A Survey of the Improper Uses Of ∇ in Vector Analysis

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1 Introduction

In a separate paper entitled " An Historical Study of Vector Analysis" to be published shortly [1] we have criticized the improper uses of ∇, the nabla or del operator, in some early works and in a few contemporary writings. To support that claim a survey of these improper uses has been conducted. The result is contained in this report.

2 Books in Calculus, Applied Mathematics, and Vector analysis

1. Arfken, George. *Mathematical methods for Physicists*, (Second Edition), Academic Press, NY, 1970

> p.32: In section 1.6 ∇ was defined as a vector operator. Now, paying careful attention to both its vector and its differential properties, we let it operate on a vector. First, as a vector we dot it into a second vector to obtain
>
> $$\nabla \cdot \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$
>
> p.35: Another possible operation with the vector operator ∇ is to cross it into a vector.

The presentation makes the 'formal' scalar product and vector product so firmly established as legitimate products.

2. Borisenko, A. I. and I. E. Tarapov, *Vector and Tensor Analysis*, Revised English Edition of the Russian text (Moscow, 1966), translated by Richard A Silverman, Dover Publications, NY, 1979

> p.180: Find $\nabla(\boldsymbol{A} \cdot \boldsymbol{B})$ Solution. Clearly
>
> $$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \nabla(\boldsymbol{A}_c \cdot \boldsymbol{B}) + \nabla(\boldsymbol{A} \cdot \boldsymbol{B}_c)$$
>
> where the subscript has the same meaning as on p.170 ($\boldsymbol{A}_c$ is constant in $\nabla(\boldsymbol{A}_c \cdot \boldsymbol{B})$). According to formula (1.30)
>
> $$\boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$
>
> hence, setting $\boldsymbol{a} = \boldsymbol{A}_c, \boldsymbol{b} = \boldsymbol{B}, \boldsymbol{c} = \nabla$ we have
>
> $$\nabla(\boldsymbol{A}_c \cdot \boldsymbol{B}) = (\boldsymbol{A}_c \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}_c \times (\nabla \times \boldsymbol{B})$$

In vector algebra $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{a} \times (\boldsymbol{c} \times \boldsymbol{b})$ but $\boldsymbol{a} \times (\boldsymbol{b} \times \nabla) \neq -\boldsymbol{a} \times (\nabla \times \boldsymbol{b})$. The author is playing a game. We must remember that the 'vector product' $\nabla \times \boldsymbol{B}$ does not exist. It is an assembly. $\nabla \times \boldsymbol{B}$ is merely a notation due to Gibbs'.

3. Chambers, L. G., *A Course in Vector Analysis*, Chapman and Hall, London, 1969 p.90: It may be regarded as the scalar product of the vector operator ∇ and the vector $\boldsymbol{A}$.

The curl is treated in a similar manner. The author is not too sure about the concept of this scalar product because he used the words "may be regarded".

4. **Coburn, Nathaniel.** *Vector and Tensor Analysis*, Macmillan, NY, 1955

p.47: From the definition of ∇ in (17.2) and the definition of the star product we may define $\nabla * \boldsymbol{a}$ in cartesian orthogonal coordinates by

$$\nabla * \boldsymbol{a} = i\frac{\partial}{\partial x} * \boldsymbol{a} + j\frac{\partial}{\partial y} * \boldsymbol{a} + k\frac{\partial}{\partial z} * \boldsymbol{a}.$$

If we assume that differentiation and the star product are interchangeable (a result which is easily verified when $\boldsymbol{a}$ is scalar or vector field) we may write the equation for $\nabla * \boldsymbol{a}$ in the form

$$\nabla * \boldsymbol{a} = i * \frac{\partial \boldsymbol{a}}{\partial x} = j * \frac{\partial \boldsymbol{a}}{\partial y} + k * \frac{\partial \boldsymbol{a}}{\partial z}.$$

Coburn's treatment is the same as Wilson's. The result thus obtained is indeed the expression for the divergence but the assumption of the interchange between $*$ and the derivative sign (Wilson's pass by proposition) can never be proved.

5. **Cole, R. J.**, *Vector Methods*, Van Nostrand, NY, 1944

p.64: The divergence of a differential vector field is a scalar field defined by

$$\begin{aligned}\nabla \cdot \boldsymbol{f} &= i \cdot \frac{\partial \boldsymbol{f}}{\partial x} + j \cdot \frac{\partial \boldsymbol{f}}{\partial y} + k \cdot \frac{\partial \boldsymbol{f}}{\partial z} \\ &= \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot (if_x + \cdots) \\ &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.\end{aligned}$$

The first line and the third line can be used as the definition for the divergence as Gibbs originally suggested. But the second line is not meaningful.

6. **Eisenman, Richard**, *Matrix Vector Analysis*, McGraw-Hill, NY, 1963

p.99: ... by definition,

$$\begin{aligned}\nabla \cdot \boldsymbol{F} &= (\partial_x, \partial_y, \partial_z)\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &= \partial_x F_1 + \partial_y F_2 + \partial_z F_3\end{aligned}$$

$\nabla \cdot \boldsymbol{F}$ is similar to a dot product. It is not a dot product because ∇ is not a vector and, e.g., $\nabla \cdot \boldsymbol{F} \neq -\boldsymbol{F} \cdot \nabla$.

The word 'similar' was not explained. In fact, it cannot be explained mathematically.

7. Franklin, Philip, *Methods of Advanced Calculus*, McGraw-Hill, NY, 1944

> p.308: Since the volume of the box is $dx\,dy\,dz$, the divergence or rate of flow outward per unit volume is
>
> $$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = \nabla \cdot \boldsymbol{Q}.$$
>
> The designation of $\nabla \cdot \boldsymbol{Q}$ as $\operatorname{div} \boldsymbol{Q}$ or the divergence anticipated this interpretation of $\nabla \cdot \boldsymbol{Q}$ as divergence per unit time per unit volume.

In the last sentence the words 'as divergence per' presumably should be 'as charge per'. The main misinterpretation is to designate $\nabla \cdot \boldsymbol{Q}$ as the sum of the partial derivatives instead of as a notation for the divergence.

8. Gans, Richard, *Einführung in die Vektor-Analysis*, B.G. Teubner, Leipizig, 1905. English translation of the sixth edition by W.M.Deans, Blakie, London, 1932. The seventh German edition was published in 1950

> p.47(English edition): Thus, the operator ∇ denotes a differentiation, Seeing that $\nabla V \equiv \operatorname{grad} V$ has the components $\partial V/\partial x, \partial V/\partial y, \partial V/\partial z$, that $(\nabla \cdot \boldsymbol{A}) \equiv \operatorname{div} \boldsymbol{A} = (\partial/\partial x)A_x + (\partial/\partial y)A_y + (\partial/\partial z)A_z$, and that $[\nabla, \boldsymbol{A}] \equiv \operatorname{curl} \boldsymbol{A}$ has the components $\partial A_z/\partial y - \partial A_y/\partial z$, etc. We may formally regard the operator as a vector with components $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

Gans used only linguistic notations ($\operatorname{grad} f, \operatorname{div} \boldsymbol{f}$ and $\operatorname{curl} \boldsymbol{f}$) in the previous editions of his book. This is the first time Gibbs's notations were used except that he used $[\nabla, \boldsymbol{A}]$ instead of $\nabla \times \boldsymbol{A}$ to denote a vector product. Nevertheless, he also consider ∇ as a constituent of the divergence and the curl.

9. Hay, George E., *Vector and Tensor Analysis*, Dover Publications, NY, 1953

> p.108: If $\boldsymbol{b}$ is a vector then
>
> $$\nabla \cdot \boldsymbol{b} = \left(\sum i_r \frac{\partial}{\partial x_r}\right) \cdot \boldsymbol{b} = \sum i_r \cdot \frac{\partial \boldsymbol{b}}{\partial x_r} = \cdots = \sum \frac{\partial b_r}{\partial x_r}$$

The curl is treated in a similar manner.

10. Haskell, Richard E., *Introduction to Vector and Cartesian Tensors*, Prentice Hall, Englewood, NY, 1972

> p.229: If we write $\boldsymbol{F} = F_j u_{(j)}$, then
>
> $$\nabla \cdot \boldsymbol{F} = u_{(i)} \frac{\partial}{\partial x_i} \cdot F_j u_{(j)} = \frac{\partial F_j}{\partial x_i} u_{(i)} \cdot u_{(j)} = \frac{\partial F_j}{\partial x_i} \delta_{ij} = \frac{\partial F_i}{\partial x_i}$$

11. Hassani, Sadri, *Foundations of Mathematical Physics*, Allyn and Bacon, Needham Heights, MA, 1991

p.48: The divergence can be written more compactly if we recall that the vector operator ∇ has components $(\partial/\partial x, \cdots)$ and note that the expression in parentheses $(\partial A_x/\partial x + \cdots)$ looks like a dot product of this operator with the vector $\boldsymbol{A}$. Thus,

$$\nabla \cdot \boldsymbol{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

p.56: In fact, using the mnemonic determinant form of vector product, we can write

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \cdots$$

12. Hollingsworth, Charles A., *Vectors, Matrices, and Group Theory for Scientists and Engineers*, McGraw-Hill, NY, 1967

p.36:

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u_x, u_y, u_z) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

13. Kaplan, Wilfred, *Advanced Calculus*, Addison Wesley, Cambridge, MA, 1952

p.145: Formula (3.15) can be written in the symbolic form

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v};$$

for, treating ∇ as a vector, one has

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \cdots\right) \cdot (v_x \boldsymbol{i} + \cdots) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \operatorname{div} \boldsymbol{v} \end{aligned}$$

In the first place ∇ should not be written as $\nabla = (\partial/\partial x)\boldsymbol{i} + \cdots$ because $(\partial/\partial x)\boldsymbol{i}$ is the conventional notation for the derivative of $\boldsymbol{i}$ with respect to x or $\partial \boldsymbol{i}/\partial x$ which happens to be equal to zero in this case. Secondly, the 'formal' scalar product model of Wilson has been adopted. On p.146, the curl is treated in a similar way. The presentation remains the same in the last edition of the book published in 1991.

14. Kemmer, N., *Vector Analysis*, Cambridge University Press, Cambridge, 1977

p.84: Thus the Cartesian form of curl $\boldsymbol{f}$ is the cross product of the ∇ symbol with $\boldsymbol{f}$. Just as for the gradient, we see that here again ∇ behaves like a vector.

p.95: ...We insert this result to (1), go to the limit and find that

$$\operatorname{div} \boldsymbol{f} = \nabla \cdot \boldsymbol{f}$$

in terms of the Cartesian operator ∇, and to be quite explicit

$$\operatorname{div} \boldsymbol{f} \equiv \nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3.$$

∇ is truly a differential operator for the gradient. But in Gibbs's notation for the divergence, $\nabla \cdot \boldsymbol{f}$, ∇ is not a differential operator, the same for $\nabla \times \boldsymbol{f}$.

15. Korn, Granino and Therasa M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, NY, 1961

p.154: The linear operator ∇ (del or nabla) is defined by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Its application to a scalar point function $\Phi(\boldsymbol{r})$ or a vector function $\boldsymbol{E}(\boldsymbol{r})$ corresponds formally to a noncommutative multiplication operation with a vector having the rectangular Cartesian "components" $\partial/\partial x, \partial/\partial y, \partial/\partial z$; thus in terms of right-handed rectangular Cartesian coordinates x, y, z,

$$\nabla\Phi \doteq \operatorname{grad}\Phi \equiv \frac{\partial}{\partial x} + \cdots$$

$$\nabla \cdot \boldsymbol{F} \equiv \operatorname{div} \boldsymbol{F} \equiv \frac{\partial F_x}{\partial x} + \cdots$$

$$\nabla \times \boldsymbol{F} \equiv \operatorname{curl} \boldsymbol{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \cdots$$

16. Kovach, Ladis D., *Advanced Engineering Mathematics*, Addison Wesley, Reading, MA, 1982

p.312: Another way in which we can use the del operator is in applying it to a vector function. Since the operator has the form of a vector we must use either dot or cross multiplication. Given the vector function $\boldsymbol{v}$, we define

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

This scalar is called the divergence of $\boldsymbol{v}$ and is also written $\operatorname{div} \boldsymbol{v}$.

The author evidently treats $\nabla \cdot \boldsymbol{v}$ as a dot multiplication. The curl is treated as a cross product on p.313. Then on p.320, ∇ in cylindrical coordinate system is written as

$$\nabla = \frac{\partial}{\partial \rho} e_\rho + \frac{1}{\rho}\frac{\partial}{\partial \Phi} e_\Phi + \frac{\partial}{\partial z} e_z$$

No distinction has been made between a differential operator and a differential function.

17. Kreyszig, Erwin. *Advanced Engineering Mathematics* (Fifth Edition), John Wiley, NY, 1983

> p.397: The function
>
> $$\operatorname{div} \boldsymbol{v} = \sum \frac{\partial v_i}{\partial x_i}$$
>
> is called the divergence of $\boldsymbol{v}$ or the divergence of the vector field defined by $\boldsymbol{v}$.
>
> Another common notation for the divergence of $\boldsymbol{v}$ is $\nabla \cdot \boldsymbol{v}$,
>
> $$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \left(\frac{\partial}{\partial x} i + \cdots\right) \cdot (v_i i + \cdots) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
>
> with the understanding that the 'product' $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$ etc. This is a conventional notation but nothing more. Note that $\nabla \cdot \boldsymbol{f}$ means the scalar $\operatorname{div} \boldsymbol{v}$ where ∇f means the vector grad f defined in Sec.8.8.

The more the author tries to explain these operations the more a student gets confused. It is impossible to make an assembly meaningful. By writing ∇ in the form of $(\partial/\partial x)i + (\partial/\partial y)j + (\partial/\partial z)k$ it is in conflict with the form of ∇ in ∇f, i.e., $i\partial/\partial x + j\partial/\partial y + k\partial/\partial z$

18. Krishnamurtz, Karamcheti, *Vector Analysis and Cartesian Tensor*, Holden-Day, San Francisco, 1967

> p.50: The quantity $\nabla \cdot \boldsymbol{A}$ is a scalar, whereas $\nabla \times \boldsymbol{A}$ is a vector. For reasons that will become known later, $\nabla \cdot \boldsymbol{A}$ is known as the divergence of $\boldsymbol{A}$ and denoted by div $\boldsymbol{A}$, $\nabla \times \boldsymbol{A}$ is known as the curl of $\boldsymbol{A}$ and denoted by curl $\boldsymbol{A}$. Thus we have
>
> $$\operatorname{div} \boldsymbol{A} \equiv \nabla \cdot \boldsymbol{A}$$
>
> $$\operatorname{curl} \boldsymbol{A} \equiv \nabla \times \boldsymbol{A}$$
>
> It may be verified that in Cartesian system we obtain
>
> $$\operatorname{div} \boldsymbol{A} \equiv \nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \cdots$$
>
> p.53:
>
> $$\begin{aligned} \nabla^2 \boldsymbol{A} &= \operatorname{grad}\operatorname{div} \boldsymbol{A} - \operatorname{curl}\operatorname{curl} \boldsymbol{A} \\ &= \nabla\nabla \cdot \boldsymbol{A} - \nabla \times (\nabla \times \boldsymbol{A}) \end{aligned}$$
>
> This identity may be verified by expansion in Cartesian system or by expand $\nabla \times (\nabla \times \boldsymbol{A})$ according to the formula for a vector triple product.

The author appears to be very sure $\nabla \cdot \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ are two valid products. For the Laplacian of $\boldsymbol{A}$ he is using a mixed language and treats the identity as one in vector algebra.

19. Lagally, Max. *Vorlesungen über Vektor-rechnung*, Akademische Verlagsgesellschaft, Leipzig, 1928

p.123: Divergence of

$$\boldsymbol{b} = \nabla \cdot \boldsymbol{b} = \left(i \frac{\partial}{\partial x} + \cdots \right) \cdot (i b_x + \cdots)$$

or

$$\operatorname{div} \boldsymbol{b} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

As the rotation of $\boldsymbol{b}$ one define the vector product of ∇ with a field function $\boldsymbol{b}$:

$$\operatorname{rot} \boldsymbol{b} = \nabla \times \boldsymbol{b} = \left(i \frac{\partial}{\partial x} + \cdots \right) \times (iu + \cdots) = i \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \cdots$$

The above passage is an English translation of the original text in German.

20. Lass, Harry, *Vector and Tensor Analysis*, McGraw-Hill, NY, 1950

p.45: We postpone the physical meaning of the curl and define

$$\operatorname{curl} \boldsymbol{f} = \nabla \times \boldsymbol{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

21. Moon, P. and D. E. Spencer, *Vectors*, Van Notrand, Princeton, NJ, 1965

The two authors interpreted $\nabla \cdot \boldsymbol{f}$ in a curvilinear system, pp. 325–326, as

$$\nabla \cdot \boldsymbol{f} = \sum \frac{1}{h_i} \frac{\partial}{\partial u_i} (\boldsymbol{u}_i \cdot \boldsymbol{f})$$

and then concluded that it does not yield the correct result. This observation actually shows the evidence that the 'scalar product' between ∇ and $\boldsymbol{f}$ does not exist. If such a product does exist then it would be invariant to the choice of the coordinate system. On the other hand, ∇, ∇ and $\boldsymbol{\nabla}$ are three independent invariant differential operators[2].

22. Pipes, Louis A., *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, NY, 1946

p.343: The scalar product of the vector operator ∇ and a vector $\boldsymbol{A}$ gives a scalar that is called the divergence of $\boldsymbol{A}$; that is,

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \text{divergence of } \boldsymbol{A}$$

p.349: The curl is defined as the vector function of space obtained by taking the vector product of the operator ∇ and $\boldsymbol{A}$.

The same presentation is found in the Third Edition published in 1970.

23. Pomey, J. B., *Éléments de calcul vectoriel*, Ganthier-Villars, Paris, 1934

p.33: One calls divergence the scalar quantity which is the scalar product of ∇ and a vector ($\boldsymbol{X}$, for example)

$$\operatorname{div} \boldsymbol{X} = (\nabla \boldsymbol{X}) = \frac{\partial \xi^1}{\partial x_1} + \frac{\partial \xi^2}{\partial x_2} + \frac{\partial \xi^3}{\partial x_3}$$

rot $\boldsymbol{X}$: It is given by the vector product of ∇ and $\boldsymbol{X}$.

This is the English translation of the French text.

24. Porter, Merle C., *Mathematical Methods in the Physical Sciences*, Prentice Hall, Englewood, NJ, 1978

p.221: The dot product of ∇ operator with $\boldsymbol{u}$ is written in rectangular coordinates as

$$\nabla . \boldsymbol{u} = \left(\frac{\partial}{\partial x} i + \cdots\right) \cdot (u_x i + \cdots) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}.$$

It is known as the divergence of the vector field $\boldsymbol{u}$.

The author treats the curl as the cross product between ∇ and $\boldsymbol{u}$ and the Laplacian as the dot product between two dels.

25. Rektorys, Karel (Editor), *Survey of Applicable Mathematics*, English translation of a Czechoslovakian book by Rudolf Vyborny et al. edited by Staff of the Department of Mathematics, University of Survey, M.I.T. Press, Cambridge, MA, 1969

p.272: The divergence of a vector $\boldsymbol{a}$ is the scalar

$$\operatorname{div} \boldsymbol{a} = \nabla \cdot \boldsymbol{a} = \sum \frac{\partial a_i}{\partial x_i}$$

p.275: ...we note that the operator ∇ is given in vector form by (20); for example,

$$\nabla^2 = \nabla \cdot \nabla = \left(\sum \boldsymbol{a}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum \boldsymbol{a}_j \frac{\partial}{\partial x_j}\right) = \sum \frac{\partial^2}{\partial x_i^2} = \Delta$$

When a Czechoslovakian book is translated into English by members of an English university and published in the U.S.A. nobody would question its qualification.

26. Schey, H. M., *Div. Grad. Curl, and All That*, W. W. Norton & Company, NY, 1973

p.43: If we take the dot product of $\boldsymbol{V}$ and some vector function $\boldsymbol{F}$.... we get

$$\nabla \cdot \boldsymbol{F} = \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot (iF_x + \cdots) = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z$$

Now we interpret the 'product' of $\partial/\partial x$ anf F_x as a partial derivative, that is,

$$\frac{\partial}{\partial x}F_x \equiv \frac{\partial F_x}{\partial x}$$

$\partial/\partial x$ is a differential operator. When it operates on F_x it yields $\partial F_x/\partial x$. The problem is the formal product manipulation not the subsequent interpretation.

p.82 You can verify for your self that

$$\text{curl}\,\boldsymbol{F} = \nabla \times \boldsymbol{F}$$

which is read "del cross $\boldsymbol{F}$".

27. Sokolnikoff, I. S. and R. M. Redheffer, *Mathematics of Physics and Modern Engineering* (Second Edition), McGraw-Hill, NY, 1966

p.395: In terms of the differential operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

introduced in Sec.2, we can consider a symbolic scalar product

$$\nabla \boldsymbol{v} = \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot (iv_x + \cdots) \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

On comparing this with (6–6) we see that

$$\text{div}\,\boldsymbol{v} = \nabla \cdot \boldsymbol{v}.$$

We can define the Laplacian operator ∇^2 by the formula

$$\nabla^2 = \nabla \cdot \nabla = \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot \left(i\frac{\partial}{\partial x} + \cdots\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and observe that if $\boldsymbol{v} = \nabla u$

$$\text{div}\,\nabla u = \nabla \cdot \nabla u = \nabla^2 u.$$

$\nabla \times \boldsymbol{v}$ is defined by the rule for computing vector products.

28. Thomas, George B. Jr., *Calculus and Analytical Geometry* (Third Edition), Addison-Wesley, Reading, MA, 1960

p.719: The 'curl' of a vector $\boldsymbol{F} = if + jg + kh$ is defined to be del cross $\boldsymbol{F}$, that is,

$$\operatorname{curl} \boldsymbol{F} \equiv \nabla \times \boldsymbol{F}$$

and the 'divergence' of a vector $\boldsymbol{V} = iu + jv + kw$ is defined to be del dot $\boldsymbol{V}$, that is,

$$\operatorname{div} \boldsymbol{V} \equiv \nabla \cdot \boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

When a popular textbook makes such a positive assertion it becomes an unquestionable truth.

29. Wilson, E. B., *Vector Analysis*, Charles Scribner's Sons, NY, 1901

p.150: Although the operation $\nabla \boldsymbol{V}$ has not been defined and cannot be at present, two formal combinations of the vector operator ∇ and a vector function $\boldsymbol{V}$ may be treated. These are the (formal) scalar product and the (formal) vector product of ∇ and $\boldsymbol{V}$. They are

$$\nabla \cdot \boldsymbol{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \boldsymbol{V}$$

$$\nabla \times \boldsymbol{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times \boldsymbol{V}.$$

$\nabla \cdot \boldsymbol{V}$ is read del dot $\boldsymbol{V}$; and $\nabla \times \boldsymbol{V}$, del cross $\boldsymbol{V}$.

The differentiators $\partial/\partial x, \partial/\partial y, \partial/\partial z$, being scalar operators, pass by the dot and the cross. That is

$$\nabla \cdot \boldsymbol{V} = i \cdot \frac{\partial \boldsymbol{V}}{\partial x} + j \cdot \frac{\partial \boldsymbol{V}}{\partial y} + k \cdot \frac{\partial \boldsymbol{V}}{\partial z}$$

$$\nabla \times \boldsymbol{V} = i \times \frac{\partial \boldsymbol{V}}{\partial x} + j \times \frac{\partial \boldsymbol{V}}{\partial y} + k \times \frac{\partial \boldsymbol{V}}{\partial z}$$

These may be expressed in terms of the components V_1, V_2, V_3 of $\boldsymbol{V}$. ...

From some standpoints objections may be brought forward against treating ∇ as a symbolic vector and introducing $\nabla \cdot \boldsymbol{V}$ and $\nabla \times \boldsymbol{V}$ respectively as the symbolic scalar and vector products of ∇ into $\boldsymbol{V}$. These objections may be avoided by simply laying down the definition that the symbol $\nabla\cdot$ and $\nabla\times$, which may be looked upon as entirely new operation operators quite distinct from ∇, shall be

$$\nabla \cdot \boldsymbol{V} = i \cdot \frac{\partial \boldsymbol{V}}{\partial x} + j \cdot \frac{\partial \boldsymbol{V}}{\partial y} + k \cdot \frac{\partial \boldsymbol{V}}{\partial z}$$

$$\nabla \times \boldsymbol{V} = i \times \frac{\partial \boldsymbol{V}}{\partial x} + j \times \frac{\partial \boldsymbol{V}}{\partial y} + k \times \frac{\partial \boldsymbol{V}}{\partial z}.$$

But for practical purposes and for remembering formulae it seems by all means advisable to regard

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

as a symbolic vector differentiator. This symbol obeys the same laws as a vector just in so far as the differentiators $\partial/\partial x, \partial/\partial y, \partial/\partial z$ obey the same laws as ordinary scalar quantities.

This is the most influential writing by an early author in vector analysis which has been adopted by many authors in this country. The operators $\partial/\partial x, \partial/\partial y, \partial/\partial z$ can operate on functions, scalar or vector, directly but they cannot pass by a dot or a cross. A simple proof of this misunderstanding is to consider the reversal operation. We start with the definition of $divV$

$$\operatorname{div} \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial Z}$$

which can be written as

$$\begin{aligned}
\operatorname{div} \boldsymbol{V} &= \frac{\partial}{\partial x}(\boldsymbol{V}\cdot\hat{x}) + \frac{\partial}{\partial y}(\boldsymbol{V}\cdot\hat{y}) + \frac{\partial}{\partial z}(\boldsymbol{V}\cdot\hat{z}) \\
&= \frac{\partial \boldsymbol{V}}{\partial x}\cdot\hat{x} + \boldsymbol{V}\cdot\frac{\partial \hat{x}}{\partial x} + \cdots \\
&= \hat{x}\cdot\frac{\partial \boldsymbol{V}}{\partial x} + \hat{y}\cdot\frac{\partial \boldsymbol{V}}{\partial y} + \hat{z}\cdot\frac{\partial \boldsymbol{V}}{\partial z}
\end{aligned}$$

because the derivatives of the unit Cartesian vector vanish identically. The last line is obviously not equal to

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\boldsymbol{V}$$

In other words, the divergence is equal to the sum of the components of the directional derivatives of a vector function; it is not the scalar product between the del operator and the vector function. Like Gibbs, Wilson did consider $\nabla\cdot$ and $\nabla\times$ as two new operators but his mistake is to create the concept of scalar and vector products between ∇ and $\boldsymbol{V}$. Even the use of the word 'formal' does not justify such a manipulation. Heaviside treated $\nabla\cdot\boldsymbol{V}$ and $\nabla\times\boldsymbol{V}$ as two legitimate products without even using the word 'formal' in his presentation. Both Heaviside and Wilson were responsible for the misinterpretation of Gibbs's notations.

30. Wylie, C. R., *Advanced Engineering Mathematics*, (Third Edition), McGraw-Hill, NY, 1966

> p.554: If $\boldsymbol{F}$ is a vector whose components are functions of x, y, and z, this leads to the combinations
>
> $$\nabla\cdot\boldsymbol{F} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \cdots\right)\cdot(F_1\boldsymbol{i} + \cdots) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
>
> which is known as the divergence of $\boldsymbol{F}$, and
>
> $$\nabla\times\boldsymbol{F} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \cdots\right)\times(F_1\boldsymbol{i} + \cdots) = \boldsymbol{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \cdots$$
>
> which is known as the curl of $\boldsymbol{F}$.

3 Books on Electromagnetic Theory and Physics

1. **Argence, Emile and Thea Kahn.** *Theory of Waveguides and Cavity Resonators.* Hart Publishing Company, NY, 1967 (English translation of an original book in French)

p.19: The scalar product $\nabla \cdot \boldsymbol{v}$ defines the divergence (div) of a vector $\boldsymbol{v}$

$$\text{div} \equiv \nabla \cdot \boldsymbol{v} = \left(e_x \frac{\partial}{\partial x} + \cdots \right) \cdot (e_x v_x + \cdots) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

The vector product $\nabla \times \boldsymbol{v}$ defines the curl....

Finally, we have the relations

$$\Delta f = \text{div}\,\text{grad}\, f \equiv \nabla \cdot \nabla f = \cdots$$

2. **Boast, Warren B.**, *Principles of Electric and Magnetic Fields*, Harper and Brothers, NY, 1956

p.391: Let the vector operator ∇ operate as a scalar product upon some vector $\boldsymbol{N}$. Letting the ∇ operator replace the vector $\boldsymbol{M}$ in Eq.21.01 ($\boldsymbol{M} \cdot \boldsymbol{N} = M_x N_x + \cdots$) gives

$$\nabla \cdot \boldsymbol{N} = \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z}.$$

3. **Chen, Hollis C.**, *Theory of Electromagnetic Waves*, McGraw-Hill, NY, 1983

p.49: The divergence of a vector function

$$\nabla \cdot \boldsymbol{\mu} \leftrightarrow \frac{\partial \mu_i}{\partial x_i} = \partial_i \mu_i$$

$$\nabla \times \boldsymbol{\mu} \leftrightarrow \epsilon_{jik} \frac{\partial \mu_k}{\partial x_j} = \epsilon_{ijk}\, \partial_j \mu_k$$

4. **Durney, C. H. and Curtis C. Johnson**, *Introduction to Modern Electromagnetics*, McGraw-Hill, NY, 1969

p.45: The dot product is formed in the obvious way. ∇ operator cannot be defined in other coordinate system.

p.57: $\nabla \cdot \nabla$ is treated as a product.

∇ is an invariant operator. It is well defined in any curvilinear coordinate system, even in a non-orthogonal curvilinear system.

5. **Elliott, Robert S.**, *Electromagnetics*, McGraw-Hill, NY, 1966

p.606: Indeed, only in cartesian coordinates, and only because in that system $h_1 = h_2 = h_3 = 1$, do the gradient and divergence operators turn out to be identical.

p.610: Comparison with (V.14) suggests the notation curl $\boldsymbol{A} = \nabla \times \boldsymbol{A}$ in which ∇ is the del operator defined by (V.67).

6. **Fano, Robert M., Lan Jen Chu, and Richard B. Adler**, *Electromagnetic Fields, Energy, and Forces*, John Wiley, NY, 1960

After the authors derived the expression of the divergence by the flux model, they commented that in the simple case of cartesian coordinates

$$\operatorname{div} \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \boldsymbol{A}$$

where it is recognized that the result expression can be interpreted as the scalar product of the del operator and the vector $\boldsymbol{A}$. A similar interpretation was given to curl $\boldsymbol{A}$ with the remark that by inspection,

$$\operatorname{curl} \boldsymbol{A} = \nabla \times \boldsymbol{A}.$$

7. **Harrington, Roger F.**, *Introduction to Electromagnetic Engineering*, McGraw-Hill, NY, 1958

p.34:

$$\operatorname{div} \boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2.40}$$

This is the differential form of divergence in rectangular coordinates and components. It is a scalar, as called to mind by the dot product symbolism of $\nabla \cdot \boldsymbol{A}$. We can view ∇ (del) as the differential operator

$$\nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \tag{2.41}$$

Eq.(2.40) results from an application of our formal rules for scalar multiplication between the operator ∇ and a vector $\boldsymbol{A}$. However, ∇ is not a vector. It has meaning only when it operates on a function according to (2.40) or according to other equations we shall consider later.

The author derived the expression for div $\boldsymbol{A}$ by the flux model first and then he added the above remarks which is the same as the 'formal' scalar product model of Wilson.

8. **Haus, Hermann A. and James R. Melcher**, *Electromagnetic Fields and Energy*, Prentice Hall, Englewood, NJ, 1989

> p.48: The definition (2) (the flux model) is independent of the choice of coordinate system. On the other hand, the del notation suggests the mechanics of the operation in cartesian coordinates. We will have it both ways by using the del notation in writing equations in cartesian coordinates, but using the name divergence in the text.

If one understands the significance of Gibbs's notation for the divergence it is perfectly all right to use it to denote the divergence in any coordinate system. It is merely a notation.

9. **Hayt, William H. Jr.**, *Engineering Electromagnetics*, McGraw-Hill, NY, 1958

> p.73: Considering $\nabla \cdot \boldsymbol{D}$, signifying
>
> $$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right) \cdot (D_x a_x + D_y a_y + D_z a_z) \\ &= \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z) \end{aligned}$$
>
> where the parentheses are now removed by operating or differentiating:
>
> $$\nabla \cdot \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$
>
> p.74:
>
> $$\begin{aligned} \nabla u &= \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right) u \\ &= \frac{\partial u}{\partial x}a_x + \frac{\partial u}{\partial y}a_y + \frac{\partial u}{\partial z}a_z \end{aligned}$$
>
> The operator ∇ does not have a specific form in other coordinate systems.

It is very unfortunate that the author writes ∇ in the form $(\partial/\partial x)a_x + \cdots$ instead of $a_x(\partial/\partial x) + \cdots$. The former is a differential function (=0) and the later is a differential operator. The differential operator ∇ is well defined in any coordinate system for the gradient. It is not a constituent for the divergence nor the curl.

10. **Heaviside, Oliver**, *Electromagnetic Theory*, Vol.I, first published in 1893, reproduced by Dover Publications, NY, 1950

> p.127: When the operand of ∇ is a vector, say, $\boldsymbol{D}$, we have both the scalar product and the vector product to consider. Taking the formula alone first, we have
>
> $$\operatorname{div} \boldsymbol{v} = \nabla_1 D_1 + \nabla_2 D_2 + \nabla_3 D_3$$
>
> This function of $\boldsymbol{D}$ is called the divergence and is a very important function in physical mathematics.

A detailed comment on Heaviside's work is given in reference [1].

11. Javid, Mansour and Philip Marshall Brown, *Field Analysis and Electromagnetics*, McGraw-Hill, NY, 1963

The authors show on pp.477–479 Appendix II that the volume-integral definition of the operator given by

$$\nabla\{\ \} = \lim_{v\to 0}\frac{1}{v}\int_s \hat{n}\{\ \}\,ds$$

leads to the expression

$$\nabla\{\ \} = \frac{1}{\Omega}\left[\frac{\partial}{\partial u_1}\left(\frac{\Omega}{h_1}\boldsymbol{a}_1\{\ \}\right) + \frac{\partial}{\partial u_2}\left(\frac{\Omega}{h_2}\boldsymbol{a}_2\{\ \}\right) + \frac{\partial}{\partial u_3}\left(\frac{\Omega}{h_3}\boldsymbol{a}_3\{\ \}\right)\right]$$

where $\Omega = h_1 h_2 h_3$, the product of the metric coefficients. Then by letting $\{\ \} = f,\ \cdot\boldsymbol{F},\ \times\boldsymbol{F}$ they found the expressions for the gradient, the divergence, and the curl in curvilinear systems and concluded that the operator may be treated as a vector quantity. Actually, the authors have already identified ∇ as the del operator. The operation $\nabla\cdot$ and $\nabla\times$ are Gibbs's notations for the divergence and the curl. ∇ is not a constituent of the divergence nor the curl. The authors fail to recognize that their definition of $\nabla\{\ \}$ is merely a general notation. When $\{\ \} = \{f\}$, we indeed have the gradient. When $\{\ \} = \{\cdot\boldsymbol{F}\}$, $\nabla\{\cdot\boldsymbol{F}\} = \nabla\cdot\boldsymbol{F}$ is merely a notation for the divergence defined by the volume-integral. Unlike the symbolic expressions ∇f, $\nabla\cdot\boldsymbol{f}$ and $\nabla\times\boldsymbol{f}$ their $\nabla\{\ \}$ for the divergence and the curl are not the same as the operators ∇ and $\boldsymbol{\nabla}$. This is one of the most delicate features of the method of symbolic vector in contrast to Javid/Brown's definition of $\nabla(\cdots)$ originally formulated by Gans in the cartesian coordinates.

12. Johnk, Carl T. A., *Engineering Electromagnetics*, John Wiley, NY, 1975

p.71: Upon taking the dot product of ∇ with $\boldsymbol{F}$ in the rectangular coordinates one finds

$$\nabla\cdot\boldsymbol{F} = \left(\sum \boldsymbol{a}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum \boldsymbol{a}_j F_j\right) = \sum\frac{\partial F_i}{\partial x_i}$$

This is the basis for the equivalent symbolisms

$$\text{div}\,\boldsymbol{F} = \nabla\cdot\boldsymbol{F}$$

Similar interpretation for the curl is found on p.82.

13. Jordan, E. C., *Electromagnetic Waves and Radiating Systems*, Prentice Hall, Englewood, NJ, 1950

p.9: 2. If $\boldsymbol{A}$ is a vector function, we can apply Eqs.(10), (12), and (18) (the formulas for $\boldsymbol{A}\cdot\boldsymbol{B}$ and ∇) and get

$$\nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The operation is called the divergence and is abbreviated $\nabla\cdot\boldsymbol{A} = \text{div}\,\boldsymbol{A}$.

3. If $\boldsymbol{A}$ is a vector function, we can apply (15), (17), and (18) (the formulas for $\boldsymbol{A} \times \boldsymbol{B}$ and ∇) to show that

$$\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \cdots$$

This operation is called the curl and can be written as

$$\nabla \times \boldsymbol{A} = \text{curl}\, \boldsymbol{A}$$

There is no change in the second edition co-authored with Keith G. Balmain, published in 1968.

14. Kraus, John D., *Electromagnetics* (Fourth Edition), McGraw-Hill, NY, 1992

After deriving correctly the expression for the divergence by the flux model it was stated on p.169 that the divergence of $\boldsymbol{D}$ can also be written as the scalar, or dot, product of the operator ∇ and $\boldsymbol{D}$. On p.249, it was stated that curl $\boldsymbol{H}$ is conveniently expressed in vector notation as the cross-product of the operator del (∇) and $\boldsymbol{H}$.

15. Marion, Jerry B., *Classical Electromagnetic Radiation*, Academic Press, NY, 1965

> p.451: The divergence of a vector $\boldsymbol{A}$ is defined by div $\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \sum \partial A_i/\partial x_i$. The curl is defined by
>
> $$\text{curl}\, \boldsymbol{A} = \nabla \times \boldsymbol{A} = \sum_{i,j,k} \epsilon_{ijk}\, e_i \frac{\partial A_k}{\partial x_j}$$

16. Mason, Max and Warren Weaver, *The Electromagnetic Field*, The University of Chicago Press, Chicago, 1929

> p.336: The differential operator ∇ can be considered formally as a vector of components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, so that its scalar and vector products with another vector may be taken.

Afterwards, the authors considered the flux model as an alternative definition of div $\boldsymbol{V}$.

17. Neff, Herbert P. Jr., *Introductory Electromagnetics*, John Wiley, NY, 1991

> p.16: The del operator must operate on some quantity, and it is easy to show
>
> $$\nabla \cdot \boldsymbol{D} = \left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \cdots\right) \cdot (\boldsymbol{a}_x D_x + \cdots)$$
>
> $$\nabla \cdot \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \boldsymbol{D}|_S \cdot d\boldsymbol{s}}{\Delta V} \qquad (1.29)$$
>
> The left and right sides of (1.29) gives us a completely general mathematical definition of del as an integral operator:
>
> $$\nabla \circ \cdots = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} d\boldsymbol{s} \circ \cdots}{\Delta V}$$
>
> If the small circle ($\circ$) becomes a dot, we obtain the divergence of a vector. If the small circle becomes a cross, we obtain the curl of a vector.

This presentation is similar to that of Gans and the one of Javid and Brown. The author did not realize that the left side of his (1.29) is a notation and the right side is truly a defining expression for the divergence.

18. Nussbaum, Allen, *Electromagnetic Theory for Engineers and Scientists*, Prentice Hall, Englewood, NJ, 1965

p.35: This operator (∇) can be used in conjunction with the definition of the scalar product to generate other fundamental operations. These are the divergence of a vector such as the field $\boldsymbol{\mathcal{E}}$, which is defined and denoted by

$$\operatorname{div} \boldsymbol{\mathcal{E}} = \nabla \cdot \boldsymbol{\mathcal{E}} = \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z}$$

p.184: In rectangular coordinates, this (the curl) is defined as

$$\operatorname{curl} \boldsymbol{A} = \nabla \times \boldsymbol{A}$$

so that by (4.26)

$$\operatorname{curl} \boldsymbol{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

19. Page, Leigh and Norman Ilsley Adams, *Electromagnetics*, Van Nostrand, NY, 1940

p.27: If we form the scalar product

$$\nabla \cdot \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

we obtain a proper function called the divergence of $\boldsymbol{V}$.

p.29: If $\boldsymbol{V}$ is a proper vector function of coordinates we may form the vector product of ∇ and $\boldsymbol{V}$ so as to obtain another proper vector function known as the curl or rotation of $\boldsymbol{V}$. This is

$$\nabla \times \boldsymbol{V} = i\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \cdots = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

p.31: The divergence of the gradient of a proper scalar function is

$$\nabla \cdot \nabla \Phi = \nabla \cdot \left(i\frac{\partial \Phi}{\partial x} + \cdots\right) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

As the same result is obtained by allowing the scalar product

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The interpretation of the Laplacian as the scalar product of two del's or nabla's is found in many books. It is, of course, not a valid product to produce another differential operator.

20. Panofsky, Wolfgang K. H. and Melba Phillips. *Classical Electricity and Megnetism* (Second Edition), Addison Wesley, Reading, MA, 1962

In p.470 the authors stated that the relations involving the vector operator ∇ may be derived formally from the vector identities (1) through (5) if one remembers that ∇ is a differential operator as well as a vector and thus does not commute with functions of the coordinates. In the example given by the authors, they wrote

$$\begin{aligned}\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) &= (\nabla \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla \cdot \boldsymbol{A})\boldsymbol{B} \\ &= (\nabla \cdot \boldsymbol{B}_c)\boldsymbol{A} + (\nabla \cdot \boldsymbol{B})\boldsymbol{A}_c - (\nabla \cdot \boldsymbol{A}_c)\boldsymbol{B} - (\nabla \cdot \boldsymbol{A})\boldsymbol{B}_c\end{aligned}$$

> where subscript c indicates that the function is held constant and may be permuted with the vector operator, with due regard to sign changes if such changes are indicated by ordinary vector relations.

It is seen that their $(\nabla \cdot \boldsymbol{B})\boldsymbol{A}$ is not equal to $(\operatorname{div} \boldsymbol{B})\boldsymbol{A}$ or $(\nabla \boldsymbol{B})\boldsymbol{A}$. Rather, it is equal to $(\nabla \cdot \boldsymbol{B}_c)\boldsymbol{A} + (\nabla \cdot \boldsymbol{B})\boldsymbol{A}_c$. If $\boldsymbol{B}_c$ is held constant $\operatorname{div} \boldsymbol{B}_c = 0$. The use of vector algebraic relations to derive vector identities has no mathematical foundation. The contradiction is similar to Shilov's work. The proper way to do this exercise is to prove first that

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \nabla \cdot (\boldsymbol{B}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{B}) \qquad \text{(Gibbs's notation)}$$

or

$$\overline{\nabla}(\boldsymbol{A} \times \boldsymbol{B}) = \nabla(\boldsymbol{B}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{B}) \qquad \text{(new notation)}$$

where $\boldsymbol{AB}$ is a dyadic and $\boldsymbol{BA}$ its transpose, then by means of dyadic analysis one finds

$$\nabla \cdot (\boldsymbol{AB}) = (\nabla \cdot \boldsymbol{A})\boldsymbol{B} + \boldsymbol{A} \cdot \nabla \boldsymbol{B}$$

$$\nabla \cdot (\boldsymbol{BA}) = (\nabla \cdot \boldsymbol{B})\boldsymbol{A} + \boldsymbol{B} \cdot \nabla \boldsymbol{A}$$

hence

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \cdot \boldsymbol{B})\boldsymbol{A} + \boldsymbol{B} \cdot \nabla \boldsymbol{A} - (\nabla \cdot \boldsymbol{A})\boldsymbol{B} - \boldsymbol{A} \cdot \nabla \boldsymbol{B}$$

A relatively simple method of deriving this identity is to apply the method of symbolic vector. The treatment is given in [2] and reproduced in [1]. We want to point out that a relation like

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla \cdot \boldsymbol{A})\boldsymbol{B}$$

as given by Panofsky and Phillips is in contradiction to the accepted rules of vector analysis. To change $(\nabla \cdot \boldsymbol{B})\boldsymbol{A}$ to $\nabla \cdot (\boldsymbol{BA})$ is similar to the 'pass by' manipulation of Wilson.
We shall mention that the formula

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \nabla \cdot (\boldsymbol{BA} - \boldsymbol{AB})$$

is listed in Appendix B of the monograph by Paul Penfield Jr., and Hermann A. Haus on *Electrodynamics of Moving Media*, M.I.T. Press, 1966, p.249 without derivation nor comment.

21. Paul, Clayton R. and Syed A. Nasar, *Introduction to Electromagnetic Fields*, McGraw-Hill, NY, 1982

> p.40: Again we may write, grad f as ∇f, but the operator is not simply defined in spherical coordinates.

> p.48: In terms of the del operator defined for a rectangular coordinate system ... we may write (90) as
> $$\text{div}\, \boldsymbol{F} = \nabla \cdot \boldsymbol{F}$$
> as a simple expression will show.

The gradient operator in sp.erical coordinate is well defined and there is no expansion for $\nabla \cdot \boldsymbol{F}$.

22. Pender, Harold and S. Reid Warren, Jr., *Electric Circuits and Fields*, McGraw-Hill, NY, 1943

> p.246: The concept of a dot product may also be expended to the vector operator ∇..., then in the notation just defined $\nabla \cdot \boldsymbol{A}$ signifies the scalar quantity
> $$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \cdots$$
> Similarly, the dot product of ∇ by itself usually written ∇^2, is the scalar operator
> $$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

23. Plonsey, Robert and Robert E. Collin, *Principles and applications of Electromagnetic Fields*, McGraw-Hill, NY, 1961

The authors derived the expression for the divergence by the flux model first then they stated on

> p.14: Since the operator (∇) defined earlier has the formal properties of a vector, we may form its product with any vector according to the usual rules. Thus, if we write the scalar product
> $$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$
> we discover a convenient representation of div $\boldsymbol{F}$ which leads to correct expression in cartesian coordinates. Accordingly, we adopt the notation
> $$\nabla \cdot \boldsymbol{F} = \text{div}\, \boldsymbol{F}$$

This is a typical example of presenting the divergence by many authors for the 'convenience' of memorizing a formula.

24. Plonus, Martin A., *Applied Electromagnetics*, McGraw-Hill, NY, 1978

> p.37: The above [div $\boldsymbol{D} = \partial D_x/\partial x + \cdots$ (1.91d)] can be written in simple form by employing the del operator which was first used in the gradient operation. Equation (1.91d) can be seen to be the dot product of the operator ∇ with $\boldsymbol{D}$; that is, div $\boldsymbol{D} = \nabla \cdot \boldsymbol{D} = ((\partial/\partial x)\hat{x} + \cdots) \cdot (D_x\hat{x} + \cdots)$.

The author wrote $\nabla = (\partial/\partial x)\hat{x} + \cdots$. This is very unfortunate because ∇ is a differential operator and $(\partial/\partial x)\hat{x} + \cdots$ is a differential function which happens to be equal to zero because $\hat{x}$ is a constant. In other words, the position of the unit vectors and the differentiators cannot be interchanged in ∇.

25. Ramo, Simon, John R. Whinnery, and Theodore von Duzer, *Fields and Waves in Communication Electronics*, John Wiley, NY, 1965

The expression for the divergence was derived by the flux model first, like many other authors, then they stated on

> p.83: Consider the expression for the dot or scalar product, Eq.2.10(1), and the definition of ∇ above. Then (5) indicates that div $\boldsymbol{D}$ can correctly be written as $\nabla \cdot \boldsymbol{D}$. It should be remembered that ∇ is not a true vector but rather a vector operator.

> p.116: It is noted that the above (curl) can logically be written as 'del cross $\boldsymbol{F}$'.

As a vector operator ∇ is only applicable to the gradient.

26. Roger, Walter E., *Introduction to Electrical Fields*, McGraw Hill, NY, 1954

The author first derived the expression for the divergence by the flux model, thus far denoted by $\nabla \cdot \boldsymbol{D}$ as

$$\nabla \cdot \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

in cartesian coordinates. Then he remarked:

> This explains the notation used for the shorthand. If ∇ is thought of as something like a vector
>
> $$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$
>
> when it operates on another vector by "scalar multiplication",
>
> $$\nabla \cdot \boldsymbol{D} = \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot (iD_x + \cdots)$$
>
> The result is Eq.(2).
>
> Unfortunately, the expression for the divergence in other coordinate systems is not so simple as this. It is very difficult to use (2) as defining equation for divergence and convert it suitably to other coordinate systems. For this reason, the "flux gain per unit volume" formulation is greatly to be preferred.

The author is very cautious in expressing his view. He seems to be not aware of the invariance property of the divergence operator ∇ which is defined by Gibbs as $\sum \hat{x}_i \cdot (\partial/\partial x_i)$ in cartesian coordinates and denoted by $\nabla\cdot$ but not as $\sum \hat{x}_i(\partial/\partial x_i)\cdot$. By means of the method of gradient[2] one can readily transfer $\sum \hat{x}_i \cdot (\partial/\partial x_i)$ to $\sum(\hat{u}_i/h_i)\cdot(\partial/\partial v_i)$ to prove the invariance property of the divergence operator. Another proof is given is [1]. The method of symbolic vector reveals the invariance property from the very definition of $T(\nabla)$.

27. Seeley, Sammuel, *Electromagnetic Fields*, McGraw-Hill, NY, 1958

> p.71: Observe that in rectangular coordinates the divergence is obtained by the vector operation
>
> $$\operatorname{div} \boldsymbol{E} = \nabla \cdot \boldsymbol{E} = \left(i\frac{\partial}{\partial x} + \cdots\right) \cdot (iE_x + \cdots).$$

28. Shen, Liang Chi and Kong Jin Au, *Applied Electromagnetism* (Second Edition), PWS Engineering, Boston, MA, 1987

> p.21: The symbol (∇) represents a vector partial-differentiation operator. The operation $\nabla \times \boldsymbol{A}$ is called the curl of $\boldsymbol{A}$, and the operation $\nabla \cdot \boldsymbol{A}$ is called the divergence of $\boldsymbol{A}$.
>
> p.22: Thus, $\nabla \cdot \boldsymbol{A}$ is the scalar product of the vector ∇ and $\boldsymbol{A}$.

Later the authors used the flux model to interpret the divergence and the curl.

29. Skitek, G. G. and S. V. Marshall, *Electromagnetic Concepts and Applications*, Prentice Hall, Englewood, NJ, 1982

> p.82: Example E-8. In rectangular coordinates, obtain the expanded form for $\nabla \cdot \boldsymbol{A}$.
>
> Solution:
>
> $$\nabla \cdot \boldsymbol{A} = \left(\hat{x}\frac{\partial}{\partial x} + \cdots\right) \cdot (\hat{x}A_x + \cdots) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The authors originally derived the expression for the divergence correctly based on the flux model but then added the formal scalar product model as an 'exercise'.

30. Smythe, William R., *Static and Dynamic Electricity* (Second Edition), McGraw-Hill, NY, 1950

> p.48: Let the components of $\boldsymbol{A}$ in rectangular coordinates as A_x, A_y, A_z so that
>
> $$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

31. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill, NY, 1941

p.49: Now in the analysis of the field we encounter frequently the operation

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla\nabla \cdot \boldsymbol{F} - \nabla \cdot \nabla \boldsymbol{F}$$

No meaning has been attributed as yet to $\nabla \cdot \nabla \boldsymbol{F}$. In rectangular cartesian system of coordinates x^1, x^2, x^3, it is clear that this operation is equivalent to

$$\nabla \cdot \nabla \boldsymbol{F} = \nabla^2 \boldsymbol{F} = \sum_{j=1}^{3} \left(\frac{\partial^2 F_j}{\partial (x^1)^2} + \frac{\partial^2 F_j}{\partial (x^2)^2} + \frac{\partial^2 F_j}{\partial (x^3)^2} \right) i_j$$

i.e., the Laplacian acting on the rectangular components of $\boldsymbol{F}$. In generalized coordinates $\nabla \times \nabla \times \boldsymbol{F}$ is represented by the determinant ... The vector $\nabla \cdot \nabla \boldsymbol{F}$ may now be obtained by substraction of (85) [the expression for $\nabla \times \nabla \times \boldsymbol{F}$] from the expression of $\nabla\nabla \cdot \boldsymbol{F}$, and the result differs from that which follows a direct application of the Laplacian operator to the curvilinear components of $\boldsymbol{F}$.

Actually, the Laplacian of a vector function, div grad $\boldsymbol{F}$, is a well defined function in any curvilinear coordinate system. The relation $\nabla \times \nabla \times \boldsymbol{F} = \nabla\nabla \cdot \boldsymbol{F} - \nabla \cdot \nabla \boldsymbol{F}$ is an identity. An analytical proof of this identity is given in [2]. The misunderstanding of the meaning of div grad $\boldsymbol{F}$ is partly due to Gibbs's notation for the Laplacian in the form of $\nabla \cdot \nabla$. There are two independent operators involved in this double operation. In terms of our new notation it is $\nabla\nabla\boldsymbol{F}$. Stratton's presentation was followed by P. Moon and D. E. Spencer ["The meaning of the vector Laplacian", J. Franklin Inst., 256, 1953, p.551]. They even introduced a special notation for the Laplacian of a vector function in the form of ✡ $\boldsymbol{F}$. The Laplacian of a vector function does not require a special symbol. Of course, one must remember that $\nabla \boldsymbol{F}$ is a dyadic function, and the divergence of a dyadic function is a well defined vector function which is the Laplacian of $\boldsymbol{F}$ in this case.

32. Ware, Lawrence A., *Elements of Electromagnetic Waves*, Pitnam, NY, 1949

p.26: If ∇ is written before a vector using a dot to indicate a scalar product, a scalar results, as will be shown, and it is defined as "divergence". Given a vector $\boldsymbol{V}$, the divergence is written

$$\operatorname{div} \boldsymbol{V} = \nabla \cdot \boldsymbol{V} = \left(i\frac{\partial}{\partial x} + \cdots \right) \cdot (iV_x + \cdots) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

This is immediately seen as a scalar.

33. Weber, Ernest, *Electromagnetic Fields*, Vol.1, John Wiley, NY, 1950

p.541: As a vector ∇ can be applied to a field vector either in scalar or in vector product form in accordance with (3) and (5) ($\boldsymbol{V} \cdot \boldsymbol{W}$, $\boldsymbol{V} \times \boldsymbol{W}$), respectively. The results in these two cases are:

$$\nabla \cdot \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \operatorname{div} \boldsymbol{V}$$

$$\nabla \times \boldsymbol{V} = \left[i \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \cdots \right] = \operatorname{curl} \boldsymbol{V}.$$

34. **Whitmer, Robert M.**, *Electromagnetics* (Second Edition), Prentice Hall, Englewood, NJ, 1962

p.41: The integrand of the right side of Eq.(2.52) (Gauss Theorem) is the divergence of the vector $\boldsymbol{\Gamma}$. It is obvious that this integrand is just the scalar product of the vector operator ∇ with $\boldsymbol{\Gamma}$.

p.51: ...we find that the curl is the vector product of the operator ∇ with the vector $\boldsymbol{\Gamma}$.

35. **Zahn, Markus**, *Electromagnetic Field Theory*, John Wiley, Somerset, NJ, 1979

p.24: It $(\sum \partial A_i/\partial x_i)$ can be recognized as the dot product between the vector del operator of Sec.1-3-1 and the vector $\boldsymbol{A}$.

p.30: The partial derivatives in (4) and (5) $(\partial A_k/\partial x_j - \partial A_j/\partial x_k)$ are just components of the cross product between the vector del operator of Sec.1-3-1 and the vector $\boldsymbol{A}$.

4 Miscellaneous Engineering Books

1. **Knudsen, J.G. and D.K. Katz**, *Fluid Dynamics and Heat Transfer*, McGraw-Hill, NY, 1958

p.526: The divergence div $\boldsymbol{V}$ of the vector $\boldsymbol{V}$ is defined as the scalar product of the two vectors $\boldsymbol{V}$ and $i(\frac{\partial}{\partial x}) + j(\frac{\partial}{\partial y}) + k(\frac{\partial}{\partial z})$. The scalar product results in a scalar quantity,

$$\begin{aligned}
\operatorname{div} \boldsymbol{V} &= \boldsymbol{V} \cdot \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \\
&= (i\,u + j\,v + k\,w) \cdot \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\end{aligned}$$

The two authors consider $\nabla \cdot \boldsymbol{V}$ to be the same as $\boldsymbol{V} \cdot \nabla$. $\boldsymbol{V} \cdot \nabla$ is a meaningful operator but $\nabla \cdot \boldsymbol{V}$ is Gibbs' notation for the divergence. It is not a product.

2. **L.E. Malyern**, *Introduction to the Mechanics of a Continuous Medium*, Prentice Hall, NJ, 1969

p.55: Divergence $\nabla \cdot \boldsymbol{V}$ When the vector operator ∇ operates in a manner analogous to scalar multiplication, the result is a scalar point function div $\boldsymbol{V}$, called the divergence of the vector filed $\boldsymbol{V}$. In rectangular Cartesians,

$$\operatorname{div} \boldsymbol{V} = \nabla \cdot \boldsymbol{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (i\,v_x + j\,v_y + k\,v_z)$$

or

$$\operatorname{div} \boldsymbol{V} = \frac{\partial u_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \ldots$$

The Laplacian operator $\nabla^2 \equiv \nabla \cdot \nabla$ gives a scalar point function when it operates on a twice differentiable scalar field

$$\nabla^2 F \equiv \nabla \cdot \nabla F = F_{,kk} \equiv \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

3. Streeter, V.A. and E.B. Wylie, *Fluid Mechanics*, McGraw-Hill, NY, 1985

p.99: The dot product $\nabla \cdot \boldsymbol{q}$ is called the divergence of $\boldsymbol{q}$, ... The operator ∇ is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

and the velocity vector $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = i\,u + j\,v + k\,w.$$

Then

$$\nabla \cdot (\rho \boldsymbol{q}) = (i\frac{\partial}{\partial x} + \ldots) \cdot (i\,\rho u + \ldots) = \frac{\partial(\rho u)}{\partial x} + \ldots.$$

5 References

[1] C. T. Tai, "A Historical Study of Vector Analysis", to be published

[2] C. T. Tai, *Generalized Vector and Dyadic Analysis*, IEEE Press, Piscataway, NJ, 1992