7.68).

Despite appearances, only the periods associated with cycles b and c must be eliminated, a rather surprising assertion given the somewhat complex strip of analyticity. A comparison of the above equations with $v_0(\alpha, u)$ and $v_1(\alpha, u)$ in Section 4.4.1 shows that they are the same apart from a factor of 1/2. The conclusions reached therein therefore apply here. As discussed in Section 4.4.1, this simplification is a consequence of the beneficial symmetries in the expressions involved. The elimination can thus be carried out using the 2π periodics (4.128), which, suitably modified, are

$$v^1_{o,2\pi}(\alpha, u) = \frac{\sin\alpha}{u(\alpha)}, \quad v^3_{o,2\pi}(\alpha, u) = \frac{u(\zeta_o)}{u(\alpha)}\frac{k(\alpha)}{k(\zeta_o)}\frac{2\cos\zeta_o\sin\alpha}{\sin^2\alpha - \sin^2\zeta_o}. \tag{7.89}$$

The part of the solution with the odd integrand, $w_o(\alpha, u)$, will be

$$w_o(\alpha, u) = \exp\int_{\alpha_0}^{\alpha}\left\{v_0(\xi, u) + v_1(\xi, u) + \kappa_o v^1_{o,2\pi}(\xi, u) + \sigma_o v^3_{o,2\pi}(\xi, u)\right\}\mathrm{d}\xi \tag{7.90}$$

and the enforcement of vanishing cyclic periods leads to the equation system

$$A'_{0+1} + \kappa_o A^1_{o,2\pi} + \sigma_o A^3_{o,2\pi} = 0, \tag{7.91a}$$

$$B'_{0+1} + \kappa_o B^1_{o,2\pi} + \sigma_o B^3_{o,2\pi} = 0. \tag{7.91b}$$

Its solution requires an approach along the same lines as the one provided in Chapter 4. The inversion for ζ_o is also straightforward since $v_{o,2\pi}(\alpha, u)$ is easily mapped to an elliptic integral in standard form using the transformation

$$\tau = \frac{\cos\alpha}{\cos\delta} \tag{7.92}$$

where δ is either δ_1 or δ_2. The biggest problem with this approach is apparently the large number of ζ_o symmetric poles introduced by $v^3_{o,2\pi}(\alpha, u)$. This is the price paid in order for $v^3_{o,2\pi}(\alpha, u)$ to have the proper symmetry which requires $\cos^2\alpha - \cos^2\zeta_o$ in the denominator.

7.4.5.2 Analysis for even parity

Again, we confine our attention to the branched portion of $v_\infty(\alpha, u)$,

$$v'_\infty(\alpha, u) = -\frac{j}{4}\frac{\sin\delta_1\sin\delta_2 - \sin^2\alpha}{u(\alpha)}, \tag{7.93}$$

and unlike its counterpart in Section 6.3.1, it is not related to a circular function and does not have a simple anti-derivative. As opposed to the odd branched portion of the integrand, the even part requires the elimination of the cyclic periods associated with the four cycles illustrated in Figure 7.6. Pairs of even 2π periodic expressions leading to elliptic integrals of the first and third kind are sought. The simplest ones that apply are

$$v^1_{e,\pi}(\alpha, u) = \frac{1}{u(\alpha)}, \qquad v^3_{e,\pi}(\alpha, u) = \frac{1}{2}\frac{u(\zeta_\pi)}{u(\alpha)}\frac{k(\alpha)}{k(\zeta_\pi)}\frac{\sin 2\zeta_\pi}{\cos^2\alpha - \cos^2\zeta_\pi}, \tag{7.94}$$

and

$$v^1_{e,2\pi}(\alpha, u) = \frac{\cos\alpha}{u(\alpha)}, \qquad v^3_{e,2\pi}(\alpha, u) = \frac{1}{2}\frac{u(\zeta_{2\pi})}{u(\alpha)}\frac{k(\alpha)}{k(\zeta_{2\pi})}\frac{\sin\zeta_{2\pi}\cos\alpha}{\cos^2\alpha - \cos^2\zeta_{2\pi}}. \tag{7.95}$$

The first pair is π periodic and the second 2π periodic. Adhering to our convention, the subscript identifies the periodicity (the added e, indicating they are even, distinguishes them from the odd expressions above) while the superscript identifies the type of elliptic integral to which it gives rise. The poles of both $v^3_{e,\pi}(\alpha, u)$ and $v^3_{e,2\pi}(\alpha, u)$ have residues ± 1

and their polar periods therefore do not disrupt the single-valuedness of the path integral. The squares in the denominators are required to achieve the correct symmetry. The specific form of $w_e(\alpha, u)$ is then

$$w_e(\alpha, u) = \exp \int_{\alpha_0}^{\alpha} \Big\{ v_0(\xi, u) + v_1(\xi, u) + \kappa_\pi v^1_{e,\pi}(\xi, u) + \sigma_\pi v^3_{o,\pi}(\xi, u) \\ + \kappa_{2\pi} v^1_{e,2\pi}(\xi, u) + \sigma_{2\pi} v^3_{o,2\pi}(\xi, u) \Big\} \mathrm{d}\xi. \quad (7.96)$$

Enforcing vanishing cyclic periods results in, if we make use of the various symmetry properties,

$$A'_\infty + \kappa_\pi A^1_{e,\pi} + A^3_{e,\pi} + \kappa_{2\pi} A^1_{e,2\pi} + A^3_{e,2\pi} = 0, \quad (7.97a)$$

$$B'_\infty + \kappa_\pi B^1_{e,\pi} + B^3_{e,\pi} + \kappa_{2\pi} B^1_{e,2\pi} + B^3_{e,2\pi} = 0, \quad (7.97b)$$

$$C'_\infty + \kappa_\pi C^1_{e,\pi} + C^3_{e,\pi} = 0, \quad (7.97c)$$

$$-B'_\infty - \kappa_\pi B^1_{e,\pi} - B^3_{e,\pi} + \kappa_{2\pi} B^1_{e,2\pi} + B^3_{e,2\pi} = 0. \quad (7.97d)$$

This can be decoupled by inspection into

$$B'_\infty + \kappa_\pi B^1_{e,\pi} + B^3_{e,\pi} = 0, \quad (7.98a)$$

$$C'_\infty + \kappa_\pi C^1_{e,\pi} + C^3_{e,\pi} = 0, \quad (7.98b)$$

and

$$A'_\infty + \kappa_\pi A^1_{e,\pi} + A^3_{e,\pi} + \kappa_{2\pi} A^1_{e,2\pi} + A^3_{e,2\pi} = 0, \quad (7.99a)$$

$$\kappa_2 \pi B^1_{e,2\pi} + B^3_{e,2\pi} = 0. \quad (7.99b)$$

Equation system (7.98) provides a full decoupling of the periods of the π periodic quantities. Since the period is π, the analysis to derive the bilinear relation of Riemann is confined to a strip of width π. Judging from Figure 7.6, it should be similar to the one carried out in Chapter 4 since in both cases two cuts are captured. The biggest challenge lies with the inversion of $V^1_{e,\pi}(\alpha, u)$, the integral of $v^1_{e,\pi}(\alpha, u)$, since the expression involved is hyperelliptic. This is probably possible using expressions similar to the current ones but with the extra complication of a square root appearing in the parameter k. Once κ_π and ζ_π are known, we can turn our attention to (7.99). In this instance the analysis is similar to the one carried out for the odd integrands above and the inversion is made possible by using the transformation

$$\tau = \frac{\sin\alpha}{\sin\delta} \tag{7.100}$$

with δ set to either δ_1 or δ_2.

7.4.6 Branch-free solutions

The construction of branch-free solutions using $w_r(\alpha, u) = w_e(\alpha, u)w_o(\alpha, u)$ is a daunting task. The need for symmetry above has led to the introduction of no less than 16 poles, 8 per Riemann sheet, in the strip of width 2π alone. If we consider the original strip of width 4π, it is populated by a total of 32 ζ poles per sheet to which are added another 8 θ poles from $v_0(\alpha, u)$ and $v_1(\alpha, u)$. Experience suggests that the best way to proceed is probably to use linear combinations of the branch-free representations (3.3) such as

$$t(\alpha) = Q(\alpha)\left\{\left(1 + \frac{r(\alpha)}{u(\alpha)}\right)w(\alpha, u) + \left(1 - \frac{r(\alpha)}{u(\alpha)}\right)w(\alpha, -u)\right\} \tag{7.101}$$

where, if we proceed analytically, the only true flexibility is due to the trigonometric polynomial $r(\alpha)$ which has a total of five degrees of freedom. This shortfall in degrees of freedom prevents an analytical solution and it seems that the only way to proceed would be numerically. However, the sheer number of poles (and zeros) to eliminate now make this an impractical option. As with the direct approach, the analysis has once again reached an impasse.

7.5 Summary

The problem of the anisotropic impedance half-plane under oblique plane wave illumination was examined within the framework of the Sommerfeld-Maliuzhinets method. From the second order difference equation, a pair of first order difference equations were obtained whose solutions must be constructed in a strip of width 4π. Unfortunately, this is populated by no less than sixteen branch points as well as, initially, sixteen poles on each Riemann sheet. While the elimination of the poles is relatively straightforward, the large number of branch points engenders a large number of cyclic periods which significantly complicates their elimination. A way to avoid these complications is to approximate the square root; this was carried out and the resulting approximate solution recovers known solutions in the appropriate limits and also indicates how to complete the analysis once an exact solution is available. Two different methods by which exact solutions can be obtained were studied. The first is an attempt to carry out the analysis in a direct fashion as done in Chapters 4 and 5. The analysis remains manageable until we come to the elimination of the cyclic periods where a shortfall in degrees of freedom preempts any further progress. However, indications strongly suggest that some beneficial symmetry is being overlooked. The second attempt at obtaining an exact solution relies on the method of reduction of the period with split elimination discussed in Chapter 6. Though not explicitly provided, the required analysis to eliminate the cyclic periods is now significantly simpler and can actually be carried out. However, yet again, an exact analytical solution appears to be out of reach: the elimination of the cyclic periods leads to the introduction of a large number ζ poles in the 4π wide strip alone. Accounting for those as well as the remaining θ poles, the construction process of the branch-free solutions in the 4π wide strip using analytical means becomes infeasible given the number of degrees of freedom available. The number of poles can probably be reduced if one proceeds as suggested in Section 6.5 but the best hope may lie with unlocking a symmetry in the direct approach.

CHAPTER 8

CONCLUSION

A METHOD was presented for solving a class of functional second order difference equations that arise when analyzing scattering from certain impedance and resistive wedges using the Sommerfeld-Maliuzhinets technique. The method, conceived from the ground up, is based on expressing solutions of the second order difference equation as meromorphic linear combinations of branched solutions to subordinate first order difference equations. Standard techniques do not apply to such equations; they are solved by taking their logarithmic derivatives, and well-defined solutions are then obtained by successively eliminating offending singularities. Completion of the solution requires the derivation of specialized versions of the bilinear relations of Riemann by means of complex analysis on simply connected Riemann surfaces.

The procedure was successfully applied to a functional second order difference equation that arises in the study of a class of penetrable right-angled wedges. A pair of solutions that satisfy all prescribed analyticity requirements were neatly obtained and these recover the known limiting solutions. This is, to the best of the author's knowledge, the first time a second order difference equation of this class was solved directly in the context of a diffraction problem. Progressively proceeding towards the formidable problem of the anisotropic half-plane, two mathematical extensions (or generalizations) of the equation for the penetrable wedge were then examined. The first was obtained by fixing the width of the strip of analyticity and doubling the number of singularities; it can be solved relatively easily due to a symmetry that can be exploited. For the second extension, the strip of analyticity of the original equation was doubled. A pair of solutions was obtained in spite of the fact that the elimination of the cyclic periods and the construction of meromorphic solutions were more complicated tasks. In this particular instance, the one flaw the analytical solutions have is that they vanish in a certain limit, and attempts to circumvent this by resorting to the numerical identification of zeros were only partially successful.

Looking ahead to the problem of the anisotropic impedance half-plane, the complications encountered motivated the study of a variation of the procedure where branched first order equations with reduced period are derived. This entails a reduced strip of analyticity and, therefore, a simpler cyclic period elimination due to a reduced number of singularities. The procedure was applied to the previously solved second extension; the elimination of cyclic periods was simplified but at the cost of requiring the elimination of a large number of poles and zeros during construction of the meromorphic solutions. Despite considerable efforts, fully analytical solutions were not obtained and the best that could achieved were solutions requiring the numerical identification of zeros, substantially reducing the elegance of the approach. In essence, the method of period reduction dramatically simplifies the elimination of cyclic periods with the penalty of introducing an unwieldy number of poles in the strip of analyticity, merely relocating a bottleneck from one step of the analysis to another. The best hope to alleviate these problems is the alternative procedure suggested at the end of Chapter 6 which reduces the number of poles introduced during the analysis; this will be the subject of future research.

With this knowledge in hand, the problem of the anisotropic half-plane was examined within the framework of the proposed approach; it remains difficult despite the array of tools developed. An approximate solution where the branch point problem is side-stepped by approximating the square root was first given along with plots of the resulting backscattering echowidth. Two possible approaches to solve the problem exactly were then discussed and in both instances, given past observations, foreseeable complications arose. The first was a direct approach along the lines of the method applied to solve the equation of the penetrable wedge (Chapter 4) and its variant with doubled period (Chapter 5.) The initial steps of the analysis were easily carried out but the number of branch points coupled with an apparent lack of degrees of freedom made the elimination of cyclic periods intractable. The expressions however harbor an abundance of symmetry; it is strongly suspected that a subtle simplifying property has been overlooked. The second approach, based on the method of reduction of the period, also ran into problems of its own. While the cyclic periods could now be eliminated analytically, the construction of meromorphic solutions having the desired analyticity properties was practically impossible due to the prohibitive number of poles introduced in the strip of analyticity during the elimination of singularities. Of the two approaches, the direct one has the most potential and will be revisited with hopes of discerning a subtle simplifying feature.

There is obviously much work to be done in order to fully appreciate the potential of the method propounded. This is particularly true with respect to its generalization to more complex problems where two sources of concern distinguish themselves. The first lies with the elimination of cyclic periods: a larger number of branch points in the strip of analyticity entails a Riemann surface of higher genus when deriving the bilinear relations of Riemann. As witnessed in Chapter 5 — compare with Chapter 4 — this complicates the computation of the sum of residues and the expression of canonical periods in terms of the basic cyclic periods. Unless a simpler approach can be found for carrying out these analyses, the procedure, though it applies in principle, will be prohibitively complicated for large numbers of branch points. The second source of concern is the construction of meromorphic solutions. Unexpectedly, this turned out be as challenging, if not more so, than the elimination of cyclic periods. It is a simple matter to obtain meromorphic solutions of the second order equation but obtaining specific meromorphic solutions free of poles and zeros in the strip gets increasingly difficult as the number of singularities increases. Compare, for example, the neat analysis for the equation of the penetrable wedge solved in Chapter 4 with the corresponding analysis in Chapter 5 where the recovery of the desired limiting behavior was compromised by the necessary use of constants and linear combinations to annul poles and zeros. A more systematic procedure would provide a better appreciation of the possibilities offered by a given number of degrees of freedom and perhaps indicate how to exploit them without jeopardizing the behavior of the solution.

In addition to the above, there are many fundamental notions that need to be addressed. A prime example is the inversion process where the unknown ζ poles are determined in the course of eliminating the cyclic periods. We recall that this requires the evaluation of a Jacobian elliptic sine function with argument Λ_n (here n denotes any of the subscripts used), a problem specific quantity which is a function of the parameters of the second order equation — θ and δ in Chapter 4, for example. In order for the inversion to be possible, it was shown that Λ_n must lie within certain period parallelograms and if this is not so, the inversion process irrevocably fails. This underlines the need to quantify the range of Λ_n in terms of the equation parameters to determine when the technique applies. This is intriguing when considered from the perspective of the existence of solutions to the second order equation: it implies that solutions cannot always be obtained using this procedure. It is unclear at this point whether this corresponds to limitations in the methodology or if it is a fundamental characteristic of the equation. Within a physical context, a failure of the inversion procedure could correspond to non-physical parameter values such as surface impedances exhibiting gain instead of loss. This line of thought also suggests that equations which are not directly obtained from physical problems, such as the generalizations considered, could be more difficult to solve.

The dependence of the branched solutions on the choice of cuts used for the root $u(\alpha)$ must also be examined more closely. While the cuts may be thought of as a mathematical manifestation of the coupling between the originally obtained coupled first order difference equations — the parent equations of the second order equation — their choice should not strongly influence the character of the final meromorphic expressions which satisfy, after all, the meromorphic second order equation. However, their choice does have strong repercussions on the construction of the branched solutions. For instance, the selection of an odd symmetric branch configuration for $u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\delta}$ in Chapter 4 produces cuts that cross at the origin and $\alpha = \pm\pi$ at the edges of the strip of analyticity. The cuts on the edges then do not fully lie within the strip of analyticity. Besides the puzzling implications on the elimination of cyclic periods, the integrand is now odd instead of even and all the expressions used to eliminate the polar and cyclic periods must now change parity. It may be that this will lead to the "complement" of the currently obtained solution where two distinct limits are obtained when $\delta \to 0$ and a common one when $\delta \to \pi/2$. If this is true, we can speculate that combination of solutions based on the different complementary branch cut configuration might provide a pair of solutions that recover distinct solutions in both limits when branch points vanish.

Another area of interest is the selection of the unit periodics for the elimination of periods. Those required for the elimination of polar periods are easily identified and it was shown in the simpler case that, even though they are not unique, the different combinations are equivalent within the scope of the approach. Things are not so clear when it comes to selecting expressions — the elliptic integrands — required to eliminate the cyclic periods. An attempt was made to select the simplest ones but, without a systematic method for selecting them, there is no simple way to ascertain this. In the case of the equation of the penetrable wedge, for example, the analysis could have been carried with an expression leading to the integral of the third kind having double the number of poles, which then makes a fully analytical solution more complicated if at all possible. One cannot help but wonder if for the analysis in Chapter 5 a better set of expressions are available for the elimination of the cyclic periods. Admittedly, this is an obscure proposition but the possibility should be contemplated given the difficulty in constructing meromorphic solutions. Part of the answer might lie in a closer examination of the mappings used to express the integrals of the first kind in terms of Legendre's standard form; they could be similarly applied to the integrals of the third kind to help identify the simplest fundamental forms. It has already been verified that canonical integrals of the third kind can be recovered in this fashion.

While we were concerned mainly with solution which are $O(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$ in this work, the process should also be examined in the context of problem where the branched solutions sought have different orders. This will require the introduction of higher order terms which, predictably, must have constant coefficients (non-parameter dependent) to recover a constant asymptotic behavior as $|\mathrm{Im}\,\alpha| \to \infty$. Apart from the subtleties encountered in the method of period reduction, this was not pursued here.

The most important point to address right now is the applicability of the new technique to cases of higher complexity. While the problem of the penetrable right-angled wedge was neatly solved and now provides a good benchmark, the problem of the anisotropic impedance half-plane, which remains unsolved, may well decide whether or not the method is by its nature restricted to cases where the square root induces simple topologies. Even if it turns out to be so, the method is a valuable addition to the Sommerfeld-Maliuzhinets formalism and represents a first step towards establishing a more generalized solution technique.

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