RAYLEIGH SCATTERING--APPLICATIONS AND EXTENSIONS

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I am going to describe some of the applications and extensions of the Rayleigh scattering formulas that are not explicitly discussed in the Chapter that Ralph Kleinman has just surveyed. For brevity, I will confine my remarks to the electromagnetic case.

The Rayleigh term (proportional to $k^2$) is just the leading term in the low frequency expansion of the far field in powers of $\omega$ or the wavenumber $k$, but for a body of small electrical size, this may be sufficient to adequately approximate the field. The term is attributable to induced electric and/or magnetic dipoles whose moments can be expressed in terms of polarizability tensors which are functions of the body's geometry and electrical properties, and the tensor elements are themselves given by certain weighted integrals of static potentials.

There are several advantages to this rather simple result:

(i) analytical expressions -- since there are more geometries for which Laplace's equation can be solved than is true for the wave equation, there are more analytical expressions for the tensor elements available.

(ii) existence of bounds -- in many instances the tensor elements can be bounded and these may be sufficiently tight to avoid the need for explicit determination of the elements.
(iii) weak shape dependence -- often (but not always) the elements are only weak functions of the body's shape, which may allow us to replace the body by a simpler shape, e.g., a spheroid, for which analytical results are available.

(iv) convenience -- in many cases the ultimate objective is to determine the scattering and/or absorption by a distribution of many particles. Whether we seek to represent this dispersion by an equivalent dielectric, or use one of the numerical radiation transfer codes that are available, the basis is the scattering from a single particle. The ability, in Rayleigh scattering, to separate out the incident field, is a considerable convenience.

The Rayleigh results for dielectric bodies (usually assumed homogeneous and lossy, but non-metallic) are widely used in studying scattering and absorption by aerosols, and by atmospheric particles in general, particularly at frequencies in the microwave, infrared and, possibly, optical regimes. Some of the objectives are to examine the effects of particle shape and composition; to see how, for a given material, the shape can be chosen to maximize the absorption, and so on. In doing this, it is assumed that the particle is so small as to lie in the Rayleigh region for the band of frequencies of interest, and then to use the frequency dependent permittivity of the material to explore the scattering and absorption over that band.
Many common materials have molecular resonances in the infrared and optical frequency region, and some interesting effects come about when we examine Rayleigh scattering over a frequency band that includes a bulk resonance of the material. It is found that the particle shape now has a critical effect on the absorption—shifting the frequency at which the absorption line or peak occurs and splitting it into many lines. These occur when the real part of the relative permittivity is negative. There is now a growing body of literature concerned with this effect, and its interpretation in terms of bulk and surface polariton modes in the particle. Suffice to say that the codes which have been developed to compute the Rayleigh scattering from dielectric particles of various shapes have been found effective in examining the nature and excitation of these modes.

As previously remarked, the Rayleigh term is just the leading (zeroth order) term in the low frequency expansion of the far field. The higher order in \( k \) terms are attributable to the effect of quadrupoles and multipoles in general and a knowledge of them should provide more information about the scattering. You can easily develop certain relationships among the coefficients of successive powers of \( k \), but to determine the coefficients explicitly requires the solution of additional boundary value problems. In acoustics, the task is reasonably straightforward and in 1952 Stevenson showed that in electromagnetics the task could be reduced to the solution of certain potential problems. There were some ambiguities about the procedure which Kleinman eliminated in his 1967 paper and as a result

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of this it might appear that to obtain the successive terms required only the solution of Dirichlet and Neumann potential problems, but with ever more complicated forcing or excitation functions at each stage. Unfortunately, things are not quite what they seem, and once you get beyond the leading (or zeroth order) term, difficulties are experienced which are fundamental as well as practical. I can illustrate these by considering the task of finding the $k^3$ (first order) term for a perfectly conducting body.

If the near fields are expanded in powers of $ik$, e.g.,

$$\vec{E} = \vec{E}_0 + ik\vec{E}_1 + \ldots,$$

it is seen that

$$\nabla \cdot \vec{E}_0 = 0$$

implying the existence of a scalar potential $\phi_0$ such that

$$\vec{E}_0 = -\nabla \phi_0.$$

The boundary condition on the scattered electric field is

$$\hat{n} \cdot \vec{E}_0 = \hat{n} \cdot \vec{E}^{inc}_0$$

giving, to the zeroth order in $ik$, 

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\[ \hat{n} \cdot \nabla \phi_0 = -\hat{n} \cdot \nabla \phi_0^{\text{inc}} \]

which can be integrated to give

\[ \phi_0 = -\phi_0^{\text{inc}} + c , \]

where \( c \) is a constant determined by the condition for charge neutrality. Thus, \( \phi_0 \) can be expressed as a single layer distribution, leading to a simple Dirichlet problem for the determination of the surface charge density. Similarly, the boundary condition

\[ \hat{n} \cdot \vec{H} = -\hat{n} \cdot \vec{H}^{\text{inc}} \]

leads to a Neumann problem for the magnetostatic potential \( \psi_0 \).

Since

\[ \nabla \cdot \vec{E}_1 = Z \vec{H}_0 \neq 0 , \]

\( \vec{E}_1 \) cannot be expressed as the gradient of an exterior potential, but we can write

\[ \vec{E}_1 = \vec{F}_1 - \nabla \phi_1 , \]

where \( \vec{F}_1 \) is a particular solution of the equation

\[ \nabla \cdot \vec{F}_1 = -\nabla \psi_0 \]
Because \( \psi_0 \) is naturally expressible as a double layer distribution, a solution can be obtained as

\[
\tilde{F}_1(\tilde{r}) = \nabla \cdot \frac{1}{4\pi} \int \hat{n}' \psi_0(\tilde{r}') \frac{1}{|\tilde{r} - \tilde{r}'|} \, dS',
\]

and we remark that this is valid even for a shell or flat plate of infinitesimal thickness (and volume). It now only remains to determine \( \phi_1 \) and this can again be expressed as a single layer potential. Unfortunately, the boundary condition is

\[
\hat{n} \cdot \nabla \phi_1 = \hat{n} \cdot (\tilde{F}_1 + \tilde{E}_1^{inc})
\]

and since this cannot be integrated to yield an explicit expression for \( \phi_1 \), it is now a non-trivial task, either analytically or numerically, to solve the resulting integral equation.

The situation is not substantially better (and sometimes worse) for the first order magnetic field \( \tilde{H}_1 \). If we write

\[
\tilde{Z}H_1 = \tilde{G}_1 - \nabla \psi_1,
\]

then

\[
\nabla \cdot \tilde{G}_1 = \nabla \phi_0.
\]

Since \( \phi_0 \) is naturally expressed as a single layer distribution, one method for solving this equation is simply to write the single layer
distribution as a double layer one. This is possible for a body of non-zero volume (it needs only the solution of an interior Neumann problem), but is not possible for a shell or plate of infinitesimal thickness. Thus, for a plate (and this is certainly a case of practical interest), the determination of \( \tilde{H}_1 \) using potential theory founders on our inability to find \( \tilde{G}_1 \). Of course, one can always go back to the dynamic problem and solve it numerically, but the purpose of the low frequency expansion was to simplify the analytical and numerical tasks. Unfortunately, that simplification is not necessarily achieved once you go beyond the leading (or zeroth order) terms in the expansion.