SOME EXTENSIONS OF BABINET'S PRINCIPLE
IN ELECTROMAGNETIC THEORY

by

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Summary

The concept of resistive and conductive sheets provides a meaningful extension of Babinet's principle to surfaces which are no longer perfect. The complementary problems are described and the appropriate field relations derived.

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Babinet arrived at the principle which now bears his name by comparing the diffraction pattern of an aperture with that of a complementary disk. It was later verified for scalar waves subject to a Neumann or Dirichlet boundary condition on the screen or disk, and extended to electromagnetic waves by Booker [1] who pointed out the polarization rotation of the primary field which is necessary if the screen and disk are both perfectly conducting.

These known forms of Babinet's principle are all consequences of the symmetry of the fields radiated by planar distributions and (where appropriate) the duality of electromagnetic fields. Over the years there have been a number of attempts to extend the principle to surfaces which are not perfect, for example, by Neugebauer [2] to surfaces which are absorbing, and more recently by Lang [3] to resistive surfaces. In Lang's extension the complementary structures are a perfectly conducting screen with a resistive insert and a resistive screen with a perfectly conducting insert, but the derivation has been criticized [4] for the assumptions made concerning the normal components of the field. Nevertheless, as noted by Baum and Singaraju [5], resistive sheets and their electromagnetic duals do afford an exact extension of Babinet's principle. This fact was exploited by Senior [6] in developing some generalised forms in acoustics, and we here discuss the analogous results for electromagnetic waves.

An electrically resistive sheet is simply an electric current sheet of infinitesimal thickness whose total current strength is proportional to the tangential electric field at its surface, and in recent years the concept of such a sheet has found several useful applications. The sheet is characterised by a jump discontinuity in the tangential components of the magnetic field across the surface, but no discontinuity in the tangential electric field, and therefore supports only an electric current. Its properties are completely specified by its resistivity and in the general case when this is anisotropic, the conditions at the surface of the sheet are

\[
\begin{align*}
[\mathbf{n} \times \mathbf{E}]^- &= 0 \\
2\pi \mathbf{R} \cdot [\mathbf{n} \times \mathbf{H}]^+ &= -\mathbf{n} \times (\mathbf{R} \times \mathbf{E}) \quad (1)
\end{align*}
\]
where \( \hat{n} \) is the outward unit vector normal to the positive side, \( Z\bar{\mathbf{R}} \) is the resistivity in ohms per square and \( Z = 1/\mathbf{Y} \) is the intrinsic impedance of free space. \( \bar{\mathbf{R}} \) is therefore the resistivity normalized relative to the free space impedance. When \( R = 0 \) the sheet is perfectly conducting and when \( R = \infty \) it is no longer there. The electromagnetic dual is a "magnetically conductive" sheet having a conductivity \( Y\bar{\mathbf{R}}^* \) mhos per square. Such a sheet has a jump discontinuity in the tangential electric field but none in the tangential magnetic field, and supports only a magnetic current. The conditions at its surface are the dual of (1), namely

\[
\left[ \hat{n}_A \mathbf{H} \right]^+ = 0 \tag{2}
\]

\[
Y\bar{\mathbf{R}}^* \cdot \left[ \hat{n}_A \mathbf{E} \right]^+ = \hat{n}_A \left( \mathbf{H} \right)
\]

and though it would be difficult to realize this type of sheet in practice, we can at least conceive of one. When \( R^* = 0 \) the sheet acts like a "perfect ferrite" whose permeability is infinite, and when \( R^* = \infty \) it ceases to exist.

By writing the second of the conditions (1) and (2) in terms of the sum fields on both sides of the sheet, we can also define a combination resistive-conductive sheet at which the conditions are

\[
2Z\bar{\mathbf{R}} \cdot \left[ \hat{n}_A \mathbf{H} \right]^+ = -\hat{n}_A \left( \mathbf{E}^{(+)} + \mathbf{E}^{(-)} \right) \tag{3}
\]

\[
2Y\bar{\mathbf{R}}^* \cdot \left[ \hat{n}_A \mathbf{E} \right]^+ = \hat{n}_A \left( \mathbf{H}^{(+)} + \mathbf{H}^{(-)} \right)
\]

and for a sheet lying in the plane \( z = 0 \) these equations imply

\[
E_x(x, y, \pm 0) = -Z \left( R_x \cdot \frac{1}{4\mathbf{R}} \right) H_y(x, y, +0) + Z \left( R_x \cdot \frac{1}{4\mathbf{R}} \right) H_y(x, y, -0) \tag{4}
\]

\[
E_y(x, y, \pm 0) = Z \left( R_y \cdot \frac{1}{4\mathbf{R}} \right) H_x(x, y, +0) - Z \left( R_y \cdot \frac{1}{4\mathbf{R}} \right) H_x(x, y, -0)
\]

where

\[
\bar{\mathbf{R}} = R_x \hat{x}\hat{x} + R_y \hat{y}\hat{y}, \quad \bar{\mathbf{R}}^* = R_x^* \hat{x}\hat{x} + R_y^* \hat{y}\hat{y}.
\]
We observe that if

\[ 4R \frac{R^*}{x} = 1 = 4R \frac{R^*}{y} \]

the eq. (4) are identical to the Leontovich boundary conditions at a sheet whose relative surface impedance is

\[ \bar{\eta} = 2R \]

on both sides. With any combination sheet, both types of current contribute to the tangential electric or magnetic field, but only for the particular combination (5) is the sheet equivalent to an impedance sheet.

We consider first an electrically resistive sheet occupying the entire plane \( z = 0 \) of a Cartesian coordinate system \((x, y, z)\) and illuminated by a primary field \( E^1 = F, \ H^1 = G \) resulting from sources in \( z < 0 \). The resistivity of the sheet may vary from point to point and will be assumed anisotropic with \( R \) having the form (4). To find the total field \( E, H \), we note that in the half space \( z \geq 0 \) the field can be obtained from the electric and magnetic Hertz vectors

\[ \begin{align*}
\frac{\partial}{\partial x} = 0, & \quad \frac{\partial}{\partial y} = 0, \\
\frac{\partial}{\partial z} = 0, & \quad \frac{\partial}{\partial x'} = 0, \\
E^s(x) = \frac{2i\pi}{k} \int \int \frac{K^s(x',y')g(x,y')}{x^2 - y^2} dx' dy' 
\end{align*} \]

where

\[ g(x,y') = (4\pi \alpha)^{-1} \ e^{-\alpha D} \]

and

\[ D = (x - x')^2 + (y - y')^2 + z^2 \]

and

\[ K^s = -\hat{x} \cdot H^1 \]

is the magnetic current density on the surface \( z = +0 \) of integration. A time factor \( e^{-i\omega t} \) has been assumed and suppressed. In \( z \geq 0 \),
\[
\mathcal{E}^{(1)} = 2 \iint K^{*} \cdot \nabla g \, dx \, dy
\]

\[
\mathcal{H}^{(1)} = 2i \kappa Y \iint \left\{ K^{*} g + k^{-2} (K^{*} \cdot \nabla g) \cdots \right\} \, dx \, dy
\]

where the differentiation is with respect to the unprimed coordinates of the observation point, and since the tangential (normal) components of \( \mathcal{E}^{(1)} \), \( \mathcal{H}^{(1)} \) are symmetrical about the plane \( z = 0 \) with the other components asymmetrical, extension to the whole space gives

\[
\mathcal{E}^{(1)} = F + \left\{ \frac{F_\Gamma}{F} \right\} + \frac{1}{2} \iint K^{*} \cdot \nabla g \, dx \, dy
\]

\[
\mathcal{H}^{(1)} = G + \left\{ \frac{G_\Gamma}{G} \right\} + 2i \kappa Y \iint \left\{ K^{*} g + k^{-2} (K^{*} \cdot \nabla g) \cdots \right\} \, dx \, dy.
\]

Here and throughout the rest of the paper, the upper and lower sign alternatives apply to the regions \( z < 0 \) and \( z > 0 \) respectively, and \( F_\Gamma, G_\Gamma \) is the field reflected by a perfectly conducting plane \( z = 0 \) when the field \( F, G \) is incident upon it. Thus

\[
F_\Gamma^R, y(x, y) = - F_{x, y}(\bar{x}) \quad \text{,} \quad F_z^R(x) = F_z(\bar{x})
\]

\[
G_\Gamma^R, y(x, y) = G_{x, y}(\bar{x}) \quad \text{,} \quad G_z^R(x) = - G_z(\bar{x})
\]

where \( \bar{x} \) is the image of the point \( x \) in the plane \( z = 0 \). It only remains to enforce the second of the conditions (1) and when this is done, the following integral equation is obtained:

\[
\frac{1}{2} Y \left( \frac{K^{*}}{R} x + \frac{K^{*}}{R} y \right) = - G + 2i \kappa Y \lim_{z \to 0} \iint \left\{ K^{*} g + k^{-2} (K^{*} \cdot \nabla g) \cdots \right\} \, dx \, dy
\]
holding at all points in the plane $z = 0$.

There are two possible problems which are complementary to the above. The first of these has the same field $\mathbf{E}^1 = \mathbf{E}$, $\mathbf{H}^1 = \mathbf{G}$ incident on the dual of the electrically resistive sheet, namely, a magnetically conductive sheet occupying the plane $z = 0$. Instead of proceeding as before, we now express the field scattered by the sheet in terms of the total current it supports. Since there is no electric current $\mathbf{J}$,

$$\nabla \cdot \mathbf{E} = 0 \quad , \quad \mathbf{r}^* = \frac{iY}{k} \iint \mathbf{J}^* \mathbf{g}(\mathbf{r}, \mathbf{r}') dx'dy'$$

where

$$\mathbf{J}^* = - [\hat{\mathbf{E}} \times \mathbf{E}]^+$$

is the total magnetic current, and the entire field at all points in space is then

$$\mathbf{E}^{(2)} = \mathbf{E} + \iint \mathbf{J}^* \mathbf{g} dx'dy'$$

(8)

$$\mathbf{H}^{(2)} = \mathbf{G} + ikY \iint \{\mathbf{J}^* \mathbf{g} + k^{-2}(\mathbf{J}^* \cdot \nabla) \mathbf{g}\} dx'dy' .$$

On applying the second of the conditions (2), an integral equation from which to determine $\mathbf{J}^*$ is found to be

$$\iint \mathbf{J}^* \mathbf{g} + k^{-2}(\mathbf{J}^* \cdot \nabla) \mathbf{g} dx'dy'$$

(9)

and this is identical to (7) if

$$2 \mathbf{R}^* = \frac{1}{2} \left( \frac{1}{R_x} \right) \mathbf{A} + \frac{1}{R_x} \mathbf{A}$$

(10)
in which case
\[ J^* = -2K^* \, . \] (11)

It now follows that if the magnetic conductivity is related to the electric resistivity via (5), the total fields in the two problems are also related, and from eqs. (6) and (8)
\[
\begin{align*}
(E(2) - F) + (E(1) - F) &= - \left\{ \frac{F^*}{E} \right\} \\
(H(2) - G) + (H(1) - G) &= - \left\{ \frac{G^*}{G} \right\} .
\end{align*}
\] (12)

A particular case of this result is an aperture (where \( R = \infty \)) in a perfectly conducting screen (\( R = 0 \)), and the complementary structure for the same incident field is a perfect ferrite disk (\( R = 0 \)) coincident with the aperture. This is the direct analogue of the standard form of Babinet's principle in acoustics.

Another problem which is also complementary to the first has an electrically resistive sheet of resistivity \( ZR \) in the plane \( z = 0 \) but a different incident field. If \( E(1) = -ZG, H(1) = YF \), we can express the field scattered by the sheet in terms of the Hertz vectors
\[ \pi(r) = \frac{iZ}{k} \iint J_g(r|r')dx'dy', \quad \pi^*(r) = 0 \]

where \( J \) is the total electric current, and the complete field is then
\[
\begin{align*}
E(3) &= -ZG + ikZ \iint \{ J_g + k^{-2}(J \cdot \nabla)g \} dx' dy' \\
H(3) &= YF - \iint J \wedge g dx' dy' .
\end{align*}
\] (13)
An integral equation for $\mathbf{J}$ is

$$\mathbf{R}^{\star} \cdot \mathbf{J} = - G + i k \lim_{z \to 0} \iint \{ \mathbf{J} g + k^{-2} (\mathbf{J} \cdot \nabla) \nabla g \} \, dx' dy', \quad (14)$$

and if

$$2 \mathbf{R}^{\star} = \mathbf{I}_{x} \left( \frac{1}{\mathbf{R}} \frac{\partial}{\partial x} + \frac{1}{\mathbf{R}} \frac{\partial}{\partial y} \right). \quad (15)$$

(14) is identical to (7), in which case

$$\mathbf{J} = 2Y \mathbf{K}^{\star}. \quad (16)$$

The resistivity $\mathbf{R}^{\star}$ bears the same relationship to $\mathbf{R}$ as did the conductivity $\mathbf{R}^{\star}$ in the second problem, but we have now avoided the introduction of a magnetically conductive sheet at the expense of a change in the incident field. With the identification (16) it follows from (6) and (13) that

$$\left( Y \mathbf{E}^{(3)} + \mathbf{G} \right) \pm \left( H_{1} - \mathbf{G} \right) = \begin{cases} \mathbf{G}, \\ f \mathbf{G} \end{cases} \quad (17)$$

$$\left( Z \mathbf{H}^{(3)} - \mathbf{F} \right) \pm \left( \mathbf{E}_{1} - \mathbf{F} \right) = - \begin{cases} \mathbf{F}, \\ f \mathbf{F} \end{cases}.$$

The relation (15) between the resistivity tensors in the two problems is identical to that derived by Baum and Singaraju [5] using their combined field formulation, and is also satisfied by the driving point impedances of a dipole and its complementary slot. Since $\mathbf{R}^{\star} = \infty (0)$ when $\mathbf{R} = 0 (\infty)$, the problem complementary to that of an aperture in a perfectly conducting screen is a perfectly conducting disk coincident with the aperture. This second problem-pairing therefore includes the standard electromagnetic form of Babinet's principle as a special case, and the eqs. (17) provide the usual connection between the fields.

In contrast to the screens considered by Lang [3], those forming our complementary structures must have resistivities and conductivities satisfying
(10) or (15) at all pairs of corresponding points. Resistive sheets having (normalized) resistivities up to about 3 are readily available and have found useful application for cross section reduction purposes. A purely conductive sheet would be more difficult to realize, but it is possible that this could be done over a limited frequency range at least.

Though it would be natural to seek a Babinet principle for a combination sheet (which includes an impedance sheet as a particular case), it seems unlikely that such a principle exists. The resulting scattered fields no longer have the symmetry properties that characterize individual electric and magnetic current sheets, and as regards the type of proof presented here, the procedure fails in the deduction of $E^{(1)}$ and $H^{(1)}$ from their values in the half space $z \geq 0$. This is not surprising since an impedance sheet is opaque whereas resistive and conductive ones are not.
References


