INTEGRAL EQUATIONS FOR THE SCATTERING OF A PLANE WAVE
BY AN ELECTRICALLY AND MAGNETICALLY PERMEABLE BODY

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Abstract

The integral equations governing the scattering of a plane electromagnetic wave by an electrically and magnetically permeable body are derived with the aid of the free-space electric dyadic Green functions. In contrast to previous works, equivalent volume currents or surface currents are not introduced.
The General Formulation

The problem under consideration is illustrated in Fig. 1 where Region 1 is occupied by an electrically and magnetically permeable body with constitutive constants \( \mu_1 \) and \( \varepsilon_1 \) where \( \varepsilon_1 \) equal to \( \varepsilon_0[1 + i(\sigma/\omega\varepsilon)] \) for a lossy dielectric body. In general, \( \mu_1 \) could be complex also for a lossy magnetized body. A plane wave is impending upon the body which is placed in air with constitutive constants \( \mu_0 \) and \( \varepsilon_0 \). This problem has previously been investigated by many authors. References [I] and [II] provided a quite complete bibliography on this problem.

In this work we shall formulate the problem without introducing the concepts of equivalent electrically and magnetically polarized currents, the result appears to be simpler and the derivation more expedient. The pertinent equations are:

**Region (I):**

\[
\nabla \times \mathbf{E}_1 = i\omega \mu_1 \mathbf{H}_1
\]

\[
\nabla \times \mathbf{H}_1 = -i\omega \varepsilon_1 \mathbf{E}_1,
\]

hence

\[
\nabla \times \nabla \times \left( \begin{array}{c} \mathbf{E}_1 \\ \mathbf{H}_1 \end{array} \right) - k_1^2 \begin{array}{c} \mathbf{E}_1 \\ \mathbf{H}_1 \end{array} = 0
\]

(1)

(2)

where

\[
k_1^2 = \omega^2 \mu_1 \varepsilon_1 = \omega^2 \mu_1 \varepsilon_1 \left( 1 + \frac{i\sigma}{\omega\varepsilon_1} \right)
\]
Region (II)

\[ \nabla \times \mathbf{E}_2 = i \omega \mu_0 \mathbf{A}_2 \]

\[ \nabla \times \mathbf{A}_2 = -i \omega \epsilon_0 \mathbf{E}_2 \]

hence

\[ \nabla \times \nabla \times \left( \mathbf{E}_2 \right) - k_0^2 \left( \mathbf{E}_2 \right) = 0 \]  \hspace{1cm} (3)

where \( k_0^2 = \omega^2 \mu_0 \epsilon_0 \).

Equations (1-4) can now be integrated with the aid of the vector-
dyadic Green's theorem and the free-space electric dyadic Green's
function defined in air \((\mu_0, \epsilon_0)\) which satisfies the equation

\[ \nabla \times \nabla \times \mathbf{G}_0 - k_0^2 \mathbf{G}_0 = \mathbf{I}_0 \left( \mathbf{R} - \mathbf{R}' \right) \] \hspace{1cm} (5)

The vector-dyadic Green's theorem states

\[ \iiint_V \left[ \mathbf{P} \cdot \nabla \times \nabla \times \mathbf{G} - \left( \nabla \times \nabla \times \mathbf{P} \right) \cdot \nabla \mathbf{G} \right] dV = - \int_S \nabla \times \mathbf{G} \cdot \mathbf{n} \cdot \left[ \mathbf{P} \times \nabla \times \mathbf{G} \right. \]

\[ + \left. \left( \nabla \times \mathbf{P} \right) \times \mathbf{G} \right] dS \] \hspace{1cm} (6)

where \( \mathbf{n} \) denotes the outward normal to \( S \). Now let

\[ \mathbf{P} = \mathbf{E}_1 \] \hspace{1cm} \( \mathbf{G} = \mathbf{G}_0 \left( \mathbf{R}/\mathbf{R}' \right) \)

with \( \mathbf{R}' \) located in Region (I). Substituting (1) and (5) into (6) we obtain
\[ E_1(R') = (k_1^2 - k_0^2) \iiint_{V_1} E_1(R) \cdot \vec{\sigma}_0(R/R') \, dV \]

\[ - \oiint_{S_1} \hat{n}_1 \cdot [E_1(R) \times \nabla \times \vec{\sigma}_0(R/R') + (\nabla \times E_1(R)) \times \vec{\sigma}_0(R/R')] \, dS \quad (7) \]

Following the same procedure for \( A_1 \) we obtain

\[ A_1(R') = (k_1^2 - k_0^2) \iiint_{V} A_1(R) \cdot \vec{\sigma}_0(R/R') \, dV \]

\[ - \oiint_{S} \hat{n}_1 \cdot [A_1(R) \times \nabla \times \vec{\sigma}_0(R/R') + (\nabla \times A_1(R)) \times \vec{\sigma}_0(R/R')] \, dS \quad (8) \]

Equations (7) and (8) are compatible. In other words, Eq. (8) can be derived from Eq. (7) by using the relation \( \nabla \times E_1(R') = i\omega \mu \, A_1(R') \). The proof is omitted here.

By integrating Eq. (3) in Region II using the same \( \vec{\sigma}_0(R/R') \) with \( R' \) located in Region (I) we obtain

\[ \oiint_{S + S_{\infty}} \hat{n}_2 \cdot [E_2 \times \nabla \times \vec{\sigma}_0(R/R') + (\nabla \times E_2(R)) \times \vec{\sigma}_0(R/R')] \, dS = 0 \quad (9) \]

The surface integral evaluated on \( S_{\infty} \) has a simple interpretation. If we consider a plane wave propagating in an empty space the integration of Eq. (3) with \( \vec{\sigma}_0(R/R') \) would yield
\[ E^{(i)}(R') = - \oint_{S_\infty} \hat{n}_2 \cdot [E^{(i)} \times \nabla \times \vec{E}_0 + (\nabla \times E^{(i)}) \times \vec{E}_0] dS \quad (10) \]

because in the absence of a scattering body \( E^2 = E^{(i)} \). In the presence of the scattered body we can write

\[ E_2 = E^{(i)} + E^{(s)} \quad (11) \]

\[ E_1 = E^{(i)} + E^{(s)} \quad (12) \]

The surface integral evaluated on \( S_\infty \) in (9) can be decomposed into two parts, i.e.,

\[ \iint_{S_\infty} \hat{n}_2 \cdot [E^{(i)} \times \nabla \times \vec{E}_0 + (\nabla \times E^{(i)}) \times \vec{E}_0] dS \]

\[ + \iint_{S_\infty} \hat{n}_2 \cdot [E^{(s)}_2 \times \nabla \times \vec{E}_0 + (\nabla \times E^{(s)}_2) \times \vec{E}_0] dS \]

Because of the radiation condition of \( \vec{E}_0 \) and \( E^{(s)} \) at infinity the second integral involving \( E^{(s)} \) vanishes while the first integral, in view of Eq. (10), represents \(-E^{(i)}(R)\) hence Eq. (9) is equivalent to

\[ E^{(i)}(R') = \oint_{S} \hat{n}_2 \cdot [E_2 \times \nabla \times \vec{E}_0 + (\nabla \times E_2) \times \vec{E}_0] dS \quad (13) \]

where we have omitted the dependent variables pertaining to various terms in the integrand. Similarly it can be shown
\[ R^{(1)}(R') = \oint_S \hat{n}_1 \cdot [A_2 \times \nabla \times \vec{E}_0 + (\nabla \times A_2) \times \vec{E}_0] dS \]  (14)

Using these two relations Eqs. (7) and (8) can be changed to an alternative form involving the incident field. We consider the surface term in Eq. (7). On \( S \) the boundary conditions are

\[ \hat{n}_1 \times (E_1 - E_2) = 0 \]  (15)

\[ \hat{n}_1 \times \left[ \frac{\nabla \times E_1}{\mu_1} - \frac{\nabla \times E_2}{\mu_0} \right] = 0 \]  (16)

hence

\[ \hat{n}_1 \times \nabla \times E_1 = \left( \frac{\mu_1}{\mu_0} \right) \hat{n}_1 \times \nabla \times E_2 \]

\[ = \left[ 1 + \left( \frac{\mu_1 - \mu_0}{\mu_0} \right) \right] \hat{n}_1 \times \nabla \times E_2 \]

The surface term in Eq. (7) therefore can be written in the form

\[ - \oint_S \hat{n}_1 \cdot (\nabla \times E_2) \times \vec{E}_0 + (\nabla \times E_2) \times \vec{E}_0] dS \]

\[ = \oint_S \hat{n}_1 \cdot [(\nabla \times E_2) \times \vec{E}_0] dS \]

\[ = E^{(i)}(R') - \left( \frac{\mu_1 - \mu_0}{\mu_1} \right) \oint_S \hat{n}_1 \cdot [(\nabla \times E_1) \times \vec{E}_0] dS \]  (17)

Equation (7) thus can be transformed to
\[
E_1 (R') - E^{(1)}_1 (R') = (k^2 - k_0^2) \iiint_{V_1} E_1 (R) \cdot \vec{G}_o (R/R') \, dV
\]

\[- \left( \frac{\mu_1 - \mu_0}{\mu_1} \right) \oiint_{S_1} \hat{n}_1 \cdot \left[ (\nabla \times E_1) \times \vec{G}_o (R/R') \right] \, dS \quad \text{(18)}
\]

By switching the primed and unprimed variables and making use of the symmetrical property of dyadic Green's function Eq. (18) can finally be written in the form

\[
E^{(s)}_1 (R) = (k^2 - k_0^2) \iiint_{V_1} \vec{G}_o (R/R') \cdot E_1 (R') \, dV'
\]

\[- \left( \frac{\mu_1 - \mu_0}{\mu_1} \right) \oiint_{S_1} \vec{G}_o (R/R') \cdot \left[ \hat{n}_1 \times v' \times E_1 (R') \right] \, dS' \quad \text{(19)}
\]

where \( E^{(s)}_1 = E_1 - E^{(1)}_1 \). Carrying out the same manipulation for \( A_1 \) one finds

\[
A^{(s)}_1 (R) = (k^2 - k_0^2) \iiint_{V_1} \vec{G}_o (R/R') \cdot A_1 (R') \, dV'
\]

\[- \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1} \right) \oiint_{S_1} \vec{G}_o (R/R') \cdot \left[ \hat{n}_1 \times v' \times A_1 (R') \right] \, dS' \quad \text{(20)}
\]

where \( H^{(s)}_1 = A_1 - A^{(1)}_1 \).

For a purely permeable dielectric body the surface integral disappears in Eq. (19) and for a purely permeable magnetized body the surface integral disappears in Eq. (20). To solve these equations...
numerically one can split the volume integral into an indented part and a principal part.

Once \( E_1 \) and \( H_1 \) are determined the scattered field in Region (II) can be calculated using the formula

\[
E_2^{(s)}(R) = -\oint_S \nabla \times \vec{E}_0 \cdot \left[ \hat{n}_1 \times E_1 \right] + \frac{\mu_0}{\mu_1} \vec{E}_0 \cdot \left[ \hat{n}_1 \times \nabla \times E_1 \right] dS
\]

(21)

**Inhomogeneous Dielectric Body**

For an inhomogeneous dielectric body, nonmagnetic, the governing equations are

\[
\nabla \times E_1 = i\omega \mu_0 H_1
\]

(22)

\[
\nabla \times H_1 = -i\omega \varepsilon(R) E_1
\]

(23)

hence

\[
\nabla \times \nabla \times E_1 - k^2(R) E_1 = 0
\]

(24)

where

\[
k^2(R) = \omega^2 \mu_0 \varepsilon(R)
\]

The integration of (24) with the aid of the free space electric dyadic Green's function yields
\[ E^{(s)}_1(R) = E_1(R) - E^{(i)}_1(R) = \iiint_{V_1} \left[ k^2(R) - k_0^2 \right] \hat{E}_0(R/R') \cdot E_1(R') dV' \] 

which is an exact integral equation for \( E_1(R) \) inside an inhomogeneous dielectric body.
References


Fig. 1: Scattering of a Plane Wave by a Permeable Body.