LOOP EXCITATION OF TRAVELLING WAVES

T. B. A. Senior

Radiation Laboratory
The University of Michigan
Ann Arbor, Michigan

1. Introduction

One aspect of diffraction theory which is of practical importance at the present time is the study of bodies which are characterized by a low back-scattering cross section over a range of incidence angles. Quite frequently the basic shape is a long thin body of revolution, and when this is viewed at or near nose-on, some of the major features of the return can be attributed to the travelling waves which are excited on the surface (Peters, 1958; Goodrich and Kazarinoff, 1962). In many instances these represent the dominant portion of the current distribution, and the extent to which it is possible to reduce the cross section by, for example, small changes in shape is then determined by the degree to which the travelling waves can be reduced or, hopefully, suppressed entirely. It is therefore desirable to give some attention to the manner in which travelling waves are excited, with particular reference to the influence of any surface 'singularities' such as discontinuities in the slope or derivatives thereof.
As a contribution to this end we consider the launching of travelling waves by the effective field singularity represented by the curve on the body separating the lit region from the shadow. Since the incident field is here moving parallel to the surface, intuitive reasoning suggests that the neighbourhood of the shadow boundary could act as a source of travelling waves, but because of the inherent complication of the problem it is difficult to calculate the power going into the travelling wave as opposed to the power in the radiated field. In order to estimate the efficiency of the shadow boundary as a source of these waves, we shall therefore restrict ourselves to the simpler problem in which the primary source of energy is itself placed on the surface of the body, and is so chosen as to excite only the fundamental travelling wave. The body can then be approximated by an infinite cylinder of circular cross section and large (but finite) conductivity, allowing a direct comparison of the powers via their integral expressions.

2. **Formulation of the Problem**

Consider an infinitely long circular cylinder of radius \( a \) which is excited by a circumferential ring current located at a distance \( r_o \sim a \) from the axis. If the surface impedance of the cylinder is \( \eta \), the boundary conditions on the total field can be written as

\[
E_\theta = -\eta Z H_z
\]

(1)

\[
E_z = \eta Z H_\theta
\]

(2)

at \( r = a \), where \( (r, \theta, z) \) are cylindrical polar coordinates with the \( z \) axis coincident with the axis of the cylinder, and \( Z \) is the intrinsic impedance of free space.
The fundamental travelling wave is the one with lowest attenuation in the direction of propagation, and this is consequently the wave of most interest in practical applications. Since its magnetic vector is entirely transverse with no variation in the \( \theta \) direction it is convenient to choose as the source of excitation a magnetic ring current of constant amplitude and phase. Without loss of generality, the ring can be chosen to lie in the plane \( z = 0 \), and the incident field can then be represented by the single component (electric) Hertz vector

\[
\mathbf{\pi} = (0, 0, U^i)
\]

where

\[
U^i = \frac{2\pi i u}{2} \frac{e}{u} d\theta_o, \quad 0
\]

with

\[
u = k \left( r^2 + r_o^2 - 2 r r_o \cos(\theta - \theta_o) + z^2 \right).
\]

The time convention is here \( e^{-i\omega t} \) and the coordinates of a variable point on the ring source are denoted by the suffix 'o'.

In terms of the Hertz vector \( \mathbf{\pi} \) the components of the incident field are

\[
E^i = \frac{\partial^2 U^i}{\partial \theta \partial z}, \quad 0, \quad k^2 U^i + \frac{\partial^2 U^i}{\partial z^2}, \quad (5)
\]

\[
H^i = ik Y(0, \frac{\partial U^i}{\partial r}, 0), \quad (6)
\]

and this is a transverse magnetic field as required. From the boundary conditions it now follows that the scattered field must also be of similar type, so that only the second of the two conditions is relevant, and by introducing a single component
Hertz vector to represent the scattered field, the condition (2) becomes

\[ k^2 + \frac{\partial^2}{\partial z^2} - ik\eta \frac{\partial}{\partial r} (U^i + U^S) = 0, \quad (7) \]

which can be written alternatively as

\[ \frac{\partial}{\partial r} + ik\eta r \frac{\partial}{\partial r} (U^i + U^S) = 0 \quad (8) \]

at \( r = a \). This can be satisfied by inserting a suitable expression for \( U^S \), but in order to decide what is the appropriate form it is necessary to examine in more detail the structure of the incident field.

3. **The Incident Field**

Let us consider first the power radiated by the source. If \( R \) is the (spherical) radial variable defined as \( R = \sqrt{r^2 + z^2} \), then at large distances from the current loop

\[ u \sim kR \left( 1 + \frac{r^2 - 2r_0 r \cos(\theta - \theta_0)}{2R^2} \right), \]

giving

\[ U^i \sim \frac{1}{kR} e^{ik(R + \frac{r^2}{2R})} \int_0^{2\pi} \exp \left( -i \frac{krr_0}{R} \cos(\theta - \theta_0) \right) d\theta_0 \]

i.e.

\[ U^i \sim 2\pi J_0 \left( krr_0 / R \right) \cdot \frac{1}{kR} e^{ik(R + \frac{r_0^2}{2R})} \quad (9) \]

where \( J_0(x) \) is the Bessel function of zero order. For \( R \gg r_0 \) the incident field is
therefore

\[ E^i = - \frac{k^2 r}{R} \frac{z}{R}, 0, -\frac{r}{R} u^i \]

\[ H^i = - \frac{k^2 r}{R} (0, 1, 0) \cdot u^i \]

and the resulting Poynting vector is

\[ \frac{1}{2} E^i \cdot H^i = \frac{r}{R}, 0, \frac{z}{R} \cdot 2Y \left( \frac{\pi kr}{R^2} J_0 \left( kr_0 / R \right) \right)^2. \]

From this it follows immediately that the radial flow of power is

\[ 2Y \cdot \frac{\pi k}{R} \sin \phi J_0 \left( kr_0 \sin \phi \right)^2, \]

where \( \phi \) is defined by the relations

\[ r = R \sin \phi, \quad z = R \cos \phi, \]

and consequently the total power radiated by the current loop is

\[ P^i = 8Y \pi \frac{k^2}{3} \int_0^{\pi/2} \sin^3 \phi J_0 \left( kr_0 \sin \phi \right)^2 d\phi. \quad (10) \]

When \( kr_0 = 0 \) the integral in (10) is clearly 2/3, but for more general values of \( kr_0 \) no precise analytical evaluation is possible. Nevertheless, numerical results can be found for the smaller values of \( kr_0 \) by introducing the series expansion of \( J_0(x) \), and for larger values by numerical integration, and the data obtained in this
way is plotted in Figure 1. It will be observed that as $kr_o$ increases the total power radiated decreases in an oscillatory manner, with the first minimum occurring for $kr_o$ approximately $^+$ 2.6.

For sufficiently large $kr_o$ an alternative approach to the integral is to apply the method of steepest descents. Since the dominant contribution comes from values of $\phi$ in the neighbourhood of the upper limit, the Bessel function can be replaced by its asymptotic formula for large arguments to give

$$P^i \sim 8 \pi \pi^3 k^2 \frac{1}{2\pi kr_o} \int_0^\pi \sin^2 \phi \cdot \cos \left(2kr_o \sin \phi\right) d\phi.$$ 

Hence

$$P^i \sim 8 \pi \pi^3 k^2 \frac{1}{4kr_o} \left(1 - \frac{2}{\pi} \cos \left(2kr_o + \frac{\pi}{4}\right)\right)$$

(11)

showing a decreasing amplitude of oscillation about the mean value $\frac{1}{4kr_o}$. This is in excellent agreement with the computed points (see Figure 1.)

Having calculated the power radiated by the loop, we now turn to the question of the incident field structure. For this purpose the original expression for $U^i$ is not convenient, and it is necessary to seek an alternative form which will bring out the dependence on the coordinate $z$.

$^+$ The positions of at least the first few minima are similar to the zeros of $J_0 (kr_o)$.
To begin with we observe that

\[
\frac{e^{iu}}{u} = \frac{1}{2k} \int_C e^{iz} H_0 \left( (k^2 - z^2)^{1/2} \left( r^2 + r_o^2 - 2rr_o \cos(\theta - \theta_o) \right) \right)^{1/2} \, dz \tag{12}
\]

(see, for example, Campbell and Foster, 1948) where \( H_0(x) \) is the Hankel function of the first kind of order zero; the path \( C \) extends from \(-\infty\) to \( \infty \) passing above the branch point at \( z = -k \) and below the branch point at \( z = k \), and the chosen branch of \( (k^2 - z^2)^{1/2} \) is that which reduces to \( k \) at \( z = 0 \). But

\[
H_0 \left( (k^2 - z^2)^{1/2} \left( r^2 + r_o^2 - 2rr_o \cos(\theta - \theta_o) \right) \right)^{1/2} =
\]

\[
= \int_{-\infty}^{\infty} H_n \left( r \sqrt{k^2 - z^2} \right) J_n \left( r_o \sqrt{k^2 - z^2} \right) e^{i(\theta - \theta_o)} \quad \text{for } r > r_o \tag{13}
\]

\[
= \int_{-\infty}^{\infty} J_n \left( r \sqrt{k^2 - z^2} \right) H_n \left( r_o \sqrt{k^2 - z^2} \right) e^{i(\theta - \theta_o)} \quad \text{for } r < r_o \tag{14}
\]

moreover,

\[
\int_0^{2\pi} e^{i(\theta - \theta_o)} \, d\theta_o = 2\pi \delta(n)
\]
where \( \delta \) is the standard delta function, and using this in conjunction with equations (3), (12), (13) and (14) the formula for \( U^1 \) becomes

\[
U^1 = \frac{i\pi}{k} \int_C e^{iz} J_0(r \sqrt{k^2 - \tau^2}) H_0(r_o \sqrt{k^2 - \tau^2}) \, d\tau, \quad r = r_o \tag{15}
\]

\[
= \frac{i\pi}{k} \int_C e^{iz} H_0(r \sqrt{k^2 - \tau^2}) J_0(r_o \sqrt{k^2 - \tau^2}) \, d\tau, \quad r \geq r_o \tag{16}
\]

The dependence on the variable \( z \) is here made explicit.

4. The Scattered Field

Of the above expressions only the first is required to satisfy the boundary condition at \( r = a \), and its form suggests that for the scattered field we take

\[
U^S = \frac{i\pi}{k} \int_C e^{iz} H_0(r \sqrt{k^2 - \tau^2}) H_0(r_o \sqrt{k^2 - \tau^2}) f(\tau) \, d\tau \tag{17}
\]

where \( f(\tau) \) is to be determined. Since the radiation condition is now satisfied automatically by virtue of the Hankel function dependence on \( r \), it only remains to calculate \( f(\tau) \) using the boundary condition, and if (15) and (17) are substituted into (7) we have

\[
f(\tau) = -\frac{ik^2 \tau^2 J_0(a \sqrt{k^2 - \tau^2}) - i\kappa \tau J_0'(a \sqrt{k^2 - \tau^2})}{ik^2 \tau^2 H_0(a \sqrt{k^2 - \tau^2}) - i\kappa \tau H_0'(a \sqrt{k^2 - \tau^2})} \tag{18}
\]
in which the prime denotes differentiation with respect to the whole argument.

The Hertz vector for the scattered field is therefore

\[ \mathbf{\Pi} = (0, 0, U^S), \]

\[ U^S = -i \frac{\pi}{k} \int_C e^{iz} \frac{k^2 - z^2 \mathbf{J}_o(a \ k^2 - z^2) - i \eta \mathbf{J}'_o(a \ k^2 - z^2)}{k^2 - z^2 \mathbf{H}_o(a \ k^2 - z^2) - i \eta \mathbf{H}'_o(a \ k^2 - z^2)} H_o(r \ k^2 - z^2) H_o(r \ k^2 - z^2) \, dz, \]

(19)

which represents the formal solution of the problem, and from this the field components can be obtained by carrying out the differentiations indicated in equations (5) and (6).

For practical purposes the above solution is of little value as it stands and the main characteristics of the scattered field are by no means evident from (19).

Some simplification is therefore necessary and in the course of this one of our primary objectives is to bring out the contribution of the travelling wave; however, the separation into travelling wave and radiated field appears as a natural consequence if the path of integration is deformed into one of steepest descents, and to this end we introduce the new variable \( \alpha \), where

\[ z = k \cos \alpha. \]

The equation for \( U^S \) then becomes

\[ U^S = -i \pi \int_{S(\pi/2)} e^{ikz \cos \alpha} \frac{\sin \alpha \mathbf{J}_o(ka \sin \alpha) - i \eta \mathbf{J}'_o(ka \sin \alpha)}{\sin \alpha \mathbf{H}_o(ka \sin \alpha) - i \eta \mathbf{H}'_o(ka \sin \alpha)} H_o(kr \sin \alpha) \sin \alpha \, d\alpha \]

(20)
with $S(\pi/2)$ as a steepest descents contour passing through the real angle $\pi/2$.

Due to the logarithmic singularity of the Hankel function at the zero of
its argument, branch points exist at $\alpha = 0$ and $\pi$, and for convenience the branch
cuts are taken as shown in Figure 2. In addition the integrand has poles arising
from the zeros of

$$H_0^1(ka \sin \alpha) - \frac{i\gamma}{\sin \alpha} H_0^1(ka \sin \alpha)$$

as a function of $\alpha$, and these are in fact the sources of the travelling waves. A
detailed discussion is given later, and for the moment it suffices to say that if
$|k \cdot \eta| < 0.1$, say, the relevant zero has $ka \sin \alpha = 0(10^{-1})$ or less, with arg
$\sin \alpha \sim 110^0$. In terms of $\alpha$ we now have the two zeros $\alpha_o$ and $\pi - \alpha_o$ outside but
adjacent to the strip $0 \leq \text{Re } \alpha \leq \pi$, and these are indicated in Figure 2.

To evaluate the integral in (20) the obvious approach is to apply a steepest
descents analysis. If $ka = 0(10)$ or greater (as is true in most cases of practical
interest), it follows that $kr \sin \alpha > 1$ providing only that the dominant contribution
to the integral does not come from values of $\alpha$ in the neighbourhood of zero or $\pi$.

Proceeding on the assumption that this requirement is fulfilled, the function
$H_0^1(kr \sin \alpha)$ can be replaced by the leading term of its asymptotic expansion for large
arguments to give

$$U^s \sim \frac{2\pi}{kR \sin \phi} e^{i\pi/4} \int_{S(\pi/2)} e^{ikR \cos(\alpha-\phi)} f(k \cos \alpha) H_0(kr \sin \alpha)(\sin \alpha)^{1/2} d\alpha$$
where $R$ and $\phi$ are as previously defined, and inasmuch as the saddle point is now $\alpha = \phi$, the substitution of the asymptotic formula for $H_0(kr \sin \alpha)$ is justified if $kr^2 \gg R$. It will be observed that this condition is independent of the surface impedance $\eta$ and consequently it is not entirely a statement of the minimum distance from the cylinder at which the influence of the travelling wave can be ignored. It is therefore feasible that in a displacement of the path of integration so as to pass through the saddle point a pole of the travelling wave could be included even though the above condition is still fulfilled. Practically, however, this is unlikely, and in the cases under investigation here the magnitude of $\alpha_o$ is such that the pole can be included only by violating the condition.

If $kr^2 \gg R$ a simple displacement of the path of integration in (20) gives

$$U^s \sim \frac{2\pi}{kR \sin \phi} e^{i\pi/4} \int_{S(\phi)} e^{i kR \cos(\alpha-\phi)} f(k \cos \alpha) H_0(kr_0 \sin \alpha)(\sin \alpha)^{1/2} d\alpha$$

and since the non-exponential portion of the integrand is slowly varying in the neighbourhood of the saddle point, we have immediately that

$$U^s \sim 2\pi f(k \cos \phi) H_0(kr_0 \sin \phi) \frac{e^{i kR}}{kR}$$

which is entirely a radiating field. The polar diagram is 'spikey' with a continuous succession of peaks and near-zeros, but the average level shows little variation with $\phi$. 
This is clearly seen if \( k\alpha \) is large enough to allow \( J_0(k\alpha \sin \phi) \) and \( H_0(k\alpha \sin \phi) \) to be replaced by their asymptotic forms, in which event

\[
 f(k \cos \phi) = -\frac{1}{2} - i \left( -\sin \phi + \frac{\sin \phi}{\sin \phi} \right) e^{-i k \alpha \sin \phi} \\
 \approx e^{-i (k \alpha \sin \phi - \pi \frac{\alpha}{4}) \cos (k \alpha \sin \phi - \pi \frac{\alpha}{4})},
\]

and since \( r_o \gg a \) it follows that

\[
 U^s \sim 2 \sqrt{\frac{2\pi}{kr_o \sin \phi}} \frac{ik(r_o - a)\sin \phi}{\cos (k \alpha \sin \phi - \pi \frac{\alpha}{4})} e^{ikR \frac{R}{k}}. \tag{23}
\]

This is independent of \( \gamma \) implying that away from the cylinder the radiated field is unaffected by the surface impedance and is the same as if the cylinder had been perfectly conducting. Even if \( r_o \gg a \) the polar diagram is equivalent to that of a source on the surface apart from a phase factor determined by the projected distance between the real and image currents.

5. The Travelling Wave

When \( kR \gg 1 \) but \( (kr)^2 \) not much greater than \( kR \) the above analysis fails and a detailed evaluation of the integral in (20) is no longer possible unless \( kr \sin \alpha \) is small in the neighbourhood of the saddle point. In this case the Hankel function can be replaced by its logarithmic approximation, viz.

\[
 H_0(kr \sin \alpha) \sim \frac{2i}{\pi} \log \frac{kr \sin \alpha}{2} + \ldots \tag{24}
\]
where \( \gamma \) is Euler's constant \((0.5772157\ldots)\), and since \( r \ll a \)

\[
f(k \cos \alpha) \sim \frac{ka}{2} \left(1 + i \frac{ka}{2}\right) \sin^2 \alpha,
\]

leading to the following expression for the integrand in (20):

\[
i \frac{ka}{\pi \eta} \left(1 + i \frac{ka}{2}\right) e^{ikz \cos \alpha} \left(\log \frac{kr \sin \alpha}{2} + \gamma\right) H_0(kr \sin \alpha) \sin^3 \alpha.
\]

The restrictions on \( kR \) and \( kr \) imply \( k |z| >> 1 \) and consequently we can again think in terms of a steepest descents evaluation with \( kz \) being the large parameter. The saddle point is now \( \alpha = 0 \) for \( \phi \) small or \( \alpha = \pi \) for \( \pi - \phi \) small, and since the integrand vanishes at least as rapidly as \((\log \sin \alpha)^2 \sin^3 \alpha \) in either case, we have for the radiated field

\[
U^S \sim 0
\]

in the immediate vicinity of the cylinder (i.e. for \( \sin \phi \) sufficiently small).

On the other hand, in a displacement of the path of integration to pass through this new saddle point the pole at \( \alpha = \alpha_0 \) or \( \pi - \alpha_0 \) will be included, leading to a residue contribution which is, in fact, the travelling wave. The residues at \( \alpha = \alpha_0 \) and \( \pi - \alpha_0 \) differ only in the sign of \( kz \) and obviously correspond to waves travelling in opposite directions. If \( \alpha_0 \) is the pole adjacent to the saddle point \( \alpha = 0 \) this will be included in a negative sense in any displacement of the path to pass through the origin, whereas the residue at \( \pi - \alpha_0 \) will have a positive sign associated with it, but these things apart the two waves are identical in all respects and it is sufficient to consider
only the wave which travels in the positive z direction. In effect, therefore, we are restricting attention to the case in which \( \phi \) is small.

The pole from which the travelling wave originates is provided by the function \( f(k \cos \alpha) \) and is given by the smallest root of

\[
H_0(ka \sin \alpha) = \frac{i^\gamma}{\sin \alpha} H_1^1(ka \sin \alpha).
\]  
(26)

Unfortunately, a complete analytical solution of this equation is not possible and in order to proceed on a numerical basis it is necessary to set some bounds on the values of \( \gamma \) and \( ka \) to be considered. Inasmuch as our purpose is to investigate travelling waves as they appear in radar scattering problems, \( \gamma \) can be regarded as the surface impedance of a highly conducting metal. A typical value for \( |\gamma| \) is then \( 10^{-4} \), corresponding to the conductivity of copper at a frequency of order 10 KMc, and since the complex refractive index is now dominated by the conduction current term,

\[
\arg \gamma = -\pi/4.
\]

Under these conditions it is a relatively straightforward matter to determine \( \sin \alpha_0 \), and if \( ka |\gamma| \) is not greater than (say) 0.1, the solution of equation (26) can be found by inserting the logarithmic approximation (24) for the Hankel function. The details of the derivation are given in Goubau (1950) and it is there shown that

\[
\sin \alpha_0 = \frac{2i^b}{ka} e^{-\gamma + i(5\pi/8 - \beta/2)}
\]

where the real quantities \( b \) and \( \beta \) are related by the equations
\[ b \log b = -\frac{ka}{2} |\gamma| \cos \beta, \]
\[ \tan \beta = \frac{\pi/4 - \beta}{\log b}. \]

These can be solved numerically, and in Figures 3 and 4 the resulting values of \( b \) and \( \beta \) are plotted as functions of \( ka |\gamma| \) for \( 10^{-6} \leq ka |\gamma| \leq 10^{-1} \). It now only remains to specify \( |\gamma| \) to determine \( \alpha_0 \) for different \( ka \), and taking \( |\gamma| = 10^{-4} \) some values for \( \alpha_0 \) are as follows:

\[
\begin{array}{ccc}
ka & 10^{-1} & 10 & 10^3 \\
\alpha_0 & 6.5 \times 10^{-3} & 7.9 \times 10^{-4} & 1.2 \times 10^{-4} \\
\arg \alpha_0 & 110.9^\circ & 110.0^\circ & 106.1^\circ \\
\end{array}
\]

At the pole \( \alpha = \alpha_0 \) the residue of the integrand in equation (20) is

\[ e^{ikz \cos \alpha_0} \frac{\sin \alpha_0 J_0(ka \sin \alpha_0) - i \gamma J_1'(ka \sin \alpha_0)}{\frac{\partial}{\partial \alpha} \sin \alpha \bar{H}_0(ka \sin \alpha) - i \gamma \bar{H}_0'(ka \sin \alpha)} \]

\[ \cdot \bar{H}_0(k r \sin \alpha_0) \bar{H}_0(k r \sin \alpha_0) \sin \alpha_0. \]

Using equation (26) and the differential equation for the Hankel function the denominator becomes

\[ \frac{\cos \alpha_0}{\gamma} \left\{ 2\gamma + ika (\gamma^2 - \sin^2 \alpha_0) \right\} \bar{H}_0(ka \sin \alpha_0) \]

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Legends for Figures

Fig. 1  Radiated power as function of loop radius
Fig. 2  Steepest descents path in complex $\alpha$ plane
Fig. 3
Fig. 4