ON A CLASS OF INTEGRAL EQUATIONS AND ITS APPLICATIONS
TO THE THEORY OF LINEAR ANTENNAS

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ADDENDUM TO

ON A CLASS OF INTEGRAL EQUATIONS AND
ITS APPLICATIONS TO THE THEORY OF
LINEAR ANTENNAS.
PREFACE

An exact solution of Hallén's integral equation has long been a goal of many researchers in the theory of linear antennas. This thesis presents a method of solution for a class of integral equations which, in particular, yields a closed form solution to Hallén's equation.

The germ of the idea leading to the solution is to be found in P. M. Morse and H. Feshbach's "Methods of Theoretical Physics" in the chapter on integral equations. Professor Feshbach was most kind in communicating his solution of the infinite system of equations, and his assistance is kindly acknowledged.

The contents of Professor E. D. Rainville's course "Special Functions" form most of the foundation of this thesis, and to him I express my most sincere gratitude.

To Professor C. M. Chu, chairman of the doctoral committee, I am especially indebted. His constant encouragement and guidance provided the continuing impetus for the completion of this work.

I also wish to acknowledge the assistance of Professors L. Cesari and N. D. Kazarinoff of the Mathematics Department, and C. B. Sharpe and H. Weil of the Electrical Engineering Department. From them I obtained many valuable suggestions during the preparation of the manuscript.
An error has been brought to my attention by Dr. Olov Einarsson of the Radiation Laboratory, who has pointed out that one cannot hope to solve the system of equations (3.1-7) of Chapter I when a or b is infinite.

If one proceeds as indicated in the thesis it will always be found that

\[ C_m = \int_a^b \exp(-y^2) H_m(y) g(y) dy, \quad (3.1-6) \]

is independent of the limits a and b. However, this equation gives us the m\textsuperscript{th} coefficient in the expansion of the function

\[ g(y) \begin{cases} \ a \leq y \leq b \\ 0 \text{ elsewhere} \end{cases} \]

in series of Hermite polynomials; and

\[ C_m = \int_{-\infty}^{\infty} \exp(-y^2) H_m(y) h(y) dy \quad (3.1-12) \]

is the m\textsuperscript{th} coefficient in the expansion of

\[ h(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-is) \frac{\hat{f}(s)}{K(s)} ds \quad (3.1-11) \]

in series of Hermite polynomials. Since the Hermite polynomials form a complete set it would follow that \( g(y) = h(y) \) almost everywhere, which is certainly incorrect.

This shortcoming of the procedure can, however, be overcome by operating directly on the system of equations (3.1-7) and the corresponding one for the equation with infinite limits without making any attempt to eliminate the \( C \)'s.

That is, one has

\[ f^{(k)}(0) = \sum_{n=0}^{\infty} (-)^n \frac{a_n}{n} C^{n+k} \]
\[
C_0 + \frac{-i\pi}{(b-a)!} \quad C_1 + \frac{-i\pi}{(b-a)^2} \quad C_2 - a''C_3 + \ldots
\]

\[= C_1 + \frac{-i\pi}{(b-a)!} \quad C_1 + \frac{-i\pi}{(b-a)^2} \quad C_2 - a''C_3 + \ldots\]

It is obvious that by continuing this procedure one can again recover Equation (3.1-18)

\[
\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2/4) \exp\left(-\frac{\pi s}{b-a}\right) \frac{\tilde{f}(s)}{K(s)} \, ds
\]

\[
= \int_{a}^{b} \exp(-y^2) g(y) \exp\left(-2n \frac{iy}{b-a} + \frac{n^2 \pi^2}{(b-a)^2}\right) \, dy .
\]

It should be noted that from an infinite set of relations between the C's and the C' 's we have obtained one relation at the best. It would seem a more satisfactory procedure to begin the above scheme with the k'th equation, i.e. multiply the (k+1)th equation by a certain constant and add it to the k'th. Multiply the (k+2)th equation by a certain constant and add it to the k'th, etc. In this manner one would obtain one such relation for each k and hope to determine a g(y) which would satisfy each of them. I must confess that if I proceed in this fashion I find myself unable to solve the resulting set.

It is my belief that because of the form of the solution

\[
g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-is) \frac{\tilde{f}(s)}{K(s)} \theta_3 \left[ (b-a) \left( \frac{s}{2} - iy \right) e^{-(b-a)^2} \right] \, ds
\]

and its limiting behavior as \(b \to \infty, \ a \to -\infty\) giving \(h(y)\), that this \(g(y)\) is at least a part of the complete solution. If an independent proof were to be found, the above expression should appear naturally in the complete solution.
ABSTRACT

In this study a systematic method of solution is presented for a class of Fredholm integral equations. The most significant result is a closed form solution, valid under very weak restrictions, for the equations

$$f(x) = \int_a^b K(x-y) g(y) \, dy, \quad a < x < b$$

and

$$f(x) + g(x) = \int_a^b K(x-y) g(y) \, dy, \quad a < x < b$$

if the interval \((a, b)\) is finite.

If the interval \((a, b)\) is infinite the solution is given in series of orthogonal polynomials with explicit coefficients.

Integral equations of the first kind whose kernels are generating functions for polynomial sets are also treated,

$$f(x) = \int_a^b K(x, y) g(y) \, dy, \quad a < x < b$$

$$K(x, y) = \sum_{n=0}^{\infty} \phi_n(y) x^n$$

where \(\phi_n(y)\) is a polynomial of degree \(n\) in \(y\).

A general solution is also obtained for the equation

$$f(x) = \int_a^b e^{-\lambda xy} g(y) \, dy$$

with \(\lambda\) generally complex. The special choice \(a = 0, b = \infty, \lambda = 1\) leading to Laplace's integral equation is illustrated by two examples.
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In light of the above remarks, the method to be expounded in the sequel possesses a definite advantage. For instance, when the kernel is of the form $K(x-y)$ it is only necessary in order for the method to be applicable that $g(x)$ be integrable and bounded, provided that $f$ and $K$ satisfy suitable conditions. For other types of kernels the conditions on $g(x)$ become more severe, but it is never necessary that the kernel be continuous.

The techniques to be presented were developed in order to solve certain integral equations which are of importance in the study of distribution of currents in linear antennas.

Section 2. Some results from the theory of distributions.

In what follows we shall have occasion to use some results of the theory of distributions. The pertinent lemmas are stated in this section and are proved in the appendix.

Lemma 1. In the sense of convergence of distributions

$$\lim_{v \to 0} \int_{-v}^{v} \exp(ix\xi) \, d\xi = 2\pi \delta(x)$$

Lemma 2. If

$$f_t(x) = \frac{1}{2\sqrt{\pi t}} \exp(-x^2/4t), \quad t > 0$$
Section 3. The kernel \( K(x-y) \).

To obtain a solution of the equation

\[
f(x) = \int_a^b K(x-y)g(y)dy \quad a<x<b \quad (3-1)
\]

we will consider two separate cases: the interval \((a,b)\) is finite and the interval \((a,b)\) is infinite.

Section 3.1. The interval \((a,b)\) is finite.

Section 3.11. Expression of the solution as an infinite integral.

We will construct a solution of Eq. (3-1) as a function of the solution of the same equation with both limits infinite. The essence of the method will be the determination of an entity related to Eq. (3-1), which is independent of the limits of the integral equation, this quantity being then evaluated in terms of the solution of the equation with infinite limits.

We will assume that \( f, K \) and \( g \) satisfy the following conditions

(i-1) \( f(x) \) has a Maclaurin expansion

(ii-1) \( f(x) \) can be continued analytically for all \( x \) and is such that its analytic continuation belongs to \( L^2(-\infty, \infty) \)

(iii-1) \( K(x-y) \) has a formal (not necessarily convergent) series expansion

\[
K(x-y) = \sum_{n=0}^{\infty} k_n(y)x^n
\]
Eq. (3.1-3) becomes

\[ K(x-y) = \exp(-y^2) \sum_{n=0}^{\infty} \frac{(-1)^n \frac{H_{n+k}(y)}{k!}} \sum_{k=0}^{\infty} \frac{a_n}{k!} x^k \]

Introducing this expansion into Eq. (3-1), and making use of condition (\(i-1\)), there follows

\[ f^{(k)}(0) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{k!} \int_{a}^{b} \exp(-y^2) H_{n+k}(y) g(y) dy \quad (3.1-5) \]

Some words are necessary regarding the validity of Eq. (3.1-5), which was obtained by means of formal series; especially, since similar arguments are to be used again. It is necessary when studying the summation of series to satisfy requirements of absolute convergence for rearrangement of infinite series and uniform convergence for term by term integration. It is to be remembered, however, that such arguments are only pertinent when one is concerned with the sum of the series. Our only need was for establishing the relationship among the coefficients expressed by Eq. (3.1-5), and the sum of the series did not enter into the discussion. To justify these remarks one can replace the infinite series by finite sums which agree with the infinite expressions through the term in \(x^s\) to obtain

\[ \sum_{k=0}^{s} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{s} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{k!} \int_{a}^{b} \exp(-y^2) H_{n+k}(y) g(y) dy. \]
\[ f(0) = C_0 - a_1 C_1 + a_2 C_2 - \ldots \]
\[ f^{(1)}(0) = C_1 - a_1 C_2 + a_2 C_3 - \ldots \]
\[ f^{(2)}(0) = C_2 - a_1 C_3 + a_2 C_4 - \ldots \]
\[ \vdots \]

In the above \( a_0 \) was set equal to one.

To find \( C_0 \) one multiplies the expression for \( f^{(1)}(0) \) by \( a_1 \) and adds the first two equations. This automatically eliminates \( C_1 \). This gives a new coefficient for \( C_2 \) which can be eliminated by multiplying the expression for \( f^{(2)}(0) \) by a suitable coefficient. In this manner one can successively eliminate all the \( C \)'s except \( C_0 \). A similar procedure obviously works for any \( C_n \). Each \( C_n \) is thus obtained as a linear combination of \( f^{(s)}(0) \)

\[ C_n = \sum_{s=0}^{n} f^{(n+s)}(0) T_s \quad (3.1-8) \]

where the coefficients \( T_s \) are given by

\[ T_s = \sum (-1)^{r_2 + \ldots + (r_1 + r_2 + r_3 + \ldots)!} \frac{r_1! \, r_2! \, r_3! \ldots}{a_1 \, a_2 \, a_3 \ldots} \quad (3.1-9) \]

summed over all combinations of the integers \( r_i \) such that

\[ r_1 + 2r_2 + 3r_3 + 4r_4 + \ldots = s. \]
\[
C_m = \int_{-\infty}^{\infty} \exp(-y^2) H_m(y) h(y) \, dy. \tag{3.1-12}
\]

In order to express \( C_m \) as a single integral let us expand \( \exp(-isx) \) in a series of Hermite polynomials.

We have

\[
\exp(-isx) = \sum_{n=0}^{\infty} \frac{(-is)^n x^n}{n!} \tag{3.1-13}
\]

but (5)

\[
x^n = \sum_{k=0}^{[n/2]} \frac{n! H_{n-2k}(x)}{k! 2^k (n-2k)!} \tag{3.1-14}
\]

where \([n/2]\) denotes the greatest integer in \(n/2\).

Replacing (3.1-14) in (3.1-13) we have

\[
\exp(-isx) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-is)^n x^n}{n! 2^k k! (n-2k)!} H_{n-2k}(x)
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-is)^n x^n}{2^{n+2k} k! n!} \frac{H_n(x)}{n!}
\]

\[
= \exp\left(-\frac{s^2}{4}\right) \sum_{n=0}^{\infty} \frac{(-i)^n s^n H_n(x)}{2^n n!} \tag{3.1-15}
\]

and from Eq. (3.1-11)
series converges uniformly for all values of its argument. Hence by condition (viii-1) and a pertinent theorem\(^{(8)}\), term by term integration is permissible.

Equation (3.1-19) can be written in the form

\[
\frac{1}{b-a} \int_{a}^{b} \exp\left(-2i \frac{\pi ny}{b-a}\right) \exp(-y^2) g(y) dy = \exp\left(-\frac{n^2 \pi^2}{(b-a)^2}\right) \int_{-\infty}^{\infty} \exp(-s^2/4) \exp\left(-\frac{n \pi s}{b-a}\right) \frac{\tilde{f}(s)}{K(s)} \, ds.
\]

This is just the Fourier coefficient in the expansion of \(\exp(-y^2) g(y)\) in a Fourier series in the interval \((a, b)\)

\[
g(y) = \frac{\exp(y^2)}{2\sqrt{\pi} (b-a)} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2n\pi y}{b-a}\right) \frac{n^2 \pi^2}{(b-a)^2} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{4} - \frac{n \pi s}{b-a}\right) \frac{\tilde{f}(s)}{K(s)} \, ds.
\]

Since condition (vii-1) corresponds to Dirichlet conditions\(^{(9)}\), the above series is uniformly convergent and it follows that

\[
g(y) = \frac{\exp(y^2)}{2\sqrt{\pi} (b-a)} \int_{-\infty}^{\infty} \exp(-s^2/4) \frac{\tilde{f}(s)}{K(s)} \left[ \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2 \pi^2}{(b-a)^2}\right) \exp i2n \left( \frac{\pi y}{b-a} + \frac{i \pi s}{2(b-a)} \right) \right] \, ds
\]

The expression in brackets will be recognized as the third theta function defined by\(^{(10)}\)

\[
\theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} \exp(i2n z)
\]

hence

\[
g(y) = \frac{\exp(y^2)}{2\sqrt{\pi} (b-a)} \int_{-\infty}^{\infty} \exp(-s^2/4) \frac{\tilde{f}(s)}{K(s)} \theta_3\left( \frac{\pi y}{b-a} + \frac{i \pi s}{2(b-a)} ; q \right) e^{-\pi^2 s^2/(b-a)^2} \, ds
\]

(3.1-19)
the interval \( a < y < b \).

Consider the equation

\[
\frac{1}{2\pi} \hat{f}(s) \exp(-isx) = \int_a^b K(x-y)G(y,s)dy, \quad a < x < b \tag{3.1-23}
\]

where \( \hat{f}(s) \) denotes the Fourier transform of \( f(x) \) and \( s \)
is to be interpreted as a parameter.

Since the left hand side of Eq. (3.1-23) has a
Maclaurin expansion it is possible to proceed as before
to Eq. (3.1-7) with

\[
C_m = \int_a^b \exp(-y^2) H_m(y)G(y,s)dy, \tag{3.1-24}
\]

Since \( \exp(-isx) \) can be continued analytically for all \( x \),
we will determine \( C_m \) by solving the equation

\[
\frac{1}{2\pi} \hat{f}(s) \exp(-isx) = \int_a^b K(x-y)h(y)dy \tag{3.1-25}
\]

By use of lemmas 7 and 8 of section 2, we have

\[
h(x) = F^{-1} \left[ \frac{\hat{f}(s) \delta(d-s)}{K(d)} \right] \tag{3.1-26}
\]

where \( F^{-1} \) denotes the inverse Fourier transform. It follows
then that

\[
\left( F^{-1} \left[ \frac{\delta(d-s)\hat{f}(s)}{K(d)} \right], \phi \right) = \frac{1}{2\pi} \left( \frac{\delta(d-s)\hat{f}(s)}{K(d)} \right) \phi
\]
Proceeding as in Eq. (3.1-17) we obtain

\[ G(y, s) = \frac{1}{2\pi} \exp\left(\sqrt{ys} \frac{f(s)}{K(s)} \right) \theta_3 \left( (b-a)(\frac{s}{2} - iy), e^{(b-a)^2} \right) \quad (3.1-28) \]

as the solution of Eq. (3.1-23).

Let us now integrate both sides of Eq. (3.1-23) with respect to \( s \) from \(-y\) to \( y\)

\[ \frac{1}{2\pi} \int_{-y}^{y} f(s) \exp\left( -isx \right) ds = \int_{-y}^{y} K(x-y) \int_{-y}^{y} G(y, s) ds \, dy \quad (3.1-29) \]

The interchange of the order of integration follows by Fubini's theorem since the left hand side is integrable.

Letting

\[ \frac{\partial}{\partial y} \int_{-y}^{y} G(y, s) ds \]

and assuming that the infinite integral

\[ \int_{-\infty}^{\infty} G(y, s) ds \]

converges, we have by virtue of lemma 4, since the interval \((a, b)\) is finite

\[ f(x) = \int_{a}^{b} k(x-y) \int_{-\infty}^{\infty} G(y, s) ds \, dy \]

In other words the solution of Eq. (3.1) is
\[ G(y, t) = \frac{1}{2\pi} \int_0^\infty \exp(-iys) \frac{\hat{f}(s)}{K(s)} \theta_3 \left( (b-a) \left( \frac{s}{2} - iy \right), e^{-(b-a)^2} \right) ds. \] (3.1-34)

Use of the result

\[ \exp(-s^2 t) = \int_0^\infty \exp(isx) \left( \frac{1}{2 \sqrt{\pi t}} \exp(-x^2/4t) \right) dx \]

in Eq. (3.1-34) gives

\[ G(y, t) = \frac{1}{2\pi} \int_0^\infty \exp(-iys) \frac{\hat{f}(s)}{K(s) - \exp(s^2 t)} \theta_3 \left( (b-a) \left( \frac{s}{2} - iy \right), e^{-(b-a)^2} \right) ds \] (3.1-35)

as the solution of Eq. (3.1-32).

Taking the limit of Eq. (3.1-32) as \( t \) approaches zero gives, in view of lemma 2

\[ f(x) = \left( K(x-y) - \delta(x-y), g(y) \right) \]

where

\[ g(y) = \lim_{t \to 0} G(y, t) \]

It follows then from Eq. (3.1-35) that the solution of Eq. (3.1-30) is

\[ g(y) = \frac{1}{2\pi} \int_0^\infty \exp(-iys) \frac{\hat{f}(s)}{K(s) - 1} \theta_3 \left( (b-a) \left( \frac{s}{2} - iy \right), e^{-(b-a)^2} \right) ds \] (3.1-36)

which is the required result.
Conditions (1-3) to (4-3) are sufficient to go from Eq. (3.1) to Eq. (3.1-5). With $C_m$ defined by Eq. (3.1-6) we have

$$C_m = \int_a^b \exp(-y^2) H_m(y) g(y) dy$$  \hspace{1cm} (3.1-6)

Multiplying both sides of this equation by $(-t)^m/2^m m!$ and summing from $m=0$ to $\infty$, we obtain

$$\sum_{m=0}^{\infty} \frac{C_m (-t)^m}{2^m m!} = \int_a^b \exp(-y^2) \exp(-yt - \frac{t^2}{4}) g(y) dy$$

or

$$\exp\left(\frac{t^2}{4}\right) \sum_{m=0}^{\infty} \frac{C_m (-t)^m}{2^m m!} = \int_a^b \exp(-yt) \exp(-y^2) g(y) dy$$

Expanding the exponentials in $t$ in a power series and collecting powers of $t$ we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{[m/2]} \frac{C_m (-t)^m}{2^m (m-2n)! n!} = \sum_{m=0}^{\infty} (-t)^m \int_a^b y^m \exp(-y^2) g(y) dy$$

comparing coefficients

$$\sum_{n=0}^{[m/2]} \frac{C_m (-t)^m}{2^m (m-2n)! n!} = \int_a^b y^m \exp(-y^2) g(y) dy$$  \hspace{1cm} (3.1-37)

We now expand $\exp(-y^2) g(y)/w(y)$ in a uniformly convergent
where

\[ g_k = \int_a^b w(y) \mathcal{M}_k^2(y) \, dy \]  

(3.1-42)

If the polynomials \( \mathcal{M}_n(y) \) are such that \( \mathcal{M}_n(y) \) is an even function of \( y \) when \( n \) is even and odd when \( n \) is odd, then \( \mathcal{M}_n(y) \) is given by an equation of the form

\[ \mathcal{M}_n(y) = \sum_{m=0}^{[n/2]} \beta(n,m) y^{n-2m} \]  

(3.1-43)

In this case we have from Eq. (3.1-37)

\[ \sum_{m=0}^{[k/2]} \sum_{n=0}^{[k/2-2m]} \frac{\beta(k,m) C_{k-2m-2n} (k-2m)!}{2^{k-2m} (k-2m-2n)! \cdot n!} = \int_a^b \mathcal{M}_k(y) \exp(-y^2) g(y) \, dy \]  

(3.1-44)

and substituting, as before, Eq. (3.1-38) in Eq. (3.1-44) we find

\[ b_k = \frac{1}{g_k} \sum_{m=0}^{[k/2]} \sum_{n=0}^{[k/2-2m]} \frac{\beta(k,m) C_{k-2m-2n} (k-2m)!}{2^{k-2m} (k-2m-2n)! \cdot n!} \]  

(3.1-45)

with \( g_k \) given by Eq. (3.1-42).

We have shown then that the solution of Eq. (3.1) is given by

\[ g(y) = \exp(y^2) w(y) \sum_{k=0}^{\infty} \sum_{m=0}^{[m/2]} \sum_{n=0}^{\infty} \frac{\alpha(k,m) m! C_{m-2n} \mathcal{M}_k(y)}{q_k^m (m-2n)! n!} \]  

(3.1-46)
where we replaced $s$ by $2s$. Substitution of this result in Eq. (3.1-46) gives

$$g(y) = \exp(y^2) w(y) \sum_{k=0}^{\infty} \sum_{m=0}^{\lfloor m/2 \rfloor} \frac{a(k, m) (-)^m m! H_k(y)}{g_k}$$

$$\frac{1}{2^{m_m} \sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\lfloor m/2 \rfloor} \frac{(-)^m (2s)^{m-2n}}{n! (m-2n)!} \exp(-s^2) \frac{\widehat{f(2s)}}{\widehat{K(2s)}} \, ds$$

(3.1-46)

but

$$H_m(s) = \sum_{n=0}^{\lfloor m/2 \rfloor} \frac{(-)^m (2s)^{m-2n}}{n! (m-2n)!}$$

(3.1-49)

and it follows immediately that

$$\frac{1}{2^{m_m} \sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2) H_m(s) \frac{\widehat{f(2s)}}{\widehat{K(2s)}} \, ds$$

is the mth coefficient in the expansion of $\frac{\widehat{f(2s)}}{\widehat{K(2s)}}$ in a series of Hermite polynomials

$$\frac{\widehat{f(2s)}}{\widehat{K(2s)}} = \sum_{m=0}^{\infty} h_m H_m(s)$$

(3.1-50)

and Eq. (3.1-48) becomes
In the event that \( f(x) \) does not have a Maclaurin expansion as assumed by condition (i-1), one can, at least in principle, replace \( f(x) \) by a sequence \( \{ f_y \} \) of \( C^\infty \) functions converging to \( f \); solve the corresponding integral equation for \( f_y \) and take the limit of the resulting solution.

As illustration let \( a = -1, b = +1 \). One possible choice for \( \{ f_y \} \) are the Legendre polynomials:\(^{14}\)

\[
P_n(y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \binom{n}{k} n-k (2y)^{n-2k}}{k! (n-2k)!}
\]

(3.1-54)

The factorial function \( (a)_n \) used in Eq. (3.1-54) is defined as

\[
(a)_n = a(a + 1)(a + 2) \ldots (a + n - 1), \quad n > 1
\]

and

\[
(a)_0 = 1, \quad a \neq 0.
\]

For the Legendre polynomials we have

\[
q_k = \frac{2}{2k + 1}, \quad w(y) = 1
\]

\[
\beta(k, m) = (-)^m \binom{1/2}{k-m} k-m^2 - 2m
\]

\[
m! (k-2m)!
\]
Also $H_n(y)$ will be assumed defined when the index $n$ varies continuously and to be an analytic function of $n$.

We will show that in this case $g(y)$ can be represented by the following integral

$$g(y) = w(y) \exp(y^2) \frac{1}{2} \int_C \frac{b_z \exp(i\pi z) H_n(y)}{\sin \pi z} \, dz \quad (3.1-58)$$

where $C$ is a contour that starts at $\infty - i\delta$ in the $z$-plane, goes below the real axis to $z = -\frac{1}{2}$ and then above the real axis to $\infty + i\delta$.

![Contour C](image)

**FIGURE 1.1: CONTOUR C**

Since $b_z$ and $H_n(z)$ are analytic functions of $z$, the only singularities of the integrand are poles at those values of $z$ inside $C$ for which $\sin \pi z = 0$, those are $z = 0, 1, 2, \ldots$. Since the contour $C$ is described in the clockwise direction, the integral $(3.1-58)$ is equal to the negative sum of the residues at the poles, so that

$$g(y) = w(y) \exp(y^2) \frac{1}{2} (-2\pi i) \sum_{n=0}^\infty \frac{b_n(-)^n H_n(y)}{\pi \cos \pi n}$$
For this set we have (17)

\[ \text{eq}_n = \frac{\Gamma(\alpha + n)}{n!} = \frac{\Gamma(1 + \alpha) (1 + \alpha)_n}{n!} \]

\[ w(y) = y^{\alpha} \exp(-y), \quad \text{Re}(\alpha) > -1 \]

\[ \varphi(k, m) = \frac{(-)^m (1 + \alpha)_k}{m! (k-m)! (1 + \alpha)_m} \]

and Eq. (3.1-51) becomes

\[ g(y) = \frac{\exp(y^2 - y)y^\alpha}{\Gamma(1 + \alpha)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{i^m k! h_m L_k^\alpha(y)}{(k-m)! (1 + \alpha)_m} \]  \hspace{1cm} (3.2-2)

corresponding to the solution of the equation

\[ f(x) = \int_0^x K (x-y) g(y) dy, \quad 0 < x < \infty \]  \hspace{1cm} (3.2-3)

In particular for \( \alpha = 0 \), Eq. (3.2-2) takes the form

\[ g(y) = \exp(y^2 - y) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} h_m L_k^\alpha(y) \]  \hspace{1cm} (3.2-4)

where \( \binom{k}{m} \) is the binomial coefficient.

In both Eqs. (3.2-2) and (3.2-4) \( h_m \) is given in Eq. (3.1-50).

An equation of the type


\[ f(x) = \int_a^b K(x,y)g(y)dy \]  \hfill (4-2)

we obtain

\[ f^{(n)}(0) = \int_a^b \varphi_n(y)g(y)dy \]  \hfill (4-3)

where we have assumed that \( f(x) \) possesses a Maclaurin expansion.

Let us also assume that \( g(y)/w(y) \) can be expanded in a series of polynomials \( \varphi_n(y) \) orthogonal in the interval \((a, b)\) with respect to the weighting function \( w(y) > 0 \).

\[ g(y) = w(y) \sum_{m=0}^{\infty} b_m \varphi_n(y) \]  \hfill (4-4)

Substituting Eq. (4-4) in Eq. (4-3) and assuming term by term integration permissible, we have

\[ f^{(n)}(0) = \sum_{m=0}^{\infty} b_m \int_a^b w(y) \varphi_n(y) \varphi_n(y)dy \]  \hfill (4-5)

We now appeal to a theorem in the theory of polynomial sets.

Theorem 2. Let \( \varphi_n(x) \) be a polynomial of degree \( n \) in \( x \) and assume there is one such polynomial for each \( n \). Let \( w(x) > 0 \) on \( a < x < b \). Then a necessary and sufficient
Then as is obvious by inspection of Eq. (4-7) \( b_0 = 1, \ b_m = 0 \) if \( m \neq 1 \). Hence if we have iterated Eq. (4-7) \( N \) times and have obtained a set of relations

\[
b_m = \sum_{n=0}^{m} a_{nm} f^{(n)}(0), \quad m = 0, 1, \ldots, N. \quad (4-9)
\]

then by letting \( f^{(n)}(0) \) take the values specified by Eq. (4-8) we must obtain \( b_0 = 1, \ b_m = 0, \ m \neq 1 \). If this result is not obtained, a mistake was made in the computations. This device can be used to check the numerical accuracy of Eqs. (4-9).

It should be noticed that the integral in Eq. (4-6)

\[
\int_{a}^{b} w(y) \varphi_n(y) \mathcal{M}_m(y) \, dy \quad (4-10)
\]

is \( g_m^{-1} \) times the \( m \)th coefficient in the expansion of \( \varphi_n(y) \) in a series of polynomials \( \mathcal{M}_m(y) \), where \( g_m \) is given by Eq. (3.1-42). Hence in order to do the integral (4-10) it is, perhaps, the simplest procedure, to expand \( \varphi_n(y) \) in series of \( \mathcal{M}_m(y) \) and pick out the coefficients. Such expansions are given for a number of classical polynomials in Professor Rainville's "Special Functions." (See References).
\[ C_m = \int_a^b \varphi_m(y) g(y) \, dy \quad (4-17) \]

Let now
\[ g(y) = w(y) \sum_{n=0}^{\infty} b_n T_n(y) \quad (4-18) \]

Substitution of Eq. (4-18) in Eq. (4-17) gives in view of theorem 2
\[ C_m = \sum_{n=0}^{\infty} b_n \int_a^b w(y) \varphi_m(y) T_n(y) \, dy \]

which are equations (4-6).

Now from Eqs. (4-13) and (4-14) there follows
\[ \frac{f^{(n)}(0)}{\gamma_n n!} = \int_a^b y^n g(y) \, dy \quad (4-19) \]

Let the polynomials \( T_k(y) \) be of the form
\[ T_k(y) = \sum_{n=0}^{\infty} \alpha(k,n) y^n \quad (4-20) \]

Multiply both sides of Eq. (4-19) by \( \alpha(k,n) \) and

sum over \( n \)
\[ \sum_{n=0}^{\infty} \frac{\alpha(k,n) f^{(n)}(0)}{\gamma_n n!} = \int_a^b T_k(y) g(y) \, dy \quad (4-21) \]

Inserting Eq. (4-18) in Eq. (4-21) we have by virtue of
\[ J_k(y) = \sum_{n=0}^{[k/2]} \beta(k,n) y^{k-2n} \quad (4-26) \]

then we have from Eq. (4-19)
\[ \sum_{n=0}^{[k/2]} \beta(k,n) f(k-2n)(0) \over g_{k-2n}(k-2n)! = \int_a^b J_k(y)g(y)dy \]

Use of Eq. (4-18) gives as before
\[ b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \beta(k,n) f(k-2n)(0) \over g_{k-2n}(k-2n)! \quad (4-27) \]

with \( g_k \) given by Eq. (4-23).

Finally, from Eq. (4-24)
\[ b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \sum_{m=0}^{k-2n} \beta(k,n) \lambda(m) \beta_k-2n-m \over g_{k-2n} \quad (4-28) \]

we have thus shown that the solution of the equation
\[ f(x) = \int_a^b K(x,y)g(y)dy \quad (4-29) \]

with
\[ K(x,y) = \sum_{n=0}^\infty \varphi_n(y) \frac{x^n}{n!} \quad (4-30) \]

and \( \varphi_n(y) \) satisfying Eqs. (4-11) to (4-13) is given by
In Eq. (4-25) replace $m$ by $n-2m$, then

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{[n/2]} \frac{\gamma(n-2m) \beta(2m) \alpha(k,m) C_{n-2m}}{\gamma_n}$$

since $\beta_{2m+1} \equiv 0$

Hence for this case

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{[n/2]} \frac{\alpha(k,n) n! C_{n-2m}}{(n-2m)! m! 2^n}$$

(4-37)

which is, of course, Eq. (3.1-41).

A similar substitution can be made for Eq. (4-34).

For the Laguerre polynomials (19)

$$\exp(t) \mathcal{F}_1 (-; 1+\alpha; -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(t)}{(1+\alpha)_n} t^n$$

(4-38)

$$\beta_n = (-)^n n! \quad \gamma_n = (-)^n/(1+\alpha)_n n! \quad \lambda(n) = 1/(1+\alpha)_n$$

and from Eq. (4-32)

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{n} \frac{(-)^m \alpha(k,n)(1+\alpha)_n n! C_m}{(1+\alpha)_m (n-m)!}$$

(4-39)

The substitution could just as easily have been made in Eq. (4-34).
\[ g(y) = w(y) \exp(\lambda x_0 y) \sum_{n=0}^{\infty} b_n \mathcal{H}_n(y) \quad (5-3) \]

\[ \mathcal{H}_n(y) \] are polynomials orthogonal in the interval \((a, b)\) with respect to the weighting function \(w(y)\).

In Eq. (5-1) replace \(x\) by \(x + x_0\)

\[ f(x + x_0) = \int_a^b \exp(-\lambda xy) \exp(-\lambda x_0 y) g(y) dy \quad (5-4) \]

Expanding the exponential in \(x\) in a power series and making use of Eq. (5-2) we have

\[ \frac{(-\lambda)^n f^{(n)}(x_0)}{\lambda^n} = \int_a^b y^n \exp(-\lambda x_0 y) g(y) dy \quad (5-5) \]

We shall assume the polynomials \(\mathcal{H}_k(y)\) to be given by an equation of the form

\[ \mathcal{H}_k(y) = \sum_{n=0}^{k} \alpha(k, n) y^n \quad (5-6) \]

Multiplying Eq. (5-5) by \(\alpha(k, n)\) and summing over \(n\) from 0 to \(k\), gives

\[ \sum_{n=0}^{k} \frac{(-\lambda)^n \alpha(k, n) f^{(n)}(x_0)}{\lambda^n} = \int_a^b \mathcal{H}_k(y) \exp(-\lambda x_0 y) g(y) dy \quad (5-7) \]

Use of Eq. (5-3) in Eq. (5-7) gives finally
A special case of Eq. (5-1) corresponding to \( \lambda = 1 \) and the interval \((a, b)\) finite

\[
f(x) = \int_{a}^{b} \exp(-ixy)g(y)\,dy
\]  

(5-13)

is of central importance in the study of antenna synthesis. (22)

Another special case is provided by the choice \( \lambda = 1 \), \( a = 0 \), \( b = \infty \), which gives Laplace's integral equation (23)

\[
f(s) = \int_{0}^{\infty} \exp(-st)F(t)\,dt
\]  

(5-14)

Selecting as the set \( \{ L_n^{(\alpha)}(y) \} \) the Laguerre polynomials \( L_n^{(\alpha)}(y) \) we have from Eq. (5-9) and the results on page 30

\[
F(t) = \frac{t \exp(-(1-s\alpha)t)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{f(k)(s\alpha)}{(1+\alpha)_k} L_n^{(\alpha)}(t)
\]  

(5-15)

as the solution of Eq. (5-14)

Suppose for example that

\[
f(s) = \frac{s^m}{(s+a)^{m+1}}
\]  

(5-16)

then

\[
\frac{s^m}{(s+a)^{m+1}} = \sum_{k=0}^{\infty} \frac{(m+1)_k (-)^k s^{k+m}}{a^{k+m+1} k!}
\]
\[ F(t) = \exp(-t) \frac{1}{a^{m+1}} \sum_{n=0}^{\infty} \frac{(1)^{n+m} L_{n+m}(t) (1 - 1/a)^n}{m! \cdot n!} \]

Use of the result (24)

\[ \sum_{n=0}^{\infty} \frac{(n+m)! \cdot L_{n+m}(t) \cdot x^n}{m! \cdot n!} = (1-x)^{-1-m} \cdot \exp\left(-\frac{tx}{1-x}\right) \cdot L_m\left(\frac{t}{1-x}\right) \]

gives

\[ F(t) = \exp(-t) \frac{1}{a^{m+1}} \exp(-t(a-1)) \cdot L_m(at) \]

\[ = \exp(-at) \cdot L_m(at) = \exp(-at) \cdot F_1(-m; l; at) \]

(5-19)

or by use of Kummer's first formula (25)

\[ \exp(-z) \cdot F_1(a; b; z) = F_1(b-a; b; -z) \]

equation (5-19) becomes

\[ F(t) = F_1(1 + m; l; -at). \]

Thus if \( \mathcal{L}^{-1} \) denotes the inverse Laplace transform we have

\[ \mathcal{L}^{-1} \left( \frac{s^m}{(s + a)^{m+1}} \right) = F_1(1 + m; l; -at) \]

(5-20)

This result is well known. (26)
CHAPTER TWO
RADIATION FROM A LINEAR ANTENNA.

Section 1. Formulation of Hallén's integral equation.

We will consider the problem of electromagnetic radiation from a perfectly conducting cylindrical antenna excited across a small air gap by an external a-c source.

It is well known that the electric field vector can be derived from a scalar and a vector potential according to the relation

\[ \vec{E} = -\nabla \phi - i\omega \vec{A} \]  

(1-1)

If the potentials are subject to the Lorentz condition

\[ \nabla \cdot \vec{A} + i\omega \mu \varepsilon \phi = 0 \]  

(1-2)

then \( \vec{E} \) can be derived from a vector potential alone

\[ \vec{E} = \frac{c^2}{i\omega} (\nabla (\nabla \cdot \vec{A}) + \frac{\omega^2}{c^2} \vec{A}) \]  

(1-3)

where \( c = 1/\sqrt{\mu \varepsilon} \).

For the radiated field \( \vec{E}(r) \) the vector potential \( \vec{A}(r) \) is related to the induced current density by means of the equation

\[ \vec{A}(r) = \frac{\mu}{4\pi} \int \frac{J(|\vec{r} - \vec{r}'|, t - \frac{r - r'}{c})}{|\vec{r} - \vec{r}'|} \, dv \]  

(1-4)

\( r \) being the distance from the origin of coordinates to the field point, and \( r' \) the corresponding distance to
As the antenna we have considered consists of two halves between which there exists a difference of potential, we should not expect the coefficients of \( \cos kz \) and \( \sin kz \) terms to be the same for both halves of the antenna. Instead it is the vector potential which is to be symmetrical with respect to the air gap when the antenna is fed at the middle. Consequently we shall put

\[
A_z = (A_1 \cos kz + A_2 \sin k|z|) \exp(i\omega t),
\]

where \( A_1 \) and \( A_2 \) are constants. For the scalar potential of the antenna we have from Eq. (1-2) \( \imath \omega \phi = -c^2 \partial A_z / \partial z \)

i.e.,

\[
\imath \omega \phi = c^2 k(A_1 \sin kz + A_2 \cos kz) \exp(i\omega t)
\]

where the upper sign corresponds to \( z > 0 \) and the lower one to \( z < 0 \). Hence the potential has two values \( \phi = \pm icA_z \exp(i\omega t) \) at \( z = 0 \). We know already, however, that there \( \phi = \pm V_o \exp(i\omega t) \) and \( A_2 \) is thus defined. The vector potential obtains then the form

\[
A_z = (A_1 \cos kz - \frac{i}{c} V_o \sin k|z|) \exp(i\omega t).
\]

It is possible to complement the last term in the parenthesis by a corresponding real cosine term to give an exponential function, the cosine part of which is taken from the first term. \( A_z \) obtains then the alternative
From Eqs. (1-6) and (1-7), the integral equation of a cylindrical transmitting antenna fed at the middle is obtained in the form

\[
\frac{4\pi}{\eta_o} V_0 \exp(-ik|z|) + A \cos k z = \int_{-\ell}^{\ell} d\xi I(\xi) \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-ikr)}{r} \, d\phi
\]

where \( A \) is a new constant and \( \eta_o = \sqrt{\mu/\varepsilon} \).

This is Hallén's integral equation.

It has been argued by Hallén (28) that to Eq. (1-9) one must add the boundary condition \( I(\ell) = I(-\ell) = 0 \) to specify the constant \( A \). We shall not follow his suggestion for, as we shall see, it is the constant \( A \) that is determined by the solution and not the other way around; that is \( I(\xi) \) is independent of \( A \).

Section 2. Solution of Hallén's integral equation.

In Eq. (1-9) replace \( z \) by \( lx \) and \( \xi \) by \( ly \), then

\[
\frac{4\pi}{\eta_o} V_0 \exp(-ikl|x|) + A \cos klx = \int_{-\ell}^{\ell} dy I(ly) \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-ikrl)}{r} \, d\phi
\]

where now

\[
r = \sqrt{(x-y)^2 + \frac{a^2}{\ell^2} \sin^2 \frac{1}{2} \phi}
\]

Multiply both sides of Eq. (2-1) by \( \frac{d^2}{dx^2} + (kl)^2 \), to obtain

\[
\frac{4\pi V_0}{\eta_o} (-2ikl) \delta(x) = \left( \frac{d^2}{dx^2} + (kl)^2 \right) \left( \int_{-\ell}^{\ell} dy I(ly) \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-ikrl)}{r} \, d\phi \right)
\]

where \( \delta(x) \) is the delta function. The factor \( (-2ikl) \) accounts for the jump discontinuity in the first derivative of \( \exp(-ikl|x|) \) at \( x = 0 \).
Let us as in page 14, consider the related equation
\[
\exp(i sx) = \frac{\ln}{4V_o} \int_{-\infty}^{\infty} \frac{dyG(y, s)(\frac{\partial}{\partial x})^2 + (k \ell)^2}{2\pi} \int_{0}^{2\pi} \exp(-ikr) \, dr \, d\phi
\]
where \( s \) is a parameter. By letting \( G(y, s) = 0 \) for \( |y| > 1 \), we have by virtue of lemma 3 of chapter one
\[
\exp(i sx) = \frac{\ln}{4V_o} \frac{k \ell}{2\pi} \int_{-\infty}^{\infty} dyG(y, s)(\frac{\partial}{\partial x})^2 + (k \ell)^2 \int_{0}^{2\pi} \exp(-ikr) dr \, d\phi
\]
but this is Eq. (3.1-23) with \( f(s) = 2\pi, a = -1, b = 1 \), \( s \) replaced by \(-s\) and
\[
K(x-y) = \frac{\ln}{4V_o} \frac{k \ell}{2\pi} \int_{0}^{2\pi} \exp(i k |x-y|) (x-y)^2 + 4 \frac{a^2}{\ell^2} \sin \frac{1}{2} \phi \, d\phi
\]
In order to satisfy the requirement that \( K(x) = o(|x|^{-n}) \) as \( |x| \to \infty \) for all \( n > 0 \), we require \( k \) to have a small negative imaginary part, corresponding to the medium having finite non-zero conductivity
\[
k = p - iq \quad p, q > 0.
\]
for substitution in Eq. (3.1-28) we need the Fourier transform of Eq. (2-6).

To obtain this consider the following integral
\[
\int_{0}^{\infty} \frac{K_{\ell\ell}}{\sqrt{t}} \left\{ \frac{a}{(t^2 + z^2)^{1/2}} \right\}^{m+1} dt = \frac{b^m}{a^m} \left\{ \frac{a^2 + b^2}{Z} \right\}_{\nu, \mu} \left\{ \frac{a^2 + b^2}{Z} \right\}^{\nu, \mu-1}
\]
In this equation let \( \nu = \frac{1}{2}, \mu = \frac{1}{2}, a = i k \ell, b = s, z = a \) and since \( J_{\frac{1}{2}}(z) = (2/\pi z) \cos z \), and \( K_{\frac{1}{2}}(z) = (\pi/2z) \exp(-z) \), there follows
Substitution of this result in Eq. (2-8) yields
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \exp(-ikl \sqrt{(x-y)^2 + 4 \frac{a^2}{l^2} \sin^2 \frac{1}{2} \phi}) \frac{1}{\sqrt{(x-y)^2 + 4 \frac{a^2}{l^2} \sin^2 \frac{1}{2} \phi}} d\phi
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-is(x-y)) 2I_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})K_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2}) ds
\]
(2-11)

By use of the result, proved in the appendix, that
\[
P(\frac{d}{ds}) \tilde{f} = P \left[ P(\frac{d}{ds}) f \right]
\]
where \(P(x)\) is a polynomial in \(x\), we have
\[
\left( \frac{\partial^2}{\partial x^2} + (kl)^2 \right) \frac{1}{2\pi} \int_{0}^{2\pi} \exp(-ikl \sqrt{(x-y)^2 + 4 \frac{a^2}{l^2} \sin^2 \frac{1}{2} \phi}) \frac{1}{\sqrt{(x-y)^2 + 4 \frac{a^2}{l^2} \sin^2 \frac{1}{2} \phi}} d\phi
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\exp(-is(x-y))(kl)^2 - s^2)I_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})K_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2}) ds
\]
(2-12)

Hence from Eq. (2-6) we have
\[
\tilde{K}(s) = \frac{\ln((kl)^2 - s^2)}{2} I_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})K_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})
\]
(2-13)

which is the desired transform.

From Eq. (3.1-28) there follows
\[
G(y,s) = \exp(iys) \frac{2V kl \Theta(3) (s + 2iy, \exp(-4))}{\ln((kl)^2 - s^2) I_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})K_{0}(\frac{a}{l} \sqrt{s^2 - (kl)^2})}
\]
(2-14)
in the real axis. The singularities of the integrand
are located in the s-plane at \( s = k \) and \( s = -k \).

Use of the relation

\[
\Theta_3(s \ell + 2i \frac{\pi}{k}, \exp(-4)) = 1 + 2 \sum_{n=1}^{\infty} \exp(-4n^2) \cos 2n(2i \frac{\pi}{k} + s \ell)
\]
in Eq. (3-1), gives

\[
I(z) = \frac{i2kV_o}{\eta_0} \int_{-\infty}^{\infty} \frac{\exp(isz) \ ds}{(s^2 - k^2)I_o(a \sqrt{s^2 - k^2})K_o(a \sqrt{s^2 - k^2})} + \frac{ivk}{\eta_0} \sum_{n=1}^{\infty} \exp(-4n(n-1)) \int_{-\infty}^{\infty} \frac{\exp(is(\ell+z)) \exp(-4n\frac{1}{k}(\ell+z)) \exp(is(2n-1)\ell) \ ds}{(s^2 - k^2)I_o(a \sqrt{s^2 - k^2})K_o(a \sqrt{s^2 - k^2})}
\]

This expression is seen to be of the form

\[
I(z) = i_o(z) + \sum_{n=1}^{\infty} i_n(1+z) + i_n(1-z)
\]

where

\[
i_o(z) = \frac{i2kV_o}{\eta_0} \int_{-\infty}^{\infty} \frac{\exp(isz) \ ds}{(s^2 - k^2)I_o(a \sqrt{s^2 - k^2})K_o(a \sqrt{s^2 - k^2})}
\]

(3-4)

\[
i_n(1+z) = \frac{ivk}{\eta_0} \exp(-4n(n-1)) \int_{-\infty}^{\infty} \frac{\exp(is(\ell+z)) \exp(-4n\frac{1}{k}(\ell+z)) \exp(is(2n-1)\ell) \ ds}{(s^2 - k^2)I_o(a \sqrt{s^2 - k^2})K_o(a \sqrt{s^2 - k^2})}
\]

(3-5)
Let us now replace the path of integration by another path (Fig.2.2)

![s-plane](image)

Figure 2.2: CONTOUR $\Gamma$

The rotation of the left half of the path of integration in Eq. (3-7) is permissible because of the factor $\exp(-is|z|)$ in the numerator, and the asymptotic behavior of $I_0 K_0$, when residues are provided for the poles. Since

$$I_0(z)K_1(z) + I_1(z)K_0(z) = 1/z$$

there follows

$$\frac{1}{(s^2 - k^2)I_0 K_0} = \frac{a}{\sqrt{s^2 - k^2}} \left( \frac{I_1}{I_0} + \frac{K_1}{K_0} \right) \bigg| a \sqrt{s^2 - k^2}$$

(3-8)

$K_0$ has no zeros in the lower half of the complex plane.

The zeros of $I_0(a \sqrt{s^2 - k^2})$ are located at $a \sqrt{s^2_m - k^2} = i \xi_{om}$, i.e.,

$$s_m = \sqrt{k^2 - \frac{\xi_{om}^2}{c_m}}$$

(3-9)
\[ i_{om}(z) = \frac{V}{\eta} \exp(-i \frac{|z|}{c} \sqrt{\omega^2 - \omega_c^2}) \]  

(3-13)

where

\[ \eta = \frac{n_0}{4\pi} \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2} \]  

(3-14)

The series terms in both cases represent guided waves within the hollow cylindrical antenna, and \( \omega_c \) has the meaning of a cutoff frequency. Equation (3-10) gives the traveling wave partly as an external antenna current expressed by the last term, and partly as a series of internal waves in wave-guides. It was pointed out before, that Hallén's integral equation was rigorously valid when the antenna consisted of a thin metal tube. It is for this reason that the sum of both the internal and the external solution is obtained as the answer to the problem.

Hallén has shown that the integral in Eq.(3-10) can be put in the form

\[ \frac{4\pi}{n_0} \int_{\eta_0}^{\Lambda} (x\z) \exp(-ik|z|) \]  

(3-15)

where \( \Lambda(x\z) \) is a non-periodic complex amplitude function given by
between the two halves of the antenna is reduced, the mutual capacitance increases without bound. In reality the air gap never vanishes since even when the separation equals a the capacitance is still very small.

The absence of a finite limit when the antenna is considered without the feeding line makes it impossible to assign to the antenna a well defined susceptance $B_0$, whereas it may be ascribed a well defined conductance $G_0$.

From Eqs. (3-3), (3-6), (3-10) and (3-16) it follows that the solution of Hallen's integral equation is

$$I(z) = \sum_{m=1}^{\infty} \frac{4\pi}{n_0} V_o \frac{k}{s_m} \exp(-is_m |z|)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp \left\{ -4n \left[ \frac{1}{l}(l+z)+n-1 \right] \right\} \frac{2\pi}{n_0} V_o \frac{k}{s_m} \exp \left\{ -is_m \left[ (l+z)+(2n-1)\ell \right] \right\}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp \left\{ -4n \left[ \frac{1}{l}(l-z)+n-1 \right] \right\} \frac{2\pi}{n_0} V_o \frac{k}{s_m} \exp \left\{ -is_m \left[ (l-z)+(2n-1)\ell \right] \right\}$$

$$+ \frac{4\pi}{n_0} V_o \varphi'(kz) \exp(-ik|z|)$$

$$+ \sum_{n=1}^{\infty} \frac{2\pi}{n_0} V_o \exp \left\{ -4n \left[ \frac{1}{l}(l+z)+n-1 \right] \right\} \varphi' (k(l+z)+k\ell(2n-1)) \exp \left\{ -ik \left[ (l+z)+\ell(2n-1) \right] \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{2\pi}{n_0} V_o \exp \left\{ -4n \left[ \frac{1}{l}(l-z)+n-1 \right] \right\} \varphi' (k(l-z)+k\ell(2n-1)) \exp \left\{ -ik \left[ (l-z)+\ell(2n-1) \right] \right\}$$

(3-18)
CHAPTER THREE

SCATTERING BY A THIN WIRE.

Section 1. Formulation of the integral equation.

We wish now to consider the problem of electromagnetic back scattering from a perfectly conducting cylindrical wire excited by a plane electromagnetic wave with harmonic time dependence.

From Eq. (1-7) of Chapter two we have for the magnitude of the vector potential on the surface of the wire

\[ A_z = \frac{\mu}{4\pi} \int_{-L}^{L} I(\xi) \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\exp(ikr)}{r} d\phi d\xi \]  

(1-1)

where as before \( k = \omega/c \) and \( r \) is the distance from the source point on the surface of the wire, to the field point also on the surface of the wire

\[ r = \sqrt{(z - \xi_i)^2 + 4a^2 \sin^2 \frac{1}{2} \phi} \]

(1-2)

The wire must be assumed thin enough so that the assumption, used in arriving at Eq. (1-1), that the current is symmetrically distributed with respect to the axis of the cylinder is satisfied.

On the surface of the wire, the tangential component of the scattered field \( \mathbf{E}^{(s)} \) must be equal to the negative
of the tangential component of the incident field $E^{(i)}$, 
that is

$$-E^{(i)}_z = \frac{c^2}{i\omega} \left( \frac{d^2}{dz^2} + k^2 \right) \frac{\mu}{2\pi} \int_{-\ell}^{\ell} \frac{2\pi}{r} \exp(-ikr) \, d\phi \, d\xi$$

With the direction of the incident $E$-vector indicated in
Fig. (3.1) we obtain at once

$$E_0 \sin \theta \cos \gamma \exp(ikz \cos \theta) = \frac{i\eta_0}{4\pi k} \frac{d^2}{dz^2} \frac{1}{2\pi} \int_{-\ell}^{\ell} \exp(-ikr) \, d\phi \, d\xi$$

where $\eta_0 = \sqrt{\mu/\varepsilon}$ and $E_0$ is the amplitude of the incident
field.

Eq. (1-4) was derived on the assumption that the end
surfaces play a negligible role and its solution will,
accordingly, bear the same limitation.

Section 2. Solution of the integral equation.

In Eq. (1-4) replace $z$ by $lx$ and $\xi$ by $ly$

$$E_0 \sin \theta \cos \gamma \exp(ikx \cos \theta) = \frac{i\eta_0}{4\pi k^2} \frac{d^2}{dx^2} \frac{1}{2\pi} \exp(-ikr) \, d\phi \, dy$$

$$r = \sqrt{(x - y)^2 + 4a^2 \sin^2 \frac{1}{2} \phi}$$

We have already solved this equation in section 2 of chapter
two. In fact it follows from Eq. (2-4) that $I(z)$ is ob-
\[ I(z) = \frac{4E_o \cos \gamma \exp(ikz \cos \theta)}{\eta_o \ k \sin \theta \ J_o(\text{aksin} \ \theta) \ H_o^{(2)}(\text{aksin} \ \theta)} \]
\[ + \sum_{n=1}^{\infty} \frac{4E_o \cos \gamma \exp(-4n(n-1))\exp(-4n \frac{1}{\lambda}(\ell+z))\exp(ik(\ell+z+(2n-1)\lambda)) \cos \theta}{\eta_o \ k \sin \theta \ J_o(\text{aksin} \ \theta) \ H_o^{(2)}(\text{aksin} \ \theta)} \]

This expression is also of the form

\[ I(z) = i_o(z) + \sum_{n=1}^{\infty} \left[ i_n(\ell + z) + i_n(\ell - z) \right] \]

(3-3)

where

\[ i_o(z) = \frac{4E_o \cos \gamma \exp(ikz \cos \theta)}{\eta_o \ k \sin \theta \ J_o(\text{aksin} \ \theta) \ H_o^{(2)}(\text{aksin} \ \theta)} \] (3-4)

\[ i_n(\ell + z) = \frac{4E_o \cos \gamma \exp(-4n(n-1))\exp(-4n \frac{1}{\lambda}(\ell+z))\exp(ik(\ell+Z+(2n-1)\lambda)) \cos \theta}{\eta_o \ k \sin \theta \ J_o(\text{aksin} \ \theta) \ H_o^{(2)}(\text{aksin} \ \theta)} \] (3-5)

\[ i_n(1-z) = \frac{4E_o \cos \gamma \exp(-4n(n-1))\exp(-4n \frac{1}{\lambda}(\ell-z))\exp(-ik(\ell-Z+2n-1)\lambda) \cos \theta}{\eta_o \ k \sin \theta \ J_o(\text{aksin} \ \theta) \ H_o^{(2)}(\text{aksin} \ \theta)} \] (3-6)

That is, \( i_o(z) \) is the induced primary current wave and the \( i_n \) are the reflected traveling currents of different orders.
The zeros of the third theta function \( \theta_3(z) \) are located at
\[
(z) = \frac{1}{2} \pi + \frac{1}{2} j \pi - \frac{m}{n} - \frac{m}{n} \tau,
\]
where \( m \) and \( n \) are any integers and \( \tau \) is defined in Eq. (3.1-20) of chapter one.

For Eq. (3-1) we have \( \tau = i4/\pi \) and \( I(z) \) vanishes at
\[
z = \frac{1}{2} (2 - 4m - i\pi)(\frac{1}{2} - m + kl \cos \theta)
\]
(3-8)
since the range of variation of \( z \) is the interval \((-1, 1)\),
\( n \) can only take the values 0 and 1, and \( m \) must be such that

\[
2\pi \cos \theta = (2m - 1) \frac{1}{2} \lambda / l.
\]
(3-9)

Equation (3-9) assures the vanishing of the imaginary part of Eq. (3-8). For angles such that Eq. (3-9) is satisfied \( I(z) \) vanishes at \(-1\) and \(+1\).

For other angles the non-vanishing of the current at the ends indicates the presence there of a displacement current in order to satisfy the equation of continuity. In fact it is well known that the radiation field of an electric dipole vanishes along the axis of the dipole,
section $\sigma(\theta, \gamma)$ is given by

$$\sigma(\theta, \gamma) = \frac{4\pi R_0^2}{E_0^2} \left| E_\theta \cos \gamma \right|^2$$  \hspace{1cm} (4-1)

where $E_\theta$ denotes the field produced by the induced current $I(z)$ at a large distance $R_0$ in the direction opposite that of the incident wave; that is,

$$E_\theta = \frac{i \hbar \sin \theta}{4\pi R_0} \int_{-\ell}^{\ell} I(z) \exp(ikz\cos \theta) \, dz$$  \hspace{1cm} (4-2)

Hence

$$\sigma(\theta, \gamma) = \frac{\hbar^2 k^2 \sin^2 \theta \cos^2 \gamma}{4\pi E_0^2} \left[ \int_{-\ell}^{\ell} I(z) \exp(ikz\cos \theta) \, dz \right]^2$$  \hspace{1cm} (4-3)

For obtaining numerical values of $\sigma(\theta, \gamma)$, it is convenient to eliminate the $\gamma$-dependence by introducing the average value of $\sigma(\theta, \gamma)$ over all values of $\gamma$, corresponding to a random distribution of dipoles.

Denoting the latter by $\sigma'(\theta)$, we have

$$\sigma'(\theta) = \frac{1}{\pi} \int_0^{\pi} \sigma(\theta, \gamma) \, d\gamma$$  \hspace{1cm} (4-4)

From Eqs. (4-3) and (4-4) there follows
\[
\frac{d(\theta)}{\lambda^2} = \frac{3\pi \ell^2}{8 \lambda^2 \left[ \left( \frac{\lambda}{2} \right)^2 + \left( \log \frac{\sqrt{\alpha k \sin \theta}}{2} \right)^2 \right]}
\]

\[
\frac{\sin(2k\ell \cos \theta)}{k\ell \cos \theta} + \frac{2}{4 + k^2 \ell^2 \cos^2 \theta}
\]

This expression is similar to that obtained by other research workers, \(^{(39,40)}\) but it predicts values much smaller than those indicated by experiment. This is not a surprising fact since we have seen that the end surfaces play a dominant role in the shaping of the current distribution.
the ordinary sense.

By a continuous linear functional \( f \) in the space \( K \) is meant a definite rule that assigns to each testing function a certain complex number \((f, \varphi)\) satisfying the following conditions:

a) For any two complex numbers \( \alpha_1, \alpha_2 \) and any two testing functions \( \varphi_1(x), \varphi_2(x) \) we have the equality

\[
(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 (f, \varphi_1) + \alpha_2 (f, \varphi_2)
\]

(linearity of the functional \( f \));

b) If the sequence of testing functions \( \varphi_1, \varphi_2, \ldots, \varphi, \ldots \) tends toward zero in \( K \), the sequence of numbers

\[
(f, \varphi_1), (f, \varphi_2), \ldots, (f, \varphi), \ldots
\]

tends toward zero. (Continuity of the linear functional \( f \)).

From here on we will designate as distributions every continuous linear functional in the space of testing functions.

If a function \( f(x) \) is absolutely integrable in every finite interval this function is said to be locally integrable. With each such function we can make correspond to each testing function \( \varphi(x) \) the number

\[
(f, \varphi) = \int_{-\infty}^{\infty} \frac{f(x)}{f(x)} \varphi(x)dx
\]

\((A-1)\)
contains the support of the functional \( f \), the functional
\( f \) is said to be concentrated in the set \( P \).

By definition, a distribution \( f \) vanishes in the open
set \( G \) if it vanishes in a certain neighborhood of each point
of this set.

Given two distributions \( f \) and \( g \), we define their sum
as the linear functional defined by

\[
(f + g, \varphi) = (f, \varphi) + (g, \varphi).
\]

The product of a distribution by a number \( \alpha \) is de-
defined by

\[
(\alpha f, \varphi) = \alpha (f, \varphi) = (f, \alpha \varphi).
\]

The multiplication by an infinitely differentiable function
\( a(x) \) is defined by

\[
(a(x)f, \varphi) = (f, a(x)\varphi).
\]  \quad (A-2)

By definition, the sequence of distributions \( f_1, f_2, \)
\( f_3, \ldots, f_\nu, \ldots \) converges to the distribution \( f \) if for any
testing function \( \varphi(x) \)

\[
\lim_{\nu \to \infty} (f_\nu, \varphi) = (f, \varphi).
\]
\[
\int_a^b \frac{\sin y}{x} \, dx = \int_{a}^{b} \frac{\sin y}{y} \, dy
\]

is uniformly bounded with respect to \(a\) and \(b\), for any \(\nu\).

Let us now consider the sequence of primitives

\[
F_{\nu}(x) = \int_{-\nu}^{\nu} f_\nu(\xi) d\xi
\]

It follows immediately from the definition of the sequence \(f_\nu(x)\), that the function \(F_{\nu}(x)\) has for limit as \(\nu \to \infty\) a constant equal to zero for \(x < 0\), and unity for \(x > 0\), and is also uniformly bounded with respect to \(\nu\) in every finite interval. It follows then that the sequence \(F_{\nu}(x)\) has for limit, in the sense of distributions, the step function \(\alpha(x)\), 0 for \(x < 0\) and 1 for \(x > 0\), hence the sequence of functions \(f_\nu(x) = F_{\nu}'(x)\) has for limit, in the sense of distributions, the function \(\alpha'(x)\)

\[
(\alpha'(x), \varphi(x)) = (\alpha(x), -\varphi'(x)) = -\int_0^\infty \varphi'(x) dx = \varphi(0)
\]

and by definition of the delta function

\[
\alpha'(x) = \delta(x)
\]
also, for any $b > 0$:

$$
\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp(-x^2/4t)\,dx < \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp(-x^2/4t) \frac{x^2}{2t} \frac{2t}{b} \,dx
$$

$$
= \frac{t}{b\sqrt{\pi}} \exp(-b^2/4t) \to 0,
$$

as $t \to 0$. The integrals over the intervals $(b, \infty)$, $b > 0$ tend to zero, and an analogous result holds for the intervals $(-\infty, a)$, $a < 0$. Constructing as in lemma 1 the sequence of primitives

$$
F_t(x) = \int_{-\infty}^{x} f_\xi(\xi)\,d\xi
$$

shows, in view of the results above, that the functions $f_\xi(x)$ form a sequence converging to the delta function. That is

$$
\frac{1}{2\sqrt{\pi t}} \exp(-x^2/4t) \to \delta(x).
$$

**Convolution of distributions.**

In classical analysis one frequently uses the operation of convolution of two functions $f(x)$ and $g(x)$

$$
f(x) * g(x) = \int f(\xi)g(x-\xi)\,d\xi.
$$

(A-3)
γ\(\phi\)(x) is finite. Hence, \(\gamma\)(x) is a testing function in the space \(K_x\) and we can apply to it the functional \(f(x)\).

Therefore the expression

\[(f(x), (g(y), \varphi(x,y)))\]  

(A-4)

is well defined. This is a certain functional over the space \(K_x\). From the continuity of the functionals \(\xi(x)\) and \(g(y)\) one can deduce the continuity of this functional.

This functional is designated by

\[h(z) = f(x) \times g(y)\]

and is called the direct product of the functional \(f(x)\) by the functional \(g(y)\).

The direct product has a particularly simple appearance when it is applied to a testing function \(\varphi(x,y)\), product of two testing functions \(\varphi_1(x)\) and \(\varphi_2(y)\). In this case, according to the definition

\[(f(x) \times g(y), \varphi_1(x) \varphi_2(y)) = (f(x), (g(y), \varphi_1(x) \varphi_2(y)))\]  

(A-5)

\[= (f(x), \varphi_1(x)(g(y), \varphi_2(y))) = (f(x), \varphi_1(x))(g(y), \varphi_2(y)).\]

If \(f(x)\) and \(g(x)\) are two absolutely integrable functions
functions of the form

\[
\sum_{j \neq 1} \mathcal{Q}_j(x) \mathcal{H}_j(y)
\]

where \( \mathcal{Q}_j(x) \), \( \mathcal{H}_j(y) \) \((j = 1, 2, \ldots, \nu; \nu = 1, 2, \ldots) \) are testing functions in their corresponding variables. We have then

\[
(f(x) \times g(y), \sum \mathcal{Q}_j(x) \mathcal{H}_j(y)) = \sum (f(x) \times g(y), \mathcal{Q}_j(x) \mathcal{H}_j(y))
\]

\[
= \sum (f(x), \mathcal{Q}_j(x))(g(y), \mathcal{H}_j(y)).
\]

In the same fashion

\[
(g(y) \times f(x), \sum \mathcal{Q}_j(x) \mathcal{H}_j(y)) = \sum (g(y), \mathcal{H}_j(y))(f(x), \mathcal{Q}_j(x)).
\]

and the result follows at once.

**Lemma 3.** If \( D \) be a differential operator then

\[
D(f \ast g) = Df \ast g = f \ast Dg.
\]

**Proof.**

We have

\[
(D(f \ast g), \varphi) = (f \ast g, D \ast \varphi)
\]
c) The supports of the functionals $f_\nu$ and $g$ are bounded on one and the same side in a manner independent of $y$.

Proof. 

According to the definition of convolution, for any testing function $\varphi$:

$$(f_\nu * g, \varphi) = (f_\nu(y), (g(x), \varphi(x + y)))$$

For the case (a) the function $(g(x), \varphi(x + y))$ can be modified to a function vanishing outside the set where all the $f_\nu(y)$ are concentrated. Hence

$$(f_\nu * g, \varphi) = (f_\nu(y), \varphi(y)) \rightarrow (f, \varphi) = (f * g, \varphi),$$

and

$$f_\nu * g \rightarrow f * g.$$  

For the case (b),

$$\varphi(y) = (g(x), \varphi(x + y))$$

is a testing function and the proof proceeds as above.

For case (c), if we suppose, to fix the ideas, that the support of the functionals $f_\nu$ and $g$ are bounded on the
\[
(f, \varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(\sigma) \exp(-ix\sigma) d\sigma \right) \left( \int_{-\infty}^{\infty} f(x) \varphi(x) dx \right) dx
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\sigma) \left( \int_{-\infty}^{\infty} f(x) \exp(ix\sigma) dx \right) d\sigma
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\sigma) \gamma(\sigma) d\sigma = \frac{1}{2\pi} (g, \gamma).
\]

This formula is valid when \( f(x) \) and \( \varphi(x) \), and hence their Fourier transforms \( g(\sigma) \) and \( \gamma(\sigma) \), are of integrable square.

Parseval's formula shows that \( g(\sigma) \) considered as a distribution, acts on a testing function according to the formula

\[
(g, \gamma) = 2\pi (f, \varphi).
\]  \hspace{1cm} (A-8)

This formula is used to define the distribution \( g \) over the space \( Z \) of testing functions \( \gamma \) corresponding to any given distribution \( f \) over the space \( K \). The functional \( g \) defined by Eq. (A-8), is called the Fourier transform of the functional \( f \) and is denoted by the symbols \( \tilde{f} \) or \( F[f] \).

The usual rules of differentiation of Fourier transforms carry over for Fourier transforms of distributions:

\[
P(\frac{d}{ds}) \tilde{f} = \tilde{F}[P(ix)\xi]
\]  \hspace{1cm} (A-9)

\[
F[P(\frac{d}{dx})f] = P(-is)\tilde{f}
\]  \hspace{1cm} (A-10)
Lemma 6. If $P(x)$ is a polynomial

$$
\hat{P}(x) = 2\pi \int \left( -i \frac{d}{ds} \right) \delta(s).
$$

Proof.

$$
\hat{P}(x) = \hat{[P(x) \cdot 1]} = \hat{P}(-i \frac{d}{ds} \delta(s)),
$$

and

$$
(1, \hat{\gamma}) = 2\pi (1, \gamma) = 2\pi \int_{-\infty}^{\infty} \gamma(x) dx = 2\pi \int_{-\infty}^{\infty} \gamma(x) \exp(-ix0) dx
$$

$$
= 2\pi \gamma'(0) = 2\pi \delta(0, \gamma)
$$

hence

$$
\hat{P}(x) = 2\pi \hat{P}(-i \frac{d}{ds} \delta(s)).
$$

Lemma 7. The Fourier transform of the exponential function $\exp(bx)$ is $2\pi \delta(s - ib)$.

Proof.

We will make use of the convergence in the complex plane of the series

$$
\exp(bx) = \sum_{n=0}^{\infty} \frac{b^n x^n}{n!}
$$
the equality

\[ \delta(s + h, \gamma(s)) = (\delta(s), \gamma(s - h)) = \gamma(-h) \quad (A-11) \]

for all (complex) \( h \).

Hence

\[ \exp(bx) = 2\pi \delta(s - ib) \quad (A-12) \]

for any complex \( b \).

**Fourier transform of a direct product.**

The Fourier transform of the functional \( f \), acting over the space \( K \) of testing functions \( \varphi(x) \) of several independent variables \( x = (x_1, x_2, \ldots, x_n) \), is defined as the functional \( g \) acting over the space \( Z \) of testing functions \( \gamma(x), s = (s_1, s_2, \ldots, s_n) \), by the formula

\[ (g, \gamma) = \int (2\pi)^n (f, \varphi) \quad (A-13) \]

where \( \gamma = \widehat{\varphi} \) is the Fourier transform of the function \( \varphi(x) \). The functional \( g \) is also designated as \( \widehat{f} \) or \( F[f] \).

Let \( f(x) \) and \( g(y) \) be given distributions over the variables \( x \) and \( y \) respectively, and let \( f(\widehat{\varphi}) \) and \( g(\widehat{\eta}) \) be
\[ \int_{-\infty}^{\infty} f(x) * g(x) \exp(i x \sigma') \, dx \]

can be considered as a distribution depending on a parameter \( \sigma' \) defined over the space of testing functions \( \exp(i x \sigma) \). According to Eq. (A-6) we have then

\[
(f * g, \exp(i x \sigma')) = (f(x) \wedge g(y), \exp(i \sigma(x + y))
\]

\[
= (f(x) \wedge g(y), \exp(i x \sigma') \exp(i y \sigma')) = (f(x), \exp(i x \sigma'))(g(x), \exp(i y \sigma'))
\]

\[
= f(\sigma')g(\sigma),
\]

where we used Eq. (A-5).

We have then

**Lemma 8.** The Fourier transform of the convolution

\[
f(x) * g(x) = \int f(\zeta) g(x - \zeta) \, d\zeta
\]

is given by

\[
F[f(x) * g(x)] = f(\sigma)g(\sigma).
\]
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