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Complementary Reciprocity Theorems in Electromagnetic Theory

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Abstract

Two complementary reciprocity theorems have been derived in this work. They are distinct from the well-known reciprocity theorems of Rayleigh-Carson and Lorentz. An application of one of the theorems to a radiation problem is given. A one-dimensional version of the theorems is introduced first by using transmission lines as the models to illustrate some of the key concepts in the theory.

1 Introduction

There are two well-known reciprocity theorems in electromagnetic theory; one due to Rayleigh and Carson and another due to Lorentz. For an infinite domain with an isotropic medium, such as free-space, the derivation of these theorems including the fictitious magnetic currents have been treated in detail by Van Bladel [1]. They are also valid for multiple media in contact. When there is an electrically perfect conductor in one of the multiple media, such as the problem of a layered dielectric placed above a ground plane, we need a reciprocity theorem involving the magnetic fields in order to derive the symmetrical relationships of the magnetic dyadic Green functions in the formulation. In this paper, we will derive such a theorem and apply it to find the symmetrical relationships.

The theorem involves the concept of two complementary sets of fields. In order to comprehend better the significance of the complementary sets we will first give a one-dimensional version of the theorem using the transmission line as the model before the full theory for a three-dimensional electromagnetic field is presented.

2 Transmission Line Model of the Complementary Reciprocity Theorems

We consider two identical sections of transmission lines ($d \geq x \geq 0$). Line (a) is short circuited at $x = 0$ and terminated by a load impedance Z_a. Line (b) is open-circuited at $x = 0$, and terminated by a load impedance Z_b. Each line is excited by a distributed current source, denoted, respectively, by $K_a(x)$ and $K_b(x)$ as shown in Fig. 1. The differential equations governing the line current and voltage of these two lines are:

$$\frac{di_a(x)}{dx} = i\omega C v_a(x) + K_a(x) \tag{1}$$

$$\frac{dv_a(x)}{dx} = i\omega L i_a(x) \tag{2}$$

$$\frac{di_b(x)}{dx} = i\omega C v_b(x) + K_b(x) \tag{3}$$

$$\frac{dv_b(x)}{dx} = i\omega L i_b(x) \tag{4}$$

We are dealing with harmonically oscillating current and voltage with a time factor $e^{-i\omega t}$ for complex quantities in time domain. In (1)-(4), L and C are the line constants. The boundary conditions for the line voltages and currents are:

$$v_a(0) = 0, \qquad v_a(d) = Z_a i_a(d) \tag{5}$$

$$i_b(0) = 0, \qquad v_b(d) = Z_b i_b(d) \tag{6}$$

By multiplying (1) by $i_b(x)$ and (3) by $i_a(x)$, adding the two resultant equations, and making use of (2) and (4) we obtain

$$\begin{aligned} &K_a(x)i_b(x) + K_b(x)i_a(x) \\ &= \frac{d}{dx}\left[i_a(x)i_b(x) - v_a(x)v_b(x)/Z_c^2\right] \end{aligned} \tag{7}$$

where $Z_C = (L/C)^{\frac{1}{2}}$, denoting the characteristic impedance of the line. An integration of (7) with respect to x from $x = 0$ to $x = d$ yields

$$\begin{aligned} &\int_0^d \left[K_a(x)i_b(x) + K_b(x)i_a(x)\right] dx \\ &= \left[i_a(x)i_b(x) - v_a(x)v_b(x)/Z_c^2\right]_0^d \end{aligned} \tag{8}$$

In view of the boundary conditions at the terminals, (8) can be written in the form

$$\int_0^d \left[K_a(x)i_b(x) + K_b(x)i_a(x)\right] dx = i_a(d)i_b(d)\left(1 - Z_a Z_b/Z_c^2\right) \tag{9}$$

Now if we impose a relationship between the load impedances such that

$$Z_a Z_b = Z_c^2 \tag{10}$$

then

$$\int_0^d [K_a(x)i_b(x) + K_b(x)i_a(x)]\, dx = 0 \tag{11}$$

Equation (11) is designated as the complementary K–i reciprocity theorem for the transmission lines and (10) as the complementary impedance condition. It should be remarked that if we apply the Rayleigh-Carson reciprocity theorem to a single section of line excited with two distributed current sources, $K_a(x)$ and $K_b(x)$, independently, we can obtain the relation

$$\int_0^d [K_a(x)v_b(x) - K_b(x)v_a(x)]\, dx = 0 \tag{12}$$

for any terminations at both ends of a single line. Equations (11) is an independent theorem; it cannot be derived from the Rayleigh-Carson theorem. The fact that the K–i theorem involves the currents on two complementary lines with different terminal conditions is an evidence of its independence.

If an integration is applied to (7) covering the regions outside of both $K_a(x)$ and $K_b(x)$ we obtain

$$\begin{aligned} &\left[i_a(x)i_b(x) - v_a(x)v_b(x)/Z_c^2\right]_{a_2}^{a_1} \\ &+ \left[i_a(x)i_b(x) - v_a(x)v_b(x)/Z_c^2\right]_{b_2}^{b_1} = 0 \end{aligned} \tag{13}$$

where a_1 and a_2 denote the extremities of the span of $K_a(x)$ and b_1 and b_2 that of $K_b(x)$. Equation (13) is designated as the complimentary v–i reciprocity theorem for the transmission lines or the $(v\text{–}i)_c$ theorem for short, in contrast to the reciprocity theorem

$$[i_a(x)v_b(x) - i_b(x)v_a(x)]_{a_2}^{a_1}$$

$$+ \left[i_a(x) v_b(x) - i_b(x) v_a(x) \right]_{b_2}^{b_1} = 0 \tag{14}$$

which can be derived by applying the Lorentz reciprocity theorem to a single section of line with arbitary terminations excited independently by $K_a(x)$ and $K_b(x)$.

When the current sources are localized such that

$$K_a(x) = I_a(x_a)\, \delta(x - x_a) \tag{15}$$

$$K_b(x) = I_b(x_b)\, \delta(x - x_b) \tag{16}$$

where $\delta(x - x_a)$ denotes a delta function defined at x_a and similarly for $\delta(x - x_b)$, the K–i theorem yields the circuit relation

$$I_a(x_a)\, i_b(x_b) = I_b(x_b)\, v_a(x_b) \tag{17}$$

while the K–v theorem of Rayleigh-Carson yields

$$I_a(x_a)\, v_b(x_b) = -I_b(x_b)\, i_a(x_b) \tag{18}$$

Historically, Rayleigh formulated the reciprocity theorem for electrically passive networks first and it was Carson who generalized it to the electromagnetic field is an isotropic medium, hence, the name of Rayleigh-Carson theorem.

In regard to the complementary impedance condition stated by (10), two special cases should be mentioned:

Case 1. $Z_a = Z_b = Z_c$ In this case, the terminal impedance would correspond to a semi-infinite line connected to the load terminals of both lines. It is the same as letting $d \to \infty$.

Case 2. $Z_a = 0$ and $Z_b \to \infty$ or $Z_b = 0, Z_a \to \infty$ These conditions show very clearly the complementary nature of the problem. It is for this reason why we adopt the word 'complementary' to describe the K–i theorem and the $(v$–$i)_c$ theorem. Our complementary lines are different from the two microwave circuits considered by Van Bladel [2] in discussing the symmetrical property of some scattering matrices in waveguide theory.

It is important to remember that the complementary theorems involve two identical sections of line with complementary impedance conditions at two ends, or two models. With the introduction of the concept of complemental models, the full theory for the three- dimensional fields would be easier to follow.

3 Complementary Reciprocity Theorems in Electromagnetic Theory

The reciprocity theorems in electromagnetic theory can be obtained most conveniently by means of Stratton's vector Green's theorem of the second kind [3] which states that for two continuous vector functions with continuous derivatives

$$\iiint_V (\overline{P} \cdot \nabla \times \nabla \times \overline{Q} - \overline{Q} \cdot \nabla \times \nabla \times \overline{P})\, dV$$
$$= \oiint_S \hat{n} \cdot [\overline{Q} \times \nabla \times \overline{P} - \overline{P} \times \nabla \times \overline{Q}]\, dS \tag{19}$$

where $\hat{n}$ denotes the outward unit normal of the surface S enclosing the volume V. We consider two sets of harmonically oscillating electromagnetic fields in an identical environment with an isotropic medium of electric and magnetic constants ϵ and μ. The wave number in such a medium is denoted by k, being equal to $\omega(\mu\epsilon)^{\frac{1}{2}}$. The two sets of fields will be labeled as $(\overline{J}_a, \overline{E}_a, \overline{H}_a)$ and $(\overline{J}_b, \overline{E}_b, \overline{H}_b)$. They are solutions of the equations

$$\nabla \times \overline{E}_a = i\omega\mu\overline{H}_a \tag{20}$$

$$\nabla \times \overline{H}_a = \overline{J}_a - i\omega\epsilon\overline{E}_a \tag{21}$$

$$\nabla \times \overline{E}_b = i\omega\mu\overline{H}_b \tag{22}$$

$$\nabla \times \overline{H}_b = \overline{J}_b - i\omega\epsilon\overline{E}_b \tag{23}$$

The wave equations for $\overline{E}_a, \overline{H}_a, \overline{E}_b$ and $\overline{H}_b$ are then given by

$$\nabla \times \nabla \times \overline{E}_a - k^2\overline{E}_a = i\omega\mu\overline{J}_a \tag{24}$$

$$\nabla \times \nabla \times \overline{H}_a - k^2\overline{H}_a = \nabla \times \overline{J}_a \tag{25}$$

$$\nabla \times \nabla \times \overline{E}_b - k^2\overline{E}_b = i\omega\mu\overline{J}_b \tag{26}$$

$$\nabla \times \nabla \times \overline{H}_b - k^2\overline{H}_b = \nabla \times \overline{J}_b \tag{27}$$

By a proper choice of the functions $\overline{P}$ and $\overline{Q}$ in (19) we can obtain the desired form of four reciprocity theorems.

Case 1) $\qquad \overline{P} = \overline{E}_a, \quad \overline{Q} = \overline{E}_b$

With this choice of the two vector functions, and with the aid of (24), (25), (20), and (22) we obtain immediately

$$\iiint (\overline{J}_b \cdot \overline{E}_a - \overline{J}_a \cdot \overline{E}_b)\, dV$$
$$= \oiint_S \hat{n} \cdot (\overline{E}_b \times \overline{H}_a - \overline{E}_a \times \overline{H}_b)\, dS \qquad (28)$$

Let us consider a specific problem where there is an electrically perfect conducting body inside V in an otherwise infinite domain then the surface S consists of two parts, one at infinity and another corresponding to the surface of the scattering body, denoted by S_d. At infinity, the fields satisfy the radiation condition and on S_d $\hat{n} \times \overline{E}_a = 0$ and $\hat{n} \times \overline{E}_b = 0$, (28) then reduces to

$$\iiint_{V_a} \overline{J}_a \cdot \overline{E}_b dV = \iiint_{V_b} \overline{J}_b \cdot \overline{E}_a dV \qquad (29)$$

where V_a denotes the volume occupied by $\overline{J}_a$ and V_b that by $\overline{J}_b$. Equation (29) represents the well-known Rayleigh-Carson reciprocity theorem. If the volume of integration <u>excludes</u> both $\overline{J}_a$ and $\overline{J}_b$, (28) reduces to

$$\oiint_{S_a + S_b} \hat{n} \cdot (\overline{E}_b \times \overline{H}_a - \overline{E}_a \times \overline{H}_b)\, dS = 0 \qquad (30)$$

where S_a denotes the surface enclosing $\overline{J}_a$ and S_a that enclosing $\overline{J}_b$. Equation (30) represents the Lorentz reciprocity theorem. Both theorems, of course, are applicable to problems in an infinite domain without a scattering body. They are also valid if the scattering body is made of an isotropic material with electric and magnetic constants different from that of the surrounding medium. The proof is straight-forward. In that case, one of the current distribution could be placed inside the scattering body.

Case 2) $\overline{P} = \overline{E}_a, \quad \overline{Q} = \overline{H}_b$

By substituting these two functions into (19), we obtain, in the first step, the equation

$$\iiint_V \left(\overline{E}_a \cdot \nabla \times \overline{J}_b - i\omega\mu \overline{J}_a \cdot \overline{H}_b\right) dV$$
$$= \oiint_S \hat{n} \cdot \left(i\omega\mu \overline{H}_b \times \overline{H}_a + \overline{J}_b \times \overline{E}_a + i\omega\epsilon \overline{E}_a \times \overline{E}_b\right) dS \,. \tag{31}$$

By means of the divergence theorem, one of the surface integrals in (31) can be split to two terms, i.e.,

$$\oiint_S \hat{n} \cdot \left(\overline{J}_b \times \overline{E}_a\right) dS$$
$$= \iiint_V \nabla \cdot \left(\overline{J}_b \times \overline{E}_a\right) dS$$
$$= \iiint_V \left(\overline{E}_a \cdot \nabla \times \overline{J}_b - \overline{J}_b \cdot \nabla \times \overline{E}_a\right) dS$$
$$= \iiint_V \left(\overline{E}_a \cdot \nabla \times \overline{J}_b - i\omega\mu \overline{J}_b \cdot \overline{H}_a\right) dV \tag{32}$$

Substituting it into (31) we obtain

$$\iiint_V \left(\overline{J}_b \cdot \overline{H}_a - \overline{J}_a \cdot \overline{H}_b\right) dV$$
$$= \oiint_S \hat{n} \cdot \left(E_a \times E_b / Z^2 - \overline{H}_a \times \overline{H}_b\right) dS \tag{33}$$

where $Z = (\mu/\epsilon)^{\frac{1}{2}}$ is the wave impedance in the isotropic medium with constants μ and ϵ. For the same problem considered in Case 1, i.e., an infinite isotropic medium with an electrically perfect conducting body inside the medium the surface integral vanishes at infinity because of the radiation condition but only the part involving $\hat{n} \cdot (\overline{E}_a \times \overline{E}_b)$ vanishes on S_d. Thus, the volume integral and the surface integral always exist simultaneously. To decouple these two integrals we can consider two complementary environments for the two sets. For the first set $(\overline{J}_a, \overline{E}_a, \overline{H}_a)$ we let the scattering body remain to be an electrically perfect conducting body with surface S_d. We call it Model (a). For the second set $(\overline{J}_b, \overline{E}_b, \overline{H}_b)$ we let the scattering body to be a magnetically perfect conducting body with the same shape, hence, the same surface S_d. We label it as Model (b). Unlike the line (b) in the transmission line theory, Model (b) is electromagnetically not physically realizable but to use it in a theoretical formulation it is entirely acceptable. Like the vector potential function in field theory which is not a measurable physical quantity. In fact, it is not unique mathematically because we can impose different gauge conditions to that function that still yield the same answer for $\overline{E}$ and $\overline{H}$ and we use these functions all the time. Having introduced the two complementary models we would like to label the surface S_d as S_e for Model (a), which is a physical model provided we accept perfect electric conductor as 'realizable', and S_m for Model (b), which is not physically realizable, then

$$\hat{n} \times \overline{E}_a = 0 \text{ on } S_e \tag{34}$$

and

$$\hat{n} \times \overline{H}_b = 0 \text{ on } S_m \tag{35}$$

Now if the volume of integration in (33) corresponding to the region outside of

S_d we obtain

$$\iiint_{V_a} \overline{J}_a \cdot \overline{H}_b dV = \iiint_{V_b} \overline{J}_b \cdot \overline{H}_a dV \tag{36}$$

where V_a and V_b denote, respectively, the volume occupied by $\overline{J}_a$ and $\overline{J}_b$. The relationship stated by (36) is designated as the <u>$\overline{J}$–$\overline{H}$ complementary reciprocity theorem</u> or the $\overline{J}$–$\overline{H}$ theorem for short, in contrast to the $\overline{J}$–$\overline{E}$ theorem of Rayleigh-Carson. The $\overline{J}$–$\overline{H}$ theorem is an independent theorem, distinct from the $\overline{J}$–$\overline{E}$ theorem. When the volume of integration excludes both V_a and V_b, we obtain the relationship

$$\oiint_{S_a+S_b} \hat{n} \cdot \left(\overline{E}_a \times \overline{E}_b / Z^2 - \overline{H}_a \times \overline{H}_b\right) dS = 0 \tag{37}$$

where S_a and S_b denote, respectively, the surface enclosing $\overline{J}_a$ and $\overline{J}_b$. Equation (37) is designated as the complementary $\overline{E}$–$\overline{H}$ reciprocity theorem, or the $(\overline{E}$–$\overline{H})_c$ theorem for short, in contrast to the $\overline{E}$–$\overline{H}$ theorem of Lorentz. Having introduced the notion of complementary models we would like to extend it to a problem involving two isotropic media in contact with an electrically conducting body in one of the media. The derivation of these reciprocity theorems with the aid of the vector Green's theorem of the second kind was first presented by this author in a conference held in China in 1987 [4] without much elaboration.

4 Two Plane Stratified Isotropic Media on a Conducting Ground Plane

For clarity, we consider a specific problem illustrated in Fig. 2 (A) where two plane stratified media are in contact and there is an electrically perfect conducting ground plane located in Region 1. The medium constants are μ_1, ϵ_1 and μ_2, ϵ_2 in the two different regions with wave number k_1 and k_2. These constants are assumed to be known. This model will be labeled as Model A. Now we create another mathematical model shown in Fig. 2 (B) where the medium constants in Region 2 are denoted by μ_2' and ϵ_2' with wave number k_2' while the constants in Region 1 remain the same as in Model A. The constants μ_2' and ϵ_2' are yet unspecified. The complementary model so created is labeled as Model B which has a magnetically perfect conducting ground plane in Region 1. The boundary conditions for the field excited by an electric current source in either region 1 or region 2 for Model A are:

$$\hat{z} \times (\overline{E}_{1A} - \overline{E}_{2A}) = 0 \text{ at } S \tag{38}$$

$$\hat{z} \times (\overline{H}_{1A} - \overline{H}_{2A}) = 0 \text{ at } S \tag{39}$$

$$\hat{z} \times \overline{E}_{1A} = 0 \text{ at } S_e \tag{40}$$

where S denotes the site of the interface. The subscript A is used to identify the $\overline{E}$ or $\overline{H}$ field in Model A. For Model B, the boundary condition for the field excited by an electric current source in either region 1 or region 2 are:

$$\hat{z} \times \left(\overline{E}_{1B} - \overline{E}_{2B}\right) = 0 \text{ at } S \tag{41}$$

$$\hat{z} \times \left(\overline{H}_{1B} - \overline{H}_{2B}\right) = 0 \text{ at } S \tag{42}$$

$$\hat{z} \times \overline{H}_{1B} = 0 \text{ at } S_m \tag{43}$$

The subscript B indicates that the fields are defined in Model B. Several cases will be considered depending upon the locations of the current sources. It should be noticed that the fields $\left(\overline{E}_{1A}, \overline{H}_{1A}\right), \left(\overline{E}_{2A}, \overline{H}_{2A}\right), \left(\overline{E}_{1B}, \overline{H}_{1B}\right)$ and $\left(\overline{E}_{2B}, \overline{H}_{2B}\right)$ satisfy the system of equations given by (20) to (27) with $\left(\overline{E}_a, \overline{H}_a, \overline{J}_A\right)$ and (μ, ϵ, k) replaced by the proper field functions and constitutional constants. For Model A, in Region 1 the replacements are $\left(\overline{E}_{1A}, \overline{H}_{1A}, \overline{J}_{1A}\right)$ and (μ_1, ϵ_1, k_1); in Region 2 they are $\left(\overline{E}_{2A}, \overline{H}_{2A}, \overline{J}_{2A}\right)$ and (μ_2, ϵ_2, k_2). For Model B, the replacements are $\left(\overline{E}_{1B}, \overline{H}_{1B}, \overline{J}_{1B}\right)$ and (μ_1, ϵ_1, k_1) in Region 1 and $\left(\overline{E}_{2B}, \overline{H}_{2B}, \overline{J}_{2B}\right)$ and $(\mu_2', \epsilon_2', k_2')$ in Region 2. The fact that $(\mu_2', \epsilon_2', k_2')$ are different from (μ_2, ϵ_2, k_2) will become clear later.

Case 1. <u>Currents $\overline{J}_{2A}$ and $\overline{J}_{2B}$ in Region 2</u>

In this case both currents are located in the region above the interface. The volume of that region will be denoted by V_2. By choosing $\overline{P} = \overline{E}_{2A}$ and $\overline{Q} = \overline{H}_{2B}$ and substituting them into (19), and with the aid of the wave equations for these functions we obtain initially

$$\begin{aligned} &\iiint_{V_2} \left[\overline{E}_{2A} \cdot \left(k_2'^2 \overline{H}_{2B} + \nabla \times \overline{J}_{2B}\right) - \overline{H}_{2B} \cdot \left(k_2^2 \overline{E}_{2A} + i\omega\mu_2 \overline{J}_{2A}\right)\right] dV \\ &= \iint_S \hat{z} \cdot \left[i\omega\mu_2 \overline{H}_{2A} \times \overline{H}_{2B} + \overline{E}_{2A} \times \left(\overline{J}_{2B} - i\omega\epsilon_2' \overline{E}_{2B}\right)\right] dS \end{aligned} \tag{44}$$

In (44), we have only the surface integral on S. The surface integral at infinity has already been dropped out as a result of the radiation condition. In view of the expression given by (32), the surface integral with integrand $\hat{z} \cdot (\overline{E}_{2A} \times \overline{J}_{2B})$ can be changed to

$$\iiint_{V_2} (\overline{E}_{2A} \cdot \nabla \times \overline{J}_{2B} - i\omega\mu_2 \overline{J}_{2B} \cdot \overline{H}_{2A})\, dV$$

hence (44) becomes

$$\iiint_{V_2} [(k_2'^2 - k_2^2)\, \overline{E}_{2A} \cdot \overline{H}_{2B} + i\omega\mu_2 (\overline{J}_{2B} \cdot \overline{H}_{2A} - \overline{J}_{2A} \cdot \overline{H}_{2B})]\, dV$$
$$= \iint_S \hat{z} \cdot (i\omega\mu_2 \overline{H}_{2A} \times \overline{H}_{2B} - i\omega\epsilon_2' \overline{E}_{2A} \times \overline{E}_{2B})\, dS\ . \tag{45}$$

Now we impose a relationship such that

$$k_2' = k_2$$

or

$$\mu_2'\epsilon_2' = \mu_2\epsilon_2\ . \tag{46}$$

Since μ_2 and ϵ_2 are given constants (46) puts a constraint on the product of μ_2' and ϵ_2' but not individually. Equation (46) will be referred to as the wave number matching condition. Under this condition (45) can be written in the form

$$\iiint_{V_2} (\overline{J}_{2B} \cdot \overline{H}_{2A} - \overline{J}_{2A} \cdot \overline{H}_{2B})\, dV$$
$$= \iint_S \hat{z} \cdot \left(\overline{H}_{2A} \times \overline{H}_{2B} - \frac{\epsilon_2'}{\mu_2} \overline{E}_{2A} \times \overline{E}_{2B} \right) dS \tag{47}$$

We now apply the vector Green's theorem to Region 1 with $\overline{P} = \overline{E}_{1A}, \overline{Q} = \overline{H}_{1B}$. Since in Region 1 there is no current source and the wave number is the same for both models, being equal to k_1, the volume integral vanishes and on the ground plane

$$\hat{z} \times \overline{E}_{1A} = 0 \text{ on } S_e$$

$$\hat{z} \times \overline{H}_{1B} = 0 \text{ on } S_m$$

the surface integral yields

$$\iint_S \hat{z} \cdot \left(\overline{H}_{1A} \times \overline{H}_{1B} - \frac{\epsilon_1}{\mu_1} \overline{E}_{1A} \times \overline{E}_{1B} \right) dS = 0 \qquad (48)$$

Now we impose a condition on the constant ϵ_2' such that

$$\epsilon_2'/\mu_2 = \epsilon_1/\mu_1$$

or

$$\epsilon_2' = (\mu_2/\mu_1)\,\epsilon_1 \qquad (49)$$

Under this condition, the surface integral in (47) is equal to the surface integral in (48) hence it also vanishes because at S, the site of the interface,

$$\hat{z} \times (\overline{E}_{1A} - \overline{E}_{2A}) = 0, \quad \hat{z} \times (\overline{H}_{1A} - \overline{H}_{2A}) = 0,$$

$$\hat{z} \times (\overline{E}_{1B} - \overline{E}_{2B}) = 0, \quad \hat{z} \times (\overline{H}_{1B} - \overline{H}_{2B}) = 0,$$

Thus, (47) becomes

$$\iiint_{V_2} (\overline{J}_{2A} \cdot \overline{H}_{2B} - \overline{J}_{2B} \cdot \overline{H}_{2A})\, dV = 0 \qquad (50)$$

This is the complementary $\overline{J}$–$\overline{H}$ reciprocity theorem for the two sets of field in Model A and Model B.

By combining (46) with (49), it can be shown readily that

$$\left(\frac{\mu_2}{\epsilon_2} \cdot \frac{\mu_2'}{\epsilon_2'}\right)^{\frac{1}{2}} = \frac{\mu_1}{\epsilon_1}$$

or

$$Z_2 Z_2' = Z_1^2 \tag{51}$$

This is the complementary condition for the wave impedances in the three media of the two models. By eliminating ϵ_2' between (46) and (49), one finds

$$\mu_2' = (\epsilon_2/\epsilon_1)\,\mu_1 \tag{52}$$

Thus, when the constitutional constants in Model A are given, the constants in Region 2 of Model B are specified by (49) and (52). For the case that $\mu_1 = \mu_2 = \mu_0, \epsilon_1 = \epsilon, \epsilon_2 = \epsilon_0$ which corresponds to the case of a dielectric layer placed on an electrically perfect conducting plane with an air medium above the layer, we have

$$\epsilon_2' = \epsilon \text{ and } \mu_2' = (\epsilon_0/\epsilon)\,\mu_0 \tag{53}$$

If the dielectric constant ϵ is complex μ_2' would be complex too. Unlike in network synthesis, the physical realizability of μ_2' is not an issue in this theory. The complementary condition for the wave impedances has the same appearance as the complementary condition for the load impedances in the transmission line model. In fact, if Model (a) of the transmission lines is terminated to a semi-infinite line with characteristic impedance Z_{c2} and the line in Model (b) is terminated to another line with characteristic impedance Z'_{c2}, (10) becomes

$$Z_{c2}Z'_{c2} = Z_{c1}^2$$

which is truly analogous to (51). The models of the complementary lines, however, are physically realizable. If we neglect the fringe field at the open end of an open-circuited line and a short-circuit is replaced by an electrically perfect conducting plane perpendicular to the line then the K–i theorem can, indeed, be derived from the $\overline{J}$–$\overline{H}$ theorem even though Model B is physically unrealizable. The situation is analogous to the derivation of the K–v theorem for the lines by the $\overline{J}$–$\overline{E}$ theorem of Rayleigh- Carson.

In addition to the case treated above, three more cases can be formulated. They are:

Case 2. $\overline{J}_{1A}$ in Region 1, $\overline{J}_{1B}$ in Region 1.

Case 3. $\overline{J}_{1A}$ in Region 1, $\overline{J}_{2B}$ in Region 2.

Case 4. $\overline{J}_{2A}$ in Region 2, $\overline{J}_{1B}$ in Region 1.

Without going through the details, we found that under the same wave number matching condition and the complementary wave impedance condition, the results are:

Case 2.
$$\iiint_{V_1} (\overline{J}_{1A} \cdot \overline{H}_{1B} - \overline{J}_{1B} \cdot \overline{H}_{1A})\, dV = 0 \tag{54}$$

where V_1 denotes the volume in Region 1

Case 3.
$$\iiint_{V_1} (\overline{J}_{1A} \cdot \overline{H}_{1B})\, dV = \iiint_{V_2} \overline{J}_{2B} \cdot \overline{H}_{2A} dV \tag{55}$$

Case 4. $$\iiint_{V_2} (\overline{J}_{2A} \cdot \overline{H}_{2B})\, dV = \iiint_{V_1} \overline{J}_{1B} \cdot \overline{H}_{1A} dV \qquad (56)$$

Formulas stated by (50), (54) - (56) have been derived using the plane stratified structure as an example. They are valid for similar canonical structures such as a conducting cylinder coated with a layer of dielectric material.

The theorem can be extended to multiple layers of isotropic media placed above an electrically conducting ground plane (Model A). It can be shown that, in general, for $n = 1, 2 \cdots N$ where the last region (N) may either extend to infinity or terminate to an electric wall in Model A and a magnetic wall in Model B, the general theorem is

$$\iiint_{V_i} \overline{J}_{iA} \cdot \overline{H}_{iB} dV = \iiint_{V_j} \overline{J}_{jB} \cdot \overline{H}_{jA} dV \qquad (57)$$

where $i, j = 1, 2 \cdots N$, derivable under the conditions:

$$k_n = k'_n, \quad n = 1, 2, \cdots N \qquad (58)$$

$$Z_n Z'_n = Z_1^2 \quad n = 1, 2, \cdots N \qquad (59)$$

The unprimed parameters are defined in Model A which are assumed to be known and the primed parameters defined in Model B are determined by (58) and (59). For $n = 1$, we have $\mu_1 = \mu'_1$ and $\epsilon_1 = \epsilon'_1$.

5 An Application of the $\overline{J}$–$\overline{H}$ Theorem

The formulation of the complementary $\overline{J}$–$\overline{H}$ theorem which we have so meticulously derived is not an academic exercise although works from educational institutions often had that tint. In our case, it was motivated by a problem which we could not resolve initially. It deals with the question of finding the transpose of some magnetic dyadic Green functions in the formulation of a radiating aperture on a ground plane coated with a layer of dielectrics, Model A in the previous section with $\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon$ (dielectric), $\epsilon_2 = \epsilon_0$ (air).

The key formulas obtained by the method of dyadic Green functions are:

$$\overline{E}_1(\overline{R}) = i\omega\mu_0 \iint_{S_A} \left[G_{m2}^{(11)}(\overline{R}', \overline{R}) \right]^T \cdot \left[\hat{z} \times \overline{E}_A(\overline{R}') \right] dS' \tag{60}$$

$$\overline{E}_2(\overline{R}) = i\omega\mu_0 \iint_{S_A} \left[G_{m2}^{(21)}(\overline{R}', \overline{R}) \right]^T \cdot \left[\hat{z} \times \overline{E}_A(\overline{R}') \right] dS' \tag{61}$$

where S_A denotes the area of a radiating aperture on a conducting ground plane with electric field $\overline{E}_A(\overline{R}')$. $\overline{\overline{G}}_{m2}^{(11)}$ and $\overline{\overline{G}}_{m2}^{(21)}$ denote two magnetic dyadic Green functions of the second kind, indicated by the subscript notation '$m2$' The superscript '11' of the function $\overline{\overline{G}}_{m2}^{(11)}(\overline{R}', \overline{R})$ means the position vector of a field point of that function, $\overline{R}'$, and that of a source point of that function, $\overline{R}$, are both located in Region 1. The superscript '21' of the function $\overline{\overline{G}}_{m2}^{(21)}(\overline{R}', \overline{R})$ means the field point of that function is in Region 2 while the source is in Region 1. The symbol $[\,]^T$ denotes the transpose of the function inside the brackets. On the other hand, the position vector $\overline{R}$ enters in the expressions for the electric field in the two regions and $\overline{R}'$ becomes the position vector for the aperture field. This is one of the characteristics of the method of the dyadic Green functions. It should be mentioned that the notation and the nomenclature for the magnetic dyadic Green function in this paper were not used in this author's original book on that subject [5], rather they were introduced later [6].

The main problem to be resolved is to find the symmetrical relationships between the transposed functions and some other functions with the roles of $\overline{R}$ and $\overline{R}'$ in the Green functions interchanged. Our original attempt encountered repeated failures that was finally conquered by invoking the $\overline{J}$–$\overline{H}$ theorem.

Let us now apply that theorem to derive the desired symmetrical relationships. When a current source $\overline{J}_{1A}$ is attributed to an infinitesimal electric dipole, with a properly normalized current moment, located in Region 1 at $\overline{R}_A$ in Model A and pointed in the $\hat{x}_i$ direction it can be written in the form

$$\overline{J}_{1A} = \hat{x}_i \delta\left(\overline{R} - \overline{R}_A\right) \ . \tag{62}$$

When another current source $\overline{J}_{1B}$, located also in Region 1 at $\overline{R}_B$ of Model B, is due to an infinitesimal electric dipole with the same current moment, but pointed in $\hat{x}_j$ direction we write

$$\overline{J}_B = \hat{x}_j \delta\left(\overline{R} - \overline{R}_B\right) \tag{63}$$

Substituting (62) and (63) into (54) we obtain

$$\hat{x}_i \cdot \overline{H}_{1B}\left(\overline{R}_A\right) = \hat{x}_j \cdot \overline{H}_{1A}\left(\overline{R}_B\right) \tag{64}$$

By definition, $\overline{H}_{1B}\left(\overline{R}_A\right)$ is the vector components of $\overline{\overline{G}}_{m1}^{(11)}\left(\overline{R}_A, \overline{R}_B\right)$ in the direction of $\hat{x}_j$, i.e.,

$$\overline{H}_{1B}\left(\overline{R}_A\right) = \overline{\overline{G}}_{m1}^{(11)}\left(\overline{R}_A, \overline{R}_B\right) \cdot \hat{x}_j \tag{65}$$

The function of the first kind with subscript '$m1$' is involved because in Model B the magnetic field satisfies the Dirichlet boundary condition, i.e.,

$$\hat{z} \times \overline{H}_{1B}\left(\overline{R}_A\right) = 0 \text{ on } S_m. \tag{66}$$

Similarly, $\overline{H}_{1A}\left(\overline{R}_B\right)$ is the vector component of $\overline{\overline{G}}_{m2}^{(11)}\left(\overline{R}_B, \overline{R}_A\right)$ in the direction of $\hat{x}_i$, i.e.,

$$\overline{H}_{1A}\left(\overline{R}_B\right) = \overline{\overline{G}}_{m2}^{(22)}\left(\overline{R}_B, \overline{R}_A\right) \cdot \hat{x}_i \tag{67}$$

The function of the second kind is involved in this case because in Model A the magnetic field satisfies the Neumann boundary condition, i.e.,

$$\hat{z} \times \nabla \times \overline{H}_{1A}\left(\overline{R}_B\right) = 0 \text{ on } S_e \tag{68}$$

The classification of the functions of the first and the second is based on the these two different boundary conditions. In view of (65) and (67), (64) is equivalent to

$$\hat{x}_i \cdot \overline{\overline{G}}_{m1}^{(11)}\left(\overline{R}_A, \overline{R}_B\right) \cdot \hat{x}_j = \hat{x}_j \cdot \overline{\overline{G}}_{m2}^{(11)}\left(\overline{R}_B, \overline{R}_A\right) \cdot \hat{x}_i \tag{69}$$

If we merely change the notation for $\overline{R}_A$ and $\overline{R}_B$ to $\overline{R}$ and $\overline{R}'$, then in the language of dyadic analysis [7], (69) corresponds to the symmetric relationship

$$\left[\overline{\overline{G}}_{m2}^{(11)}(\overline{R}', \overline{R})\right] = \overline{\overline{G}}_{m1}^{(11)}(\overline{R}, \overline{R}') \tag{70}$$

This is one of the relationships for which we are seeking. By applying the same technique to (55) with

$$\overline{J}_{1A} = \hat{x}_i \delta\left(\overline{R} - \overline{R}_A\right) \tag{71}$$

and

$$\overline{J}_{2B} = \hat{x}_j \delta\left(\overline{R} - \overline{R}_B\right) \tag{72}$$

we can find the other relationship, namely,

$$\left[\overline{\overline{G}}_{m2}^{(21)}(\overline{R}', \overline{R})\right]^T = \overline{\overline{G}}_{m1}^{(12)}(\overline{R}, \overline{R}') \,. \tag{73}$$

Attention should be called on the difference of the order of the superscripts and the different kinds of the magnetic dyadic Green functions in (73). Substituting (70) and (73) into (60) and (61) we finally obtain the desired formulas

$$\overline{E}_1(\overline{R}) = i\omega\mu_0 \iiint_{S_A} \overline{\overline{G}}_{m1}^{(11)}(\overline{R}, \overline{R}') \cdot \left[\hat{z} \times \overline{E}(\overline{R}')\right] dS \tag{74}$$

$$\overline{E}_2(\overline{R}) = i\omega\mu_0 \iiint_{S_A} \overline{\overline{G}}_{m1}^{(12)}(\overline{R}, \overline{R}') \cdot \left[\hat{z} \times \overline{E}(\overline{R}')\right] dS \tag{75}$$

This is what we tried to accomplish. In this paper, we do not intend to cover the derivation of the expressions for $\overline{\overline{G}}_{m1}^{(11)}$ and $\overline{\overline{G}}_{m1}^{(12)}$; they will be compiled in the forthcoming revised edition of the author's book currently in preparation.

In conclusion, it took us a long journey to arrive at the symmetical relationships of two magnetic dyadic Green functions encountered in a formulation. In retrospect, the challenge was rewarding as it forced us to search for a method to provide for the answers of two apparently relatively simple transformations. The complementary $\overline{J}$–$\overline{H}$ theorem is precisely the tool to uncover these formulas although the task is not an easy one. While the Rayleigh-Carson theorem can be used most conveniently to find the symmetical relationships for the electric dyadic Green functions, it is the $\overline{J}$–$\overline{H}$ theorem which is needed to find the symmetrical relationships for the magnetic dyadic Green functions. It happens that the new reciprocity theorem is more complex and sophisticated because it requires two complementary models. Aside from the application illustrated in this paper, the theorem may be useful in other occasions to formulate boundary-value problems in antenna theory and in microwave theory.

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