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THE RESPONSE OF A PANORAMIC RECEIVER TO CW AND PULSE SIGNALS

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

An analysis of the response of a panoramic receiver to cw and pulse signals is presented. The receiver's response is studied quantitatively as a function of the parameters: signal pulse length and frequency, receiver bandwidth, sweep-rate, and type of i-f amplifier. The effect of these parameters on the relative output amplitude, output pulse width, and apparent bandwidth is emphasized. Some general relations are derived. Two specific cases are considered. An electronic differential analyzer is used to study the response of a receiver with a single-tuned i-f amplifier to pulses having rectangular envelopes. Theoretically the response of a receiver with a Gaussian shaped i-f passband to pulses having Gaussian envelopes is derived. This answer is given in closed form. The agreement between these two cases justifies application of the Gaussian case to most practical design problems. Many curves are presented to aid the design engineer.

THE RESPONSE OF A PANORAMIC RECEIVER TO CW AND PULSE SIGNALS

1. INTRODUCTION

The study of the response of a linear resonant system to a sinusoidal driving function having a linear variation of frequency with time is pertinent in various fields of engineering. This problem is encountered when an engine is accelerated uniformly through a critical frequency.[1] The same situation occurs in the analysis of records of ocean waves by means of vibration galvanometers.[2] A panoramic superheterodyne receiver also presents this problem; and this is the problem studied in this report.

An analogous second problem is the response of a system whose resonant frequency varies linearly with time to a fixed frequency sinusoidal signal. This problem is encountered in various types of spectrum analyzers and in panoramic radio receivers.[3] For the high-Q or very much underdamped system, the two problems prove to be essentially equivalent.[2,4]

As indicated by the bibliography, there is a considerable amount of published work on these problems. Previous theoretical work has been confined to a single-tuned circuit or its mechanical analogue with constant amplitude driving functions. Microwave spectrum analyzers or panoramic radio receivers usually represent resonant systems having many degrees of freedom; and with the growing importance of pulse modulated communication and radar systems, the response to pulsed driving functions is important. This report is concerned with

[1] Lewis, F.M., Ref. 2

[2] Barber, N.F. and Ursel, F., Ref. 3

[3] Williams, E.M., Ref. 4; Barlow, H.M. and Cullen, A.L., Ref. 5; Montgomery, C.G., Ref. 6; Marlic, W.E., Ref. 11; Thomasson, D.W., Ref. 12

[4] Hok, G., Ref. 1

this more general case of the response of a resonant system to a constant amplitude or pulse modulated sinusoidal driving function having a linear variation of frequency with time. Throughout this report the problem is stated and discussed in terms of a panoramic radio receiver, but the results obtained here are directly applicable to other engineering problems.

2. DESCRIPTION OF THE PROBLEM

2.1 Assumptions

The receivers of this investigation are idealizations of conventional superheterodyne receivers. Diagrammatically, they are shown in Fig. 2.1. The function of the mixer is to convert an incoming signal of fixed instantaneous frequency to one with an instantaneous frequency changing linearly with time. It is assumed that the envelope of the incoming signal is not distorted by the mixer. The filter of Fig. 2.1 merely selects the desired frequencies, and the detector operates on the output of the filter to obtain the envelope. These assumptions reduce the problem to that of obtaining the response of a filter to a particular fm signal.[1]

Several types of filters and a variety of input signals are considered in this report. Two filters are examined theoretically: the one a single resonate circuit, the other a filter with a Gaussian amplitude response and a linear phase response curve. Both of these cases are examined with a cw input signal to the mixer. In addition, sinusoidal pulses with square envelopes (see Fig. 2.1) are studied in conjunction with the single circuit filter, and sinusoidal pulses with Gaussian envelopes are treated as input signals to the mixer with the Gaussian filter.

By means of a differential analyzer filters with one, two and four synchronously aligned single-tuned stages are examined with cw and pulse input signals to the mixer. The pulses studied all have square envelopes in this investigation.

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[1]Under some circumstances the response of a trf receiver which sweeps in frequency by changing the resonant elements is approximately the same as that of a receiver in which the resonant elements are fixed and the input signal sweeps in frequency. (Hok, G., Ref. 1) The analysis in the report is appropriate in such cases. (See also Barber, N.F. and Ursel, F., Ref. 3)

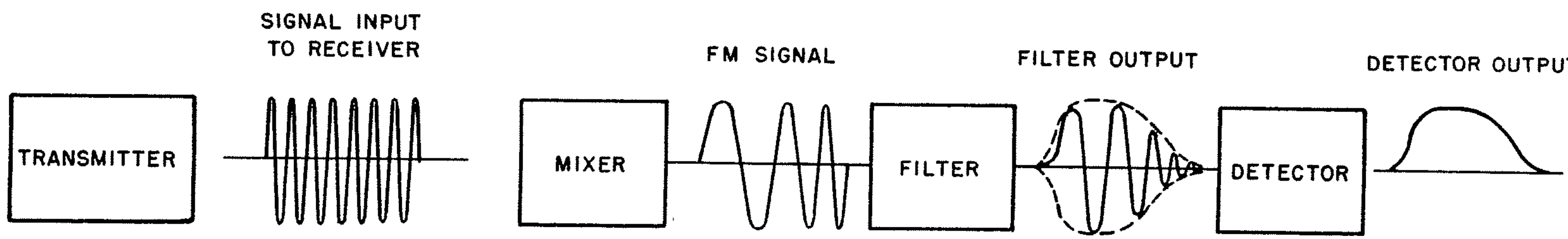

FIG. 2.1

BLOCK DIAGRAM OF IDEALIZED SUPERHETERODYNE RECEIVER

2.2 Definition of Parameters

The same notation is used throughout this report and wherever it is convenient, formulas and results are presented in normalized and dimensionless form. This facilitates the comparison of the several cases discussed.

The following parameters are used in describing the signal and the filter. The signal is assumed to have passed through the mixer, so that the filter has a constant resonant frequency and the signal frequency is varying.

a -- center frequency of the filter (radians per second)

a -- frequency of signal at the time t = 0 (radians per second)

b -- bandwidth of filter[1] (radians per second)

c -- time of center of input pulse (seconds)

d -- input pulse width[1] (seconds)

s -- sweep-rate of signal (radians per second)

These parameters are illustrated on a time-frequency diagram in Fig. 2.2. Note that the second definiton of "a" amounts to stating that the time origin is taken as the time when the signal sweeps (or would sweep) through the center frequency of the filter.

The results can be presented in terms of dimensionless variables by normalizing with respect to one of the parameters. The effect is to reduce by one the number of parameters involved. In this report the normalization is usually with respect to bandwidth. Frequencies are then in bandwidth units, e.g., $\frac{\omega}{b}$, and times in reciprocal bandwidth units, t/1/b or bt. The appearance

[1] The bandwidths and the widths of input and output pulses are generally measured between points where the amplitude drops to 0.707 of its maximum value. This convention is adopted in this report except in the treatment of the Gaussian case, where all widths are measured to the $e^{-1/4}$ points. This simplifies the formulas without seriously affecting the accuracy of the results, since $e^{-1/4}$ is only 9% larger than 0.707, the width so defined is about 18% smaller than the width between the 0.707 points. No adjustment has been made for this difference in this report.

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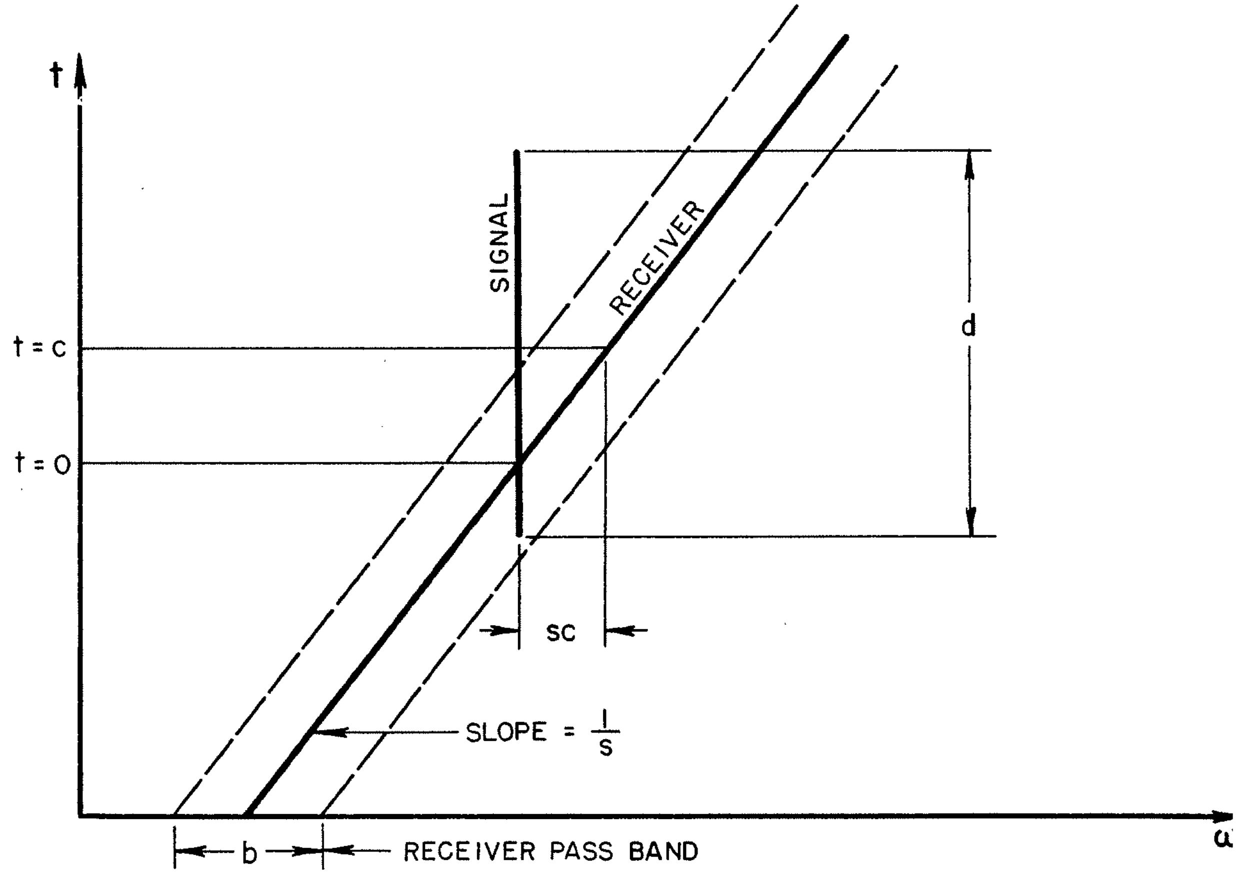

FIG. 2.2a

TIME-FREQUENCY DIAGRAM BEFORE MIXER

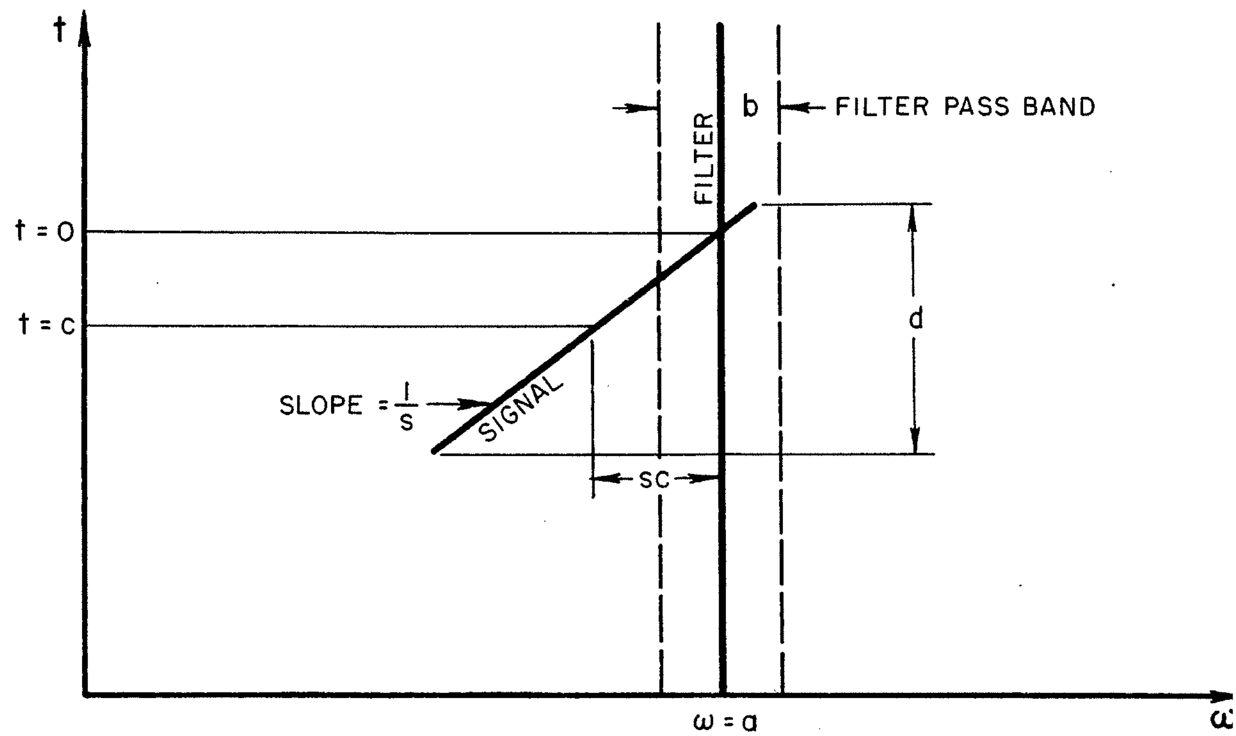

FIG. 2.2b

TIME-FREQUENCY DIAGRAM AFTER MIXER

of the time-frequency diagram in two possible sets of normalized coordinates is shown in Fig. 2.3.

2.3 Factors Describing the Response

The general nature of the response of a panoramic receiver is quite evident. The amplitude of the voltage output as a function of time is zero sufficiently early; it reaches some peak value and again approaches zero. The peak value for a pulse input depends upon the value of sc, the difference between the filter frequency and the signal frequency at the center of the pulse. The peak value is a maximum for sc approximately zero and approaches zero for large values of sc.

The most important features of the response are: (1) its peak amplitude when sc is zero, (2) the approximate width of the response in time, and (3) the width of the peak amplitude curve plotted as a function of sc. Three dimensionless measures of these features are defined below; these quantities are used to compare the cases discussed in this report.

The Relative Amplitude A

With a given input signal suppose the output voltage of the filter is g(t). Let g_o denote the steady state output voltage when the input to the filter is a cw signal with the center frequency of the filter and the same peak voltage as that of the given signal. Then the relative amplitude A is defined as:

$$A = \frac{\max |g(t)|}{|g_o|} . \qquad (2.1)$$

The value of A for sc = 0 is denoted by A_o, which is also referred to as relative amplitude. A_o is a measure of the effect of changes in bandwidth, sweep-rate, and pulse width on the amplitude of the response for a given type of filter.

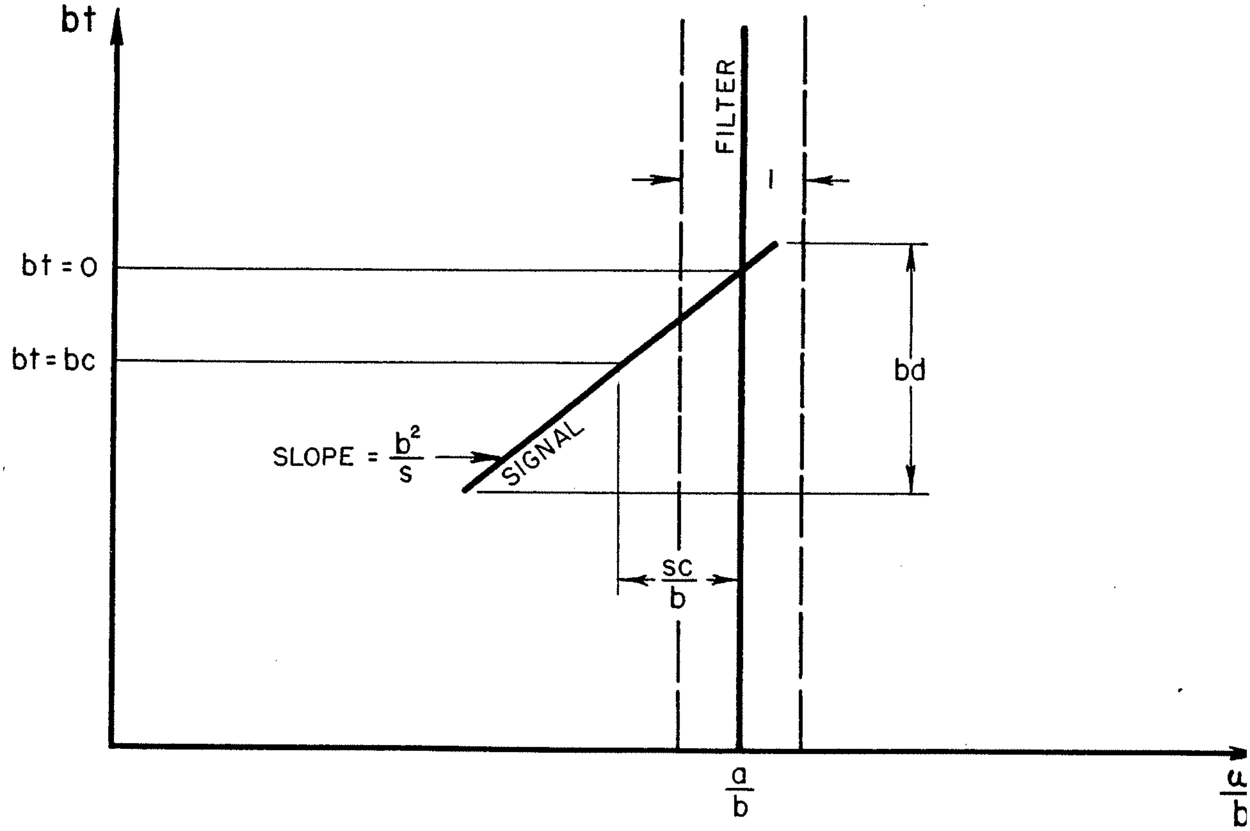

FIG. 2.3a

TIME - FREQUENCY DIAGRAM NORMALIZED TO BANDWIDTH

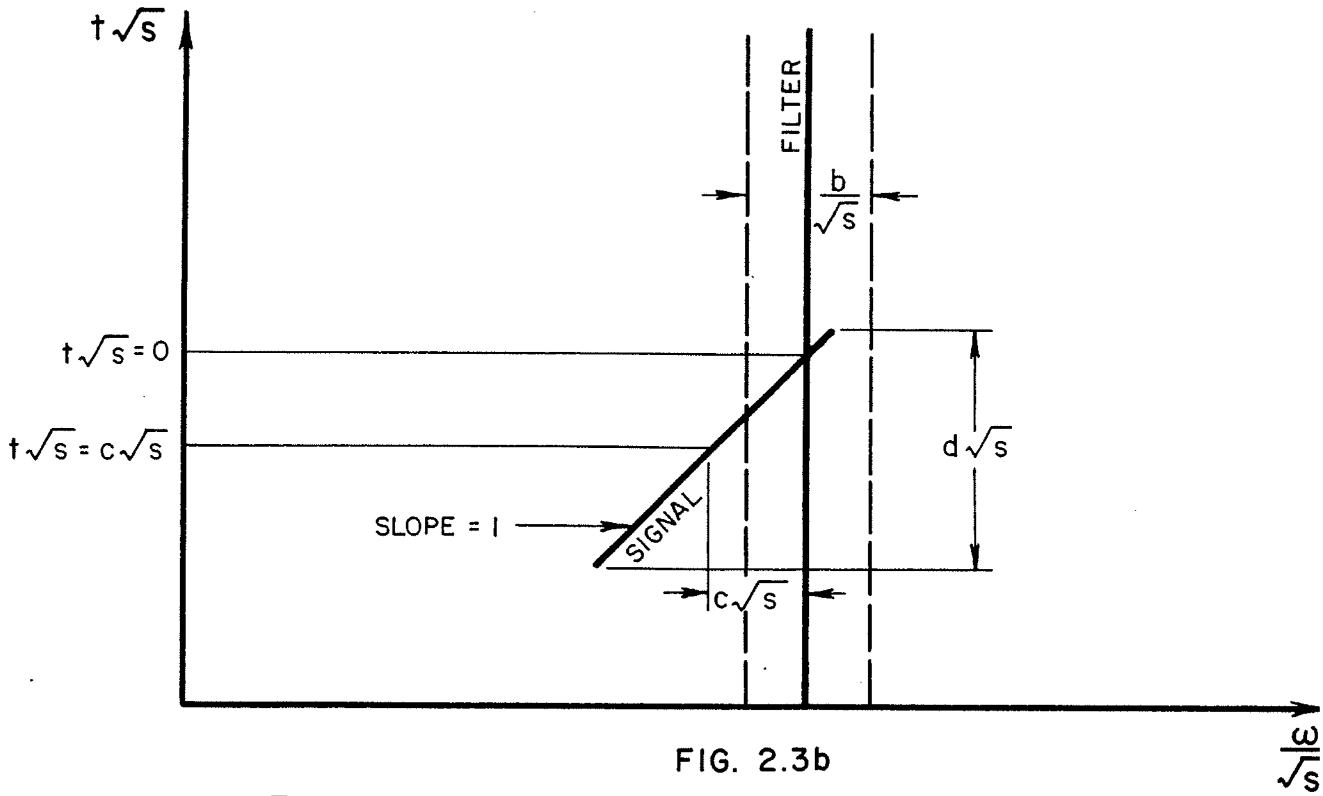

FIG. 2.3b

TIME - FREQUENCY DIAGRAM NORMALIZED TO SWEEP-RATE

The Output Pulse Width W

Suppose t_o is the width of the output pulse in seconds (between half-power points[1]). This width can conveniently be described in dimensionless form by

$$W = \frac{st_o}{b} . \tag{2.2}$$

W is the number of receiver bandwidths swept through by the signal in the duration of the output pulse. Thus, if the output of the detector is presented on an oscilloscope with the abscissa calibrated in frequency, $\frac{bW}{2\pi}$ is the width of the output pulse in cycles per second.

The Apparent Bandwidth B

Suppose for a given filter and a given type pulse input the relative amplitude A, as a function of sc, drops to 0.707 A_o at sc_2 and sc_1.[1] Then the apparent bandwidth of the receiver for pulses is by definition

$$B = \frac{sc_2 - sc_1}{b} . \tag{2.3}$$

Essentially, a signal within $\frac{bB}{2\pi}$ cycles per second of the receiver frequency is received with little attenuation.

Both W and B are important factors in the discussion of the resolution of a receiver. It was pointed out that if the output of the panoramic receiver is presented on an oscilloscope calibrated in frequency, the width of the pip on the oscilloscope would be $\frac{bW}{2\pi}$ cycles per second; the resolution could hardly be expected to be much better than this width. It was also pointed out that a

[1]See footnote p. 5

signal within $\frac{bW}{2\pi}$ cycles per second of the receiver frequency is received with little attenuation, and thus appears on an oscillograph at the receiver frequency. This discrepancy limits resolution to about $\frac{bB}{2\pi}$ (see Fig. 2.4).

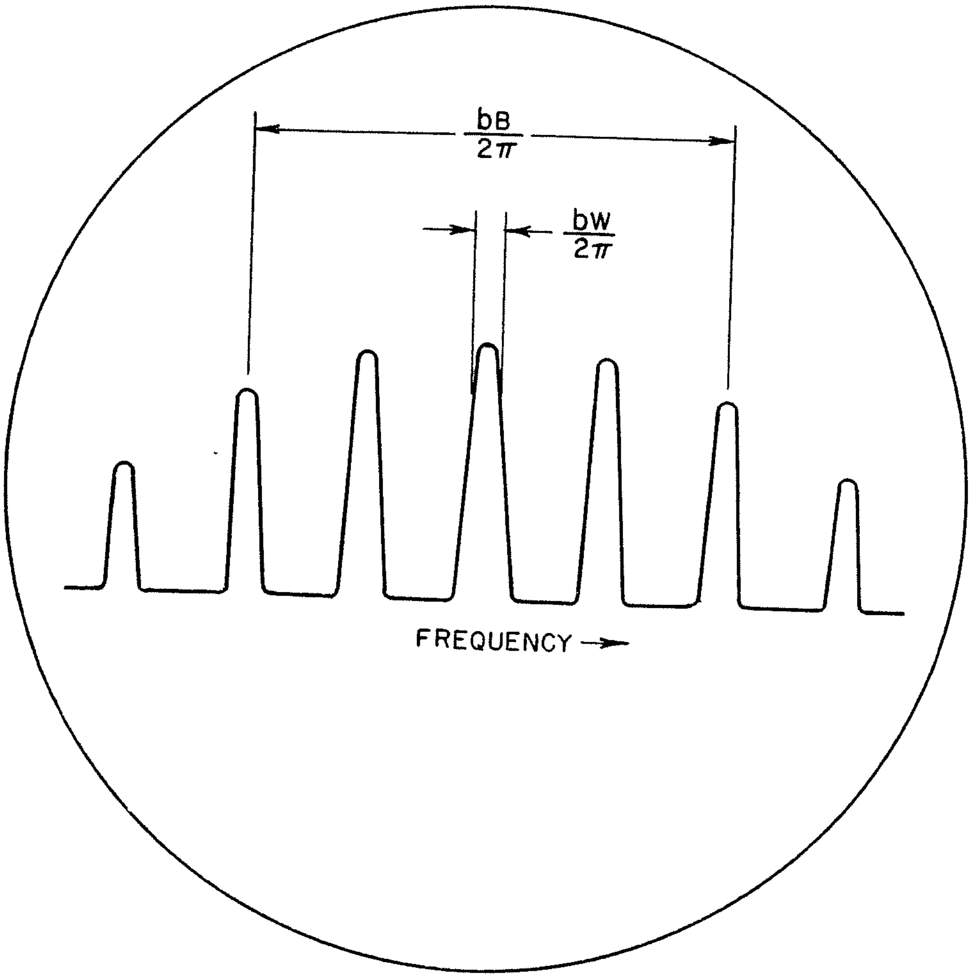

FIG. 2.4
THE RESPONSE OF A SCANNING RECEIVER TO A SERIES OF PULSES AT A FIXED FREQUENCY

3. THEORETICAL ANALYSIS

3.1 Single-Tuned Circuit

The special case of the single-tuned circuit filter, shown diagrammatically in Fig. 3.1, is examined theoretically in this section for input pulses which have square envelopes.

The assumed input current pulse to the filter is the real part of

$$i(t) = \begin{cases} \exp(jat + j\frac{s}{2}t^2), & \text{for } c - \frac{d}{2} \leq t \leq c + \frac{d}{2} \\ 0, & \text{for } t < c - \frac{d}{2} \text{ or } t > c + \frac{d}{2}; \end{cases} \qquad (3.1)$$

then the real part of the response represents the output. In this form,

a is the pulse frequency at zero time in radians per second,

s is the sweep rate in radians per second per second,

d is the pulse width in seconds, and

c is the center of the pulse in time.

The determination of the voltage that appears across the resonant circuit for this input current pulse is carried out in Appendix A. The result in complex form is given by

$$e(t) = \frac{1}{\sqrt{2s}\,C} \exp(jat + j\frac{s}{2}t^2)\, G(\gamma) \qquad (3.2)$$

where

$$\gamma = \frac{1}{2RC\sqrt{s}} + j\sqrt{s}\,t$$

and G is a function related to the error function of a complex variable and can be calculated using tables for that function. The envelope for this voltage is given by

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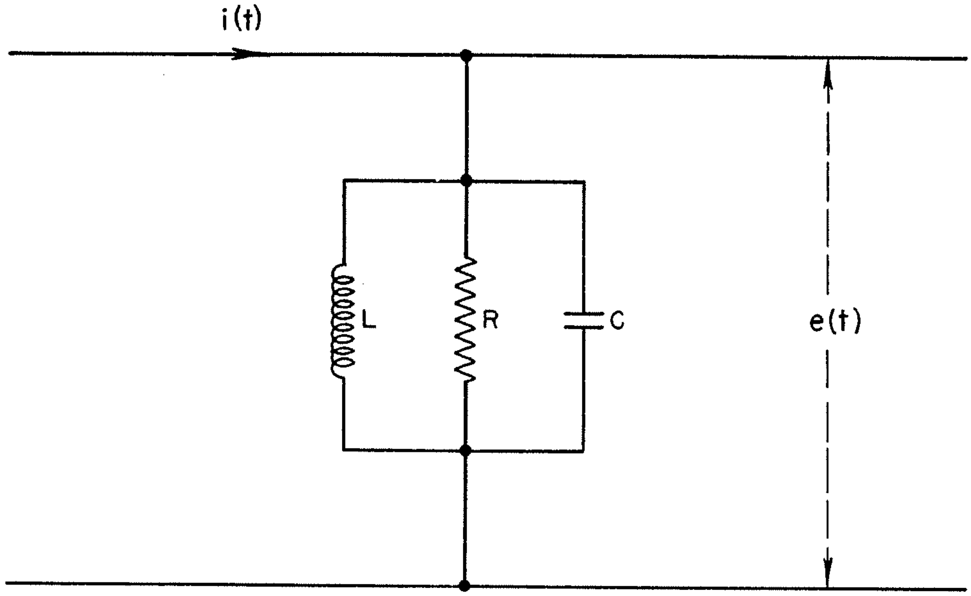

FIG. 3.1

DIAGRAM OF SINGLE-TUNED CIRCUIT FILTER

$$\left|e(t)\right| = \frac{1}{\sqrt{2s}} \left|G(\gamma)\right| . \tag{3.3}$$

The case of d $= \infty$ (cw inputs), can be found elsewhere in the literature.[1] The extension to pulses brings in two new parameters: the normalized pulse width $\sqrt{s}$ d and the normalized pulse position $\sqrt{s}$ c. Fortunately, the increase in complexity is not as severe as the two additional parameters suggest since the envelope at any time after $(c + \frac{d}{2})$ is simply given by

$$\left|e(t)\right| = \left|e(c + \tfrac{d}{2})\right| e^{\cdot \frac{t}{RC}} . \tag{3.4}$$

Therefore the starting time $(c - \frac{d}{2})$ can be taken as the only parameter, and the envelope of any pulse is readily determined from the response to the pulse which starts at the same time and continues indefinitely. Nevertheless, the complete analytic study of the normalized response with two parameters (normalized bandwidth and starting time) is tedious and, since the results can easily be obtained from a differential analyzer, only select cases have been carried out numerically. Two sets of calculated curves for particular bandwidths with the starting time as parameter are given in Figs. 3.2 and 3.3; the curves are normalized to sweep-rate. A comparison of these curves with the results obtained from the differential analyzer is made in Section 5.1 (see Figs. 5.1, 5.2 and 5.3).

3.2 The Gaussian Case

In this section the response of a filter with a Gaussian transfer function to a cw signal and to pulses with Gaussian envelopes is discussed.

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[1] Hok, Gunnar, Ref. 1; Lewis, F. M., Ref. 2; Barber, N.F. and Ursel, F., Ref. 3

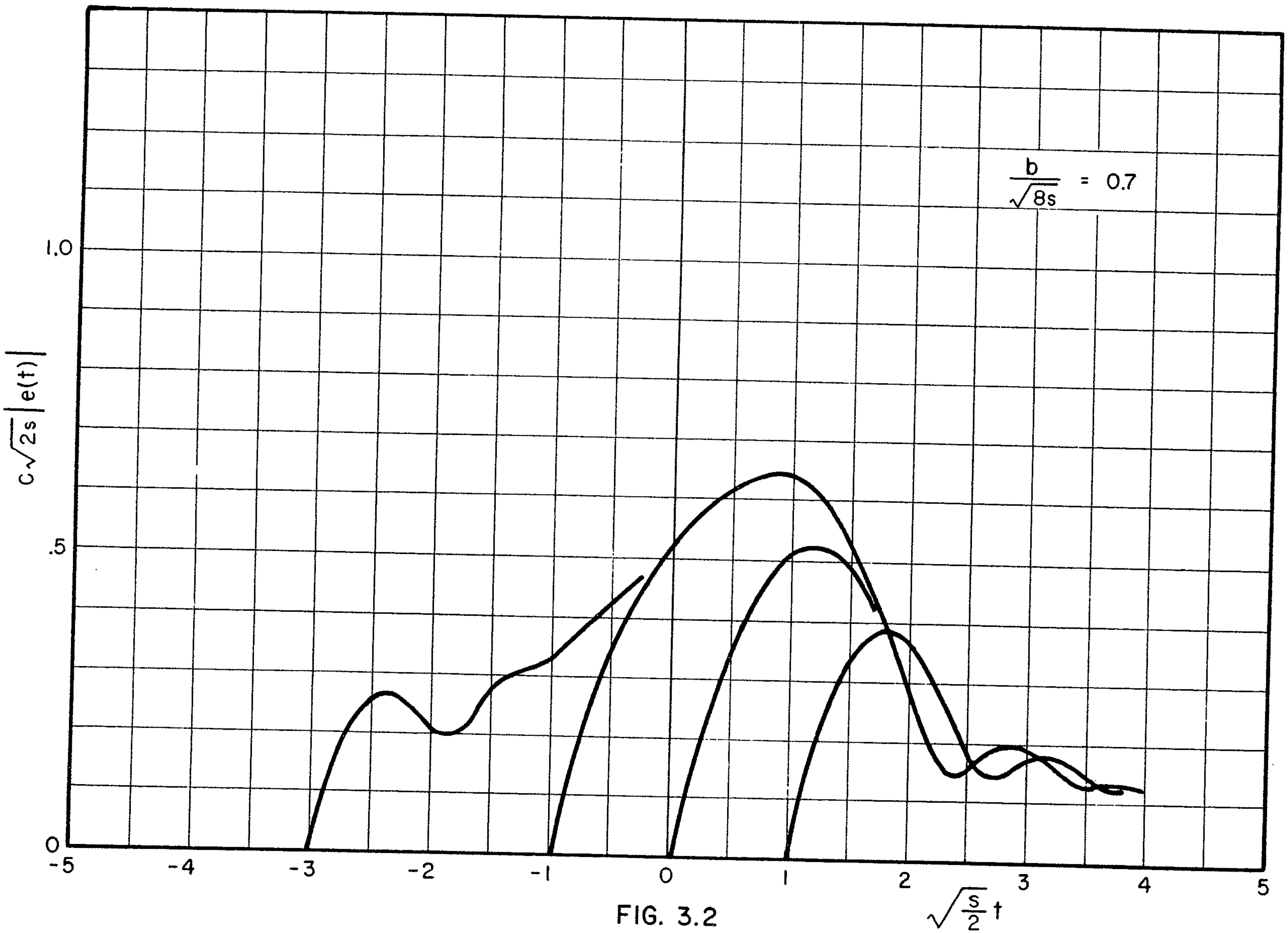

FIG. 3.2

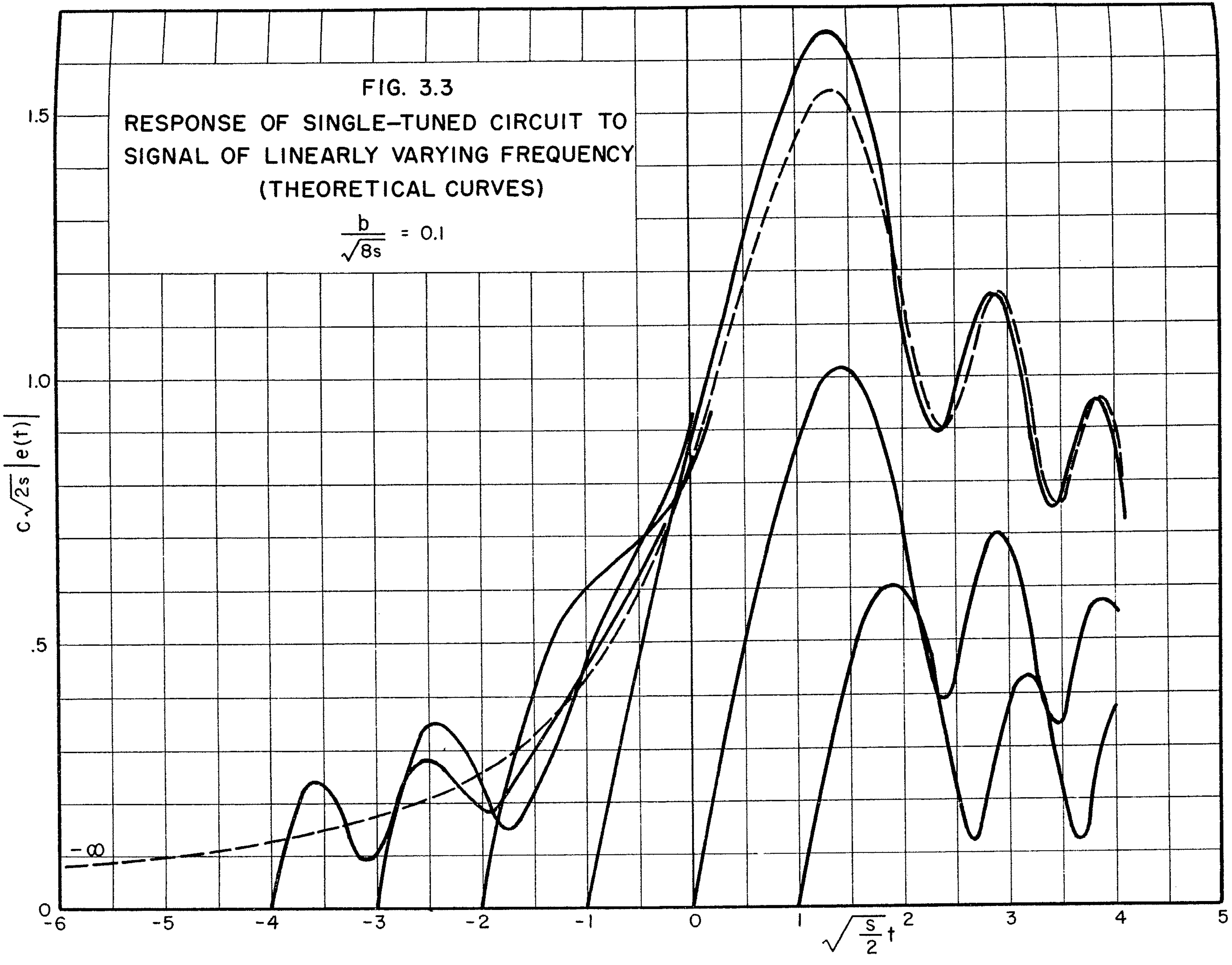

FIG. 3.3

RESPONSE OF SINGLE-TUNED CIRCUIT TO SIGNAL OF LINEARLY VARYING FREQUENCY (THEORETICAL CURVES)

$$\frac{b}{\sqrt{8s}} = 0.1$$

The most important reason for considering this case is that a closed form answer can be obtained. The Gaussian filter is not physically realizable; however, if the time delay is neglected, the transfer function of n single-tuned circuits all at the same frequency approaches the Gaussian function as n becomes large.[1] As is pointed out in Section 5.2, the envelope of the response for this hypothetical filter differs very little from that of several synchronous single-tuned circuits; therefore, a study of this case gives insight into the problem.

The transfer function assumed is

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left\{- \frac{(\omega - a)^2}{b^2}\right\}. \tag{3.5}$$

The center frequency of the filter is "a", and the bandwidth between $e^{-1/4}$ points is b.[2] Note that the phase delay is completely neglected here. The intro duction of a linear phase delay would not significantly change the answers.

The signal assumed for the cw case is,

$$f(t) = \cos\left[at + \frac{st^2}{2}\right] \tag{3.6}$$

and for the pulse case is,

$$f(t) = \exp\left[- \frac{(t - c)^2}{d^2}\right] \cos\left[at + \frac{st^2}{2}\right] . \tag{3.7}$$

The center-time of the pulse is c, and the pulse width between $e^{-1/4}$ points is d. The answer is derived first for the pulse case, and the cw case is obtained from it by letting d approach infinity.

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[1] See Section 4.1, p. 26

[2] See Footnote p. 5

The analysis is straightforward. The signal function is transformed to the ω - plane, multiplied by the transfer function $H(\omega)$, and transformed back to the t-plane. An outline of this manipulation is given in Appendix B. The envelope of the output for the pulse case is

$$|g(t)| = A_o \exp\left\{-\frac{1}{W^2}\left[\frac{s(t-t_m)}{b}\right]^2 - \frac{1}{B^2}\left(\frac{sc}{b}\right)^2\right\} \qquad (3.8)$$

where A_o, B, W, and $\frac{t_m}{c}$ are functions of s, b, and d as follows:

$$A_o = \frac{b}{\left[\left(\frac{4}{d^2}+b^2\right)^2 + 4s^2\right]^{1/4}} \qquad (3.9)$$

$$B = \frac{1}{b}\sqrt{\frac{4}{d^2}+b^2+s^2 \cdot d^2} \qquad (3.10)$$

$$W = \frac{sd}{b^2}\sqrt{\frac{\left(\frac{4}{d^2}+b^2\right)^2+4s^2}{\frac{4}{d^2}+b^2+s^2d^2}} = \frac{sd}{b} \cdot \frac{1}{A_o^2 B} \qquad (3.11)$$

$$t_m = c\left[\frac{\frac{4}{d^2}+b^2}{\frac{4}{d^2}+b^2+s^2d^2}\right] \qquad (3.12)$$

The envelope response in the cw case reduces to

$$|g(t)| = A_o \exp\left\{-\frac{1}{W^2}\left[\frac{st}{b}\right]^2\right\} \qquad (3.13)$$

where

$$A_o = \frac{b}{(b^4+4s^2)^{1/4}} \qquad (3.14)$$

$$W = \frac{1}{b^2}\sqrt{b^4+4s^2} = \frac{1}{A_o^2} \, . \qquad (3.15)$$

The definitions of A_o, W, and B given here are consistent with those in Section 2.2: A_o is the relative amplitude, W is the output pulse width, and B is the apparent bandwidth.[1]

For the Gaussian case graphs of A_o, W, and B as functions of $\frac{s}{b^2}$ are given in Figs. 3.4, 3.5, and 3.6. A_o is defined so that it has the value one for an infinite pulse and zero sweep-rate. A_o is affected little by sweeping until $\frac{s}{b^2}$ is of the order $1 + \frac{4}{(bd)^2}$, and drops off rapidly for higher sweep-rates.

By definition the output pulse width W is one for a cw signal and zero sweep-rate. The curves are never below $W = \frac{2s}{b^2}$; this is the output pulse width corresponding to the impulse response of the filter (bd = 0). For low sweep-rates the output pulse width is between the value for the cw signal and that for the impulse response. For high sweep-rates the output pulse is essentially the impulse response of the filter and is independent of bd.

The apparent bandwidth B is defined so that it is unity when the sweep-rate is very low and the pulses very long. For short pulses B is greater than one even for zero sweep-rate. As the sweep-rate increases above $\frac{1}{bd}\sqrt{1 + \frac{4}{b^2d^2}}$ the curve rises sharply and approaches $B = \frac{s}{b^2} \cdot bd$ asymptotically (see Fig. 3.6).

More curves and further discussion are included in Appendix C to show the dependence of A_o, W, and B on all the parameters. In Section 5.2, A_o, W, and B as computed from the differential analyzer data are compared with A_o, W, and B for the Gaussian case.

[1]The reader should keep in mind that A_o, W, and B are expressed in dimensionless form.

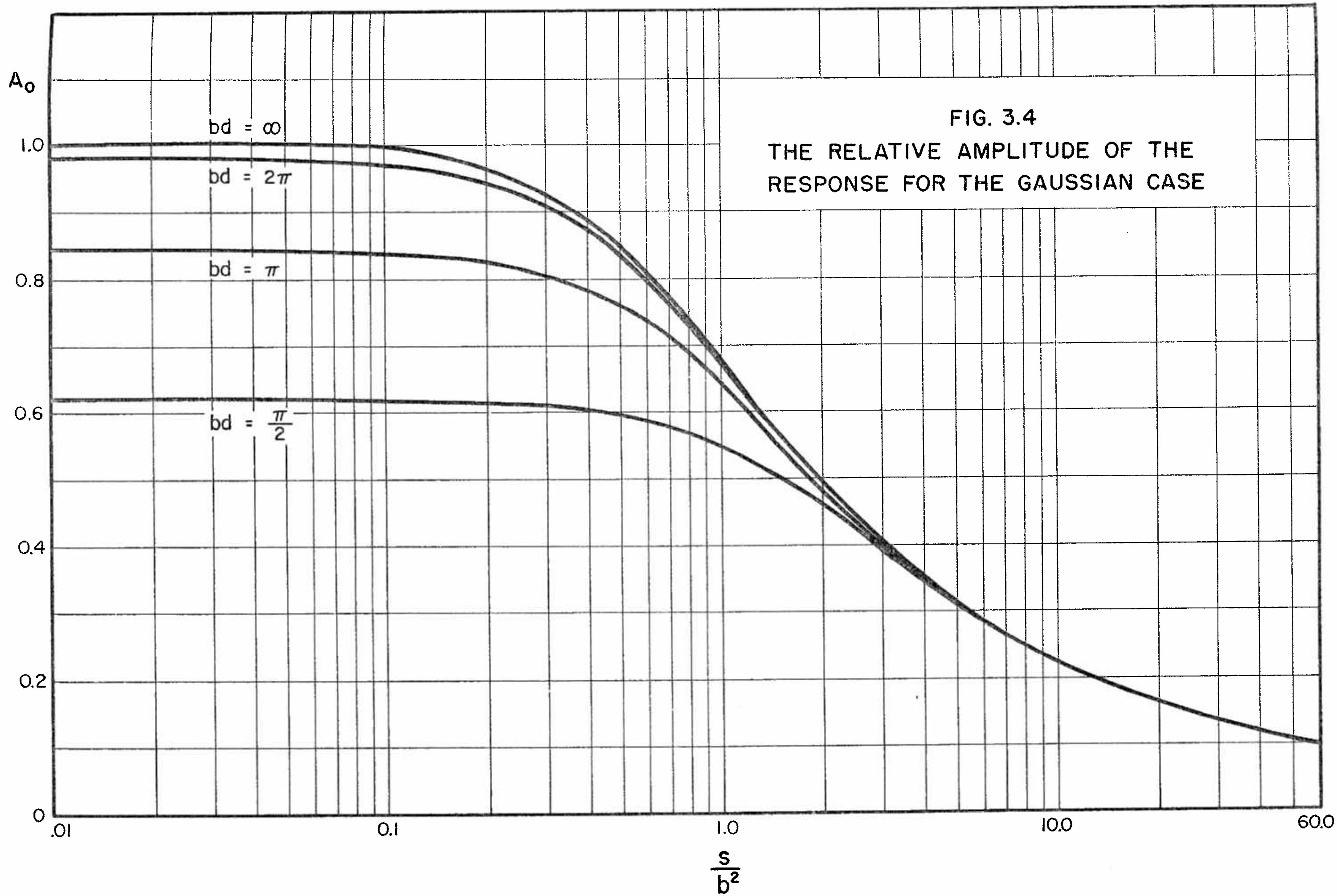

FIG. 3.4
THE RELATIVE AMPLITUDE OF THE RESPONSE FOR THE GAUSSIAN CASE

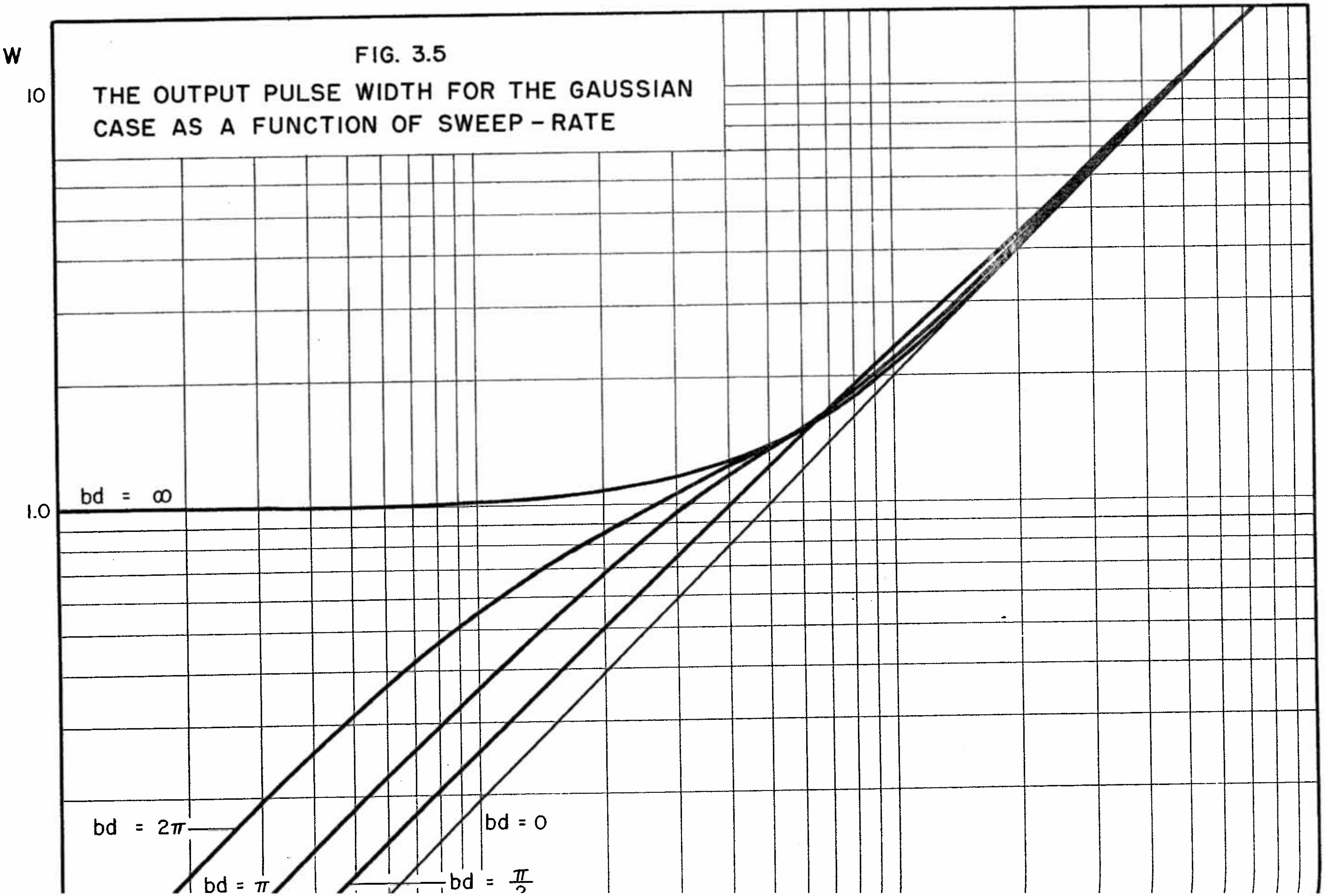

FIG. 3.5
THE OUTPUT PULSE WIDTH FOR THE GAUSSIAN CASE AS A FUNCTION OF SWEEP - RATE

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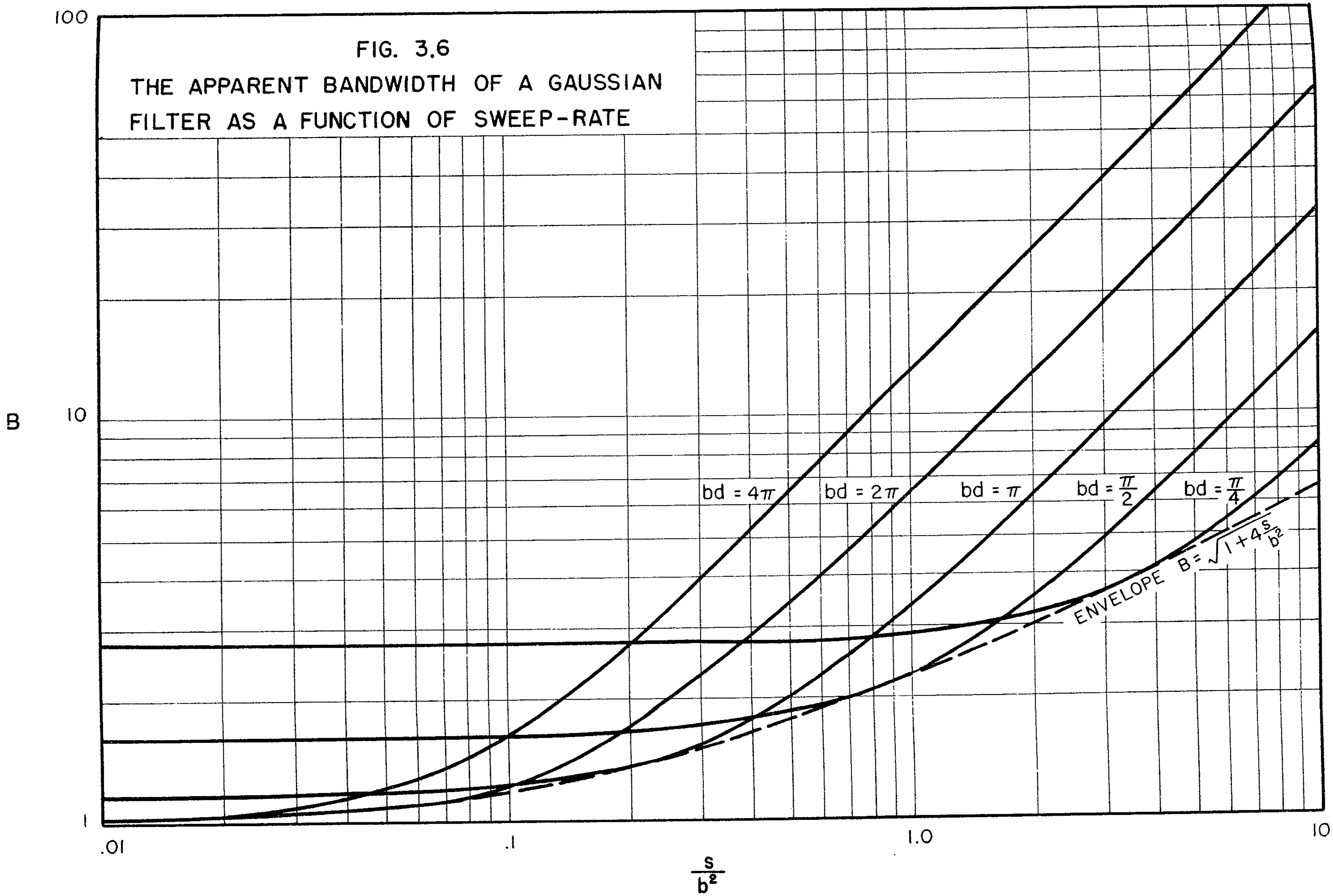


FIG. 3.6
THE APPARENT BANDWIDTH OF A GAUSSIAN FILTER AS A FUNCTION OF SWEEP-RATE

3.3 Two General Formulas

In Section 3.2 it was found that for the Gaussian case with constant amplitude input signal,

$$A_o^2 W = 1, \tag{3.16}$$

and with pulse input signals,

$$A_o^2 B \frac{Wb}{sd} = A_o^2 B \frac{t_o}{d} = 1. \tag{3.17}$$

It is natural to ask whether these formulas are more general. The answer is that they can be proved with very broad hypotheses.

The general validity of (3.16) and (3.17) depends on a different definition of bandwidths and pulse widths. The width of a pulse is defined as its total energy divided by its maximum power. The bandwidth of a filter with transfer function $H(\omega)$ is defined by

$$b = \frac{\int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega}{\max H(\omega)\, \overline{H(\omega)}} .$$

This is known as the "noise bandwidth."[1] A similar definition, in terms of the energy of pulses, is given for apparent bandwidth B.[2] Equations 3.16 and 3.17 are correct if the widths of bands and pulses are calculated in the above manner. Usually these widths differ little from the 3 db widths of curves. They coincide for square pulses. The ratio of noise bandwidth to 3 db bandwidth for synchronous single-tuned amplifiers is 1.57, 1.12, 1.13, 1.06 for one, two, four, and an infinite number of stages respectively.[1] Therefore, the formulas are still

[1]Wallman, H. and Valley, G. E., Ref. 9 , p. 169; Lawson, J. L. and Uhlenbeck, G. E., Ref. 10, pp. 176-177

[2]See Appendix D.

approximately true for 3 db bandwidths and pulse widths.

The details of the derivations are given in Appendix D. In deriving Eq 3.16, the signal

$$f(t) = f_o \cos\left[at + \frac{st^2}{2}\right]$$

is assumed to be applied to an arbitrary filter with a finite bandwidth. The energy of the output pulse is calculated using Fourier transforms and Parseval's Theorem, and the result is interpreted using the energy-type definition of bandwidth and pulse width.

The first step in deriving Eq 3.17 is to define apparent bandwidth. This is done as follows: Given any signal of finite energy, a family of signals can be constructed; each member of the family is formed by shifting the original signal in frequency. Let $f_\alpha(t)$ denote the signal which is shifted from the original by α radians per second. For each input signal $f_\alpha(t)$ the energy E_o of the output pulse is calculated. E_o is a function of α which approaches zero for large values of α (see Fig. 3.7). Then the apparent bandwidth B is defined as the width of the curve of E_o as a function of α, divided by the filter

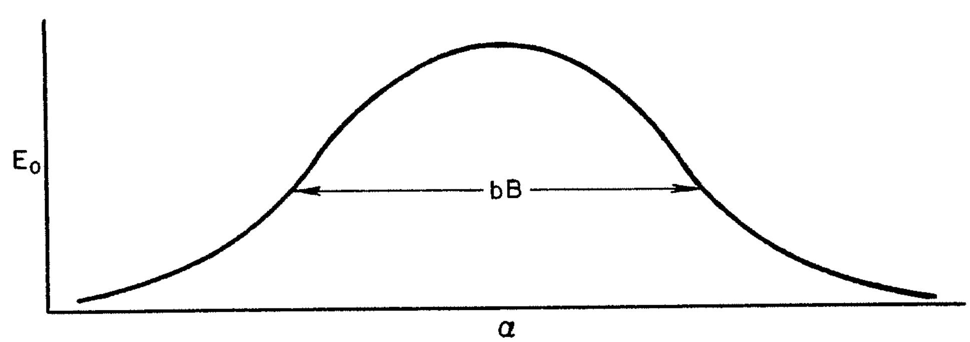


Figure 3.7

bandwidth, where the width of the curve is determined by dividing the area under the curve by its maximum height. Thus,

$$B = \frac{1}{b} \frac{\int_{-\infty}^{\infty} E_o \, d\alpha}{\max E_o} . \tag{3.18}$$

Equation 3.17 follows from this definition of B when the energy of the output pulse is calculated using Parseval's Theorem (see Appendix D).

It seems worth repeating that the derivations above assume an arbitrary filter of finite bandwidth and an arbitrary type of pulse.

It is noted in Section 3.2 that for fast sweep-rates, the apparent bandwidth B, plotted on a log log graph, has the asymptote $B^* = bd \cdot \frac{s}{b^2}$. This is true for an arbitrary filter. A proof using Eq 3.17 is given in Appendix D. By Eq 3.17,

$$B = \frac{d}{t_o A_o^2} . \tag{3.19}$$

It is shown in the appendix that the response approaches the impulse response of the filter as the sweep-rate becomes large, and thus t_o is the width of the response to an impulse, while A_o is related to the strength of the equivalent impulse. Then the behavior of B for large sweep-rates follows from Eq 3.19.

4. SOLUTIONS BY DIFFERENTIAL ANALYZER

4.1 Statement of the Problem

The physical problem is to observe the response of a bandpass filter to pulse modulated sinusoidal signal whose frequency varies linearly with time. A block diagram of the apparatus required is shown in Fig. 4.1.

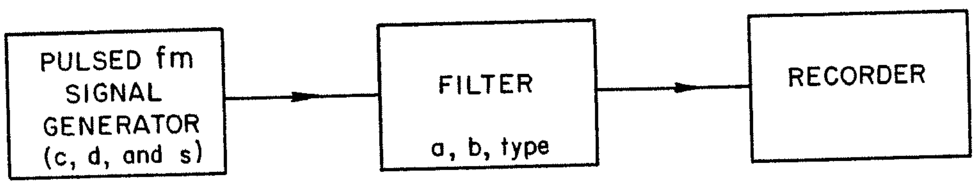


Fig. 4.1 Block diagram of differential analyzer.

This is equivalent to solving the ordinary differential equation

$$F\left[\frac{d^n y}{dt^n}, \ldots, \frac{dy}{dt}, y, t\right] = f(t) \qquad (4.1)$$

where f(t) is the time representation of the pulsed frequency-modulated signal. Since only linear filters with fixed parameters are considered in this report, the left side of Eq 4.1 is an ordinary linear differential form with constant coefficients; the order of the equation, n, depends on the complexity of the filter considered.

A differential analyzer is well suited both to the solution of Eq 4.1 and to the generation of the driving function f(t) (input pulse), through the solution of an auxiliary differential equation.[1]

[1] Bush, V. and Caldwell, S.H., Ref. 13; Ragazzini, J.R., Randall, R.H., and Russell, F.A., Ref. 14; Macnee, A.B., Ref. 15

The driving function is

$$\begin{aligned} f(t) &= 0 && \text{if } t < c - \tfrac{d}{2}, \\ f(t) &= \cos\left[at + \frac{st^2}{2}\right] && \text{if } c - \tfrac{d}{2} \le t \le c + \tfrac{d}{2}, \text{ and} \\ f(t) &= 0 && \text{if } t > c + \tfrac{d}{2}. \end{aligned} \tag{4.2}$$

Thus f(t) is a sinusoidal pulse with a square envelope. The pulse has a duration of d seconds and is centered at c seconds; the instantaneous angular frequency of the signal is

$$\omega = a + st \tag{4.3}$$

where s is the sweep-rate in radians per second per second.

The filters studied are of the sort encountered in the simplest intermediate-frequency amplifiers, a synchronous single-tuned amplifier.[1] Each filter has maximum response to sinusoidal steady-state signals at "a" radians per second, and the 3 decibel bandwidth is "b" radians per second. Filters consisting of one, two, and four single-tuned circuits are investigated. The use of one single-tuned circuit is of special interest since this case can be calculated without too much difficulty.[2] This gives a check on the analyzer solutions.[3]

The steady-state amplitude response of the n stage filter is[4]

$$|H(\omega)| = \left\{1 + \left[(2)^{\frac{1}{n}} - 1\right] \cdot \left[\frac{\omega^2 - a^2}{\omega b}\right]^2\right\}^{-\frac{n}{2}} \tag{4.4}$$

For large n

$$(2)^{\frac{1}{n}} - 1 \approx \frac{1}{n} \ln(2), \tag{4.5}$$

[1]Wallman, H. and Valley, G. E., Jr., Ref. 9
[2]See Section 3.1
[3]See Section 5.1
[4]loc. cit. p. 172

and

$$\lim_{n\to\infty} |H(\omega)| = \lim_{n\to\infty} \left[1 + \left[\frac{\omega^2 - a^2}{\omega b}\right]^2 \cdot \frac{\ln 2}{n} \right]^{-\frac{n}{2}} = e^{-\frac{\ln 2}{2}\left[\frac{\omega^2 - a^2}{\omega b}\right]^2}. \quad (4.6)$$

Further, for frequencies near resonance, $1 + \frac{a}{\omega} \approx 2$, and therefore

$$\lim_{\text{as } n\to\infty} |H(\omega)| \approx e^{-2 \ln 2 \left[\frac{\omega - a}{b}\right]^2}. \quad (4.7)$$

A filter having this Gaussian amplitude response curve can be handled by analytic means if the input pulses are assumed to have a Gaussian rather than a square envelope curve.[1] The differential analyzer data for synchronous single-tuned amplifier gives a measure of the usefulness of this analytic result.[2]

4.2 Method of Solution

4.2.1 Differential Analyzer Setup

An alternating current equivalent circuit of the intermediate frequency filter-amplifier studied is shown in Fig. 4.2. The vacuum tubes are assumed to operate in a linear fashion. The input and output capacities as well as the plate resistance r_p are lumped into the appropriate single-tuned interstage circuits which all have identical element values. The grid-plate capacities are neglected. This filter amplifier is described by the equations

$$\begin{aligned}
C\frac{de_1}{dt} + \frac{e_1}{R} + \frac{1}{L}\int e_1\,dt &= i_1(t) = f(t), \\
C\frac{de_2}{dt} + \frac{e_2}{R} + \frac{1}{L}\int e_2\,dt &= -g_m e_1, \\
C\frac{de_3}{dt} + \frac{e_3}{R} + \frac{1}{L}\int e_3\,dt &= -g_m e_2, \\
C\frac{de_4}{dt} + \frac{e_4}{R} + \frac{1}{L}\int e_4\,dt &= -g_m e_3.
\end{aligned} \quad (4.8)$$

[1] See Section 3.2

[2] See Section 5.2

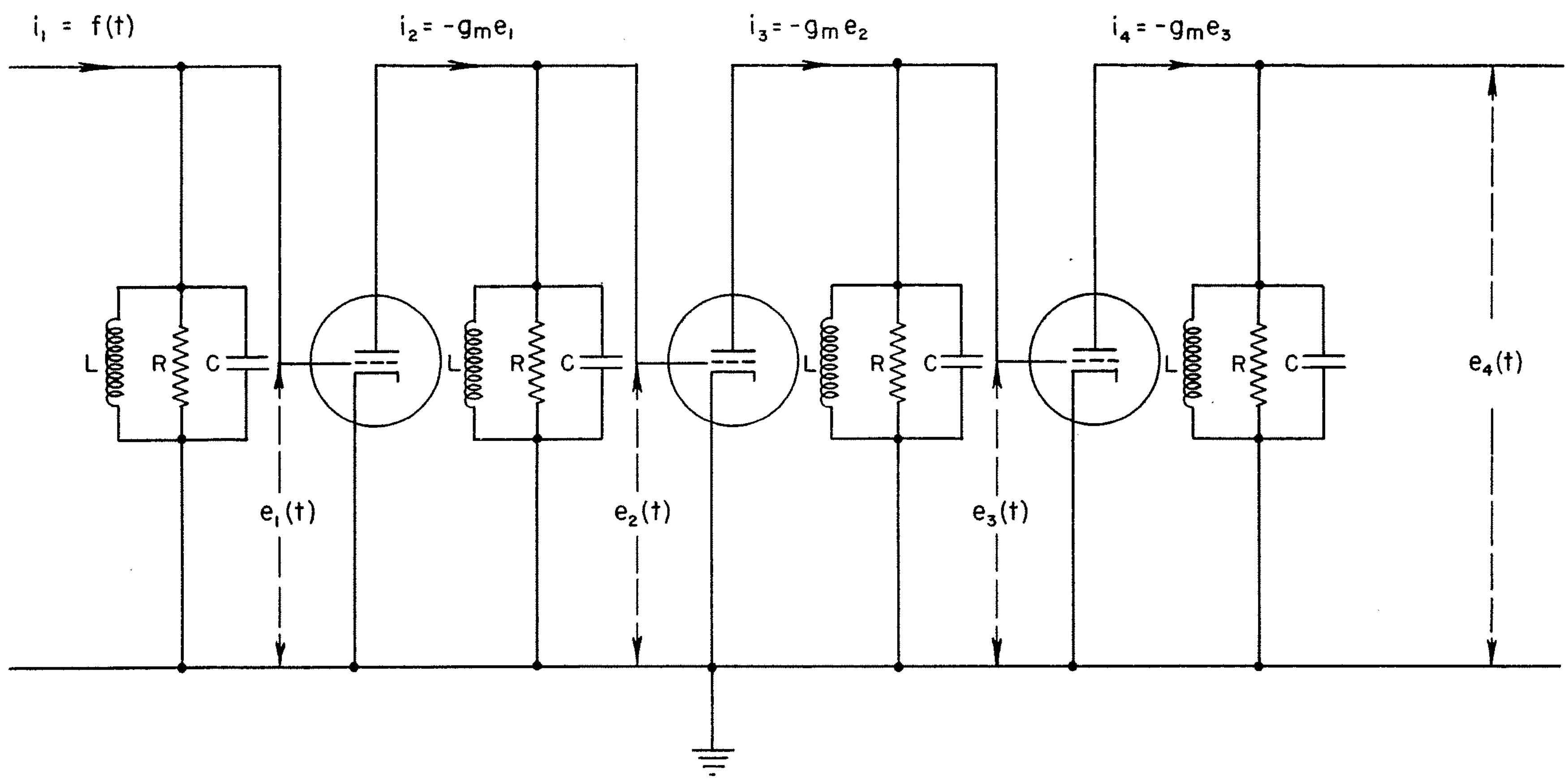


FIG. 4.2
A-C EQUIVALENT CIRCUIT OF IDEALIZED I-F AMPLIFIER
TO BE REPRESENTED BY DIFFERENTIAL ANALYZER

A block diagram of the differential analyzer setup for the solution of Eq 4.8 is shown in Fig. 4.3. Each block labeled "circuit" in Fig. 4.3 solves one of the second order differential equations of Eq 4.8. The details of the interconnection of the differential analyzer components within these blocks as well as the function generator block are given in Appendix E. Since all the circuits are tuned to a common resonant frequency and all voltages and currents are at zero at the start of every solution, there are just two parameters for each circuit box. For the jth circuit the amplitude of the driving function e_j is controlled by a potentiometer, P_{ij}, and the bandwidth b_o is controlled by a potentiometer, P_r.

In this study the bandwidths of all circuits are identical; the bandwidth of the filter consisting of n circuits is[1]

$$b = b_o\sqrt{(2)^{\frac{1}{n}} - 1} \quad . \qquad (4.9)$$

As indicated in Fig. 4.3, a four channel recorder permitted simultaneous recording of the input pulse f(t) and the responses of one, two and four single-tuned circuits. The input potentiometers were generally adjusted to give approximately full scale deflection of the recorders at the time of maximum response.

<u>4.2.2 Parameter Values</u> The function generator output was a constant amplitude sinusoidal signal that varied linearly in frequency from 10.25 to 20.19 radians per second in a period of 140 seconds. Therefore, the sweep-rate of the driving signal was always .0710 radians per second per second. The length of the input pulse d and the time of the center of the input pulse c were controlled by closing a key connecting the input signal to the first circuit box, at $c - \frac{d}{2}$ seconds; d seconds later an electronic interval timer disconnected the input

[1] loc. cit. p. 172

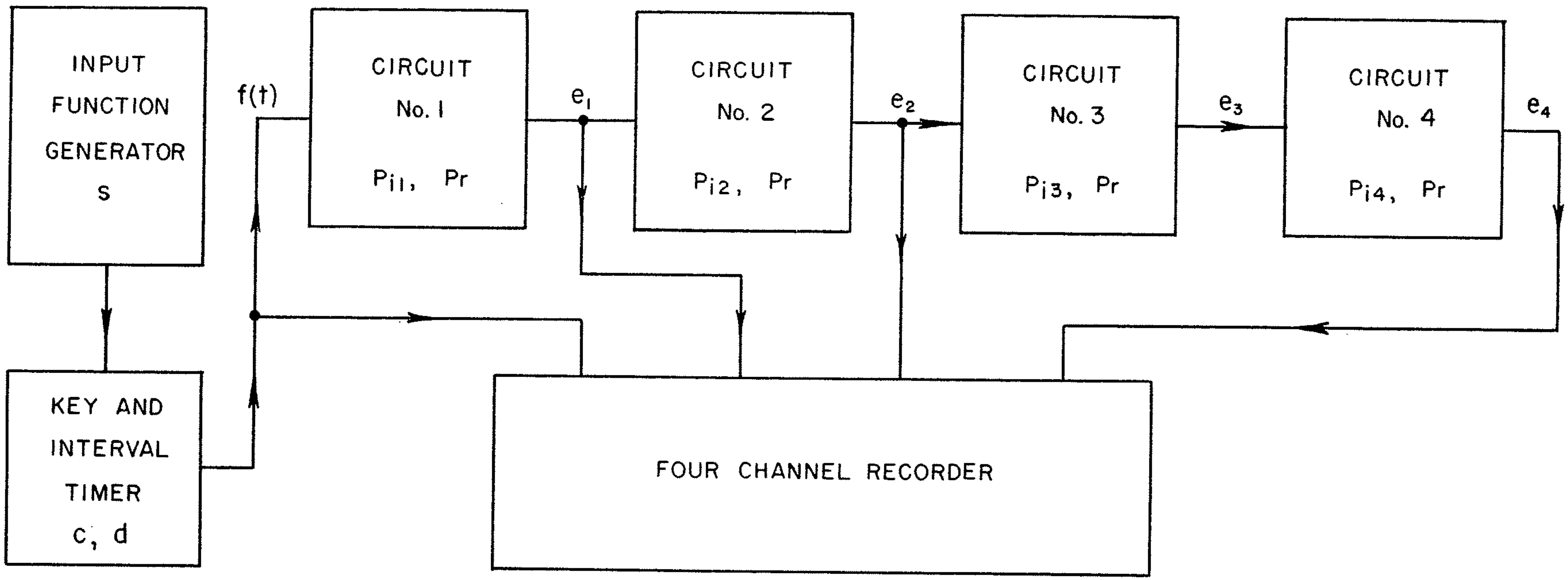


FIG. 4.3
BLOCK DIAGRAM OF DIFFERENTIAL ANALYZER SETUP FOR THE SOLUTION OF EQ. 4.8

signal. The frequency of the input signal equaled the frequency of the filter, a = 14.18 radians per second, 55.4 seconds after the start of each sweep.

Table 4.1 summarizes the ranges of parameter values. It will be noted that the Q's of the tuned circuit, $\frac{a}{b}$, range from 11.28 to 567.2.

TABLE 4.1 RANGES OF PARAMETER VALUES

a	14.18 radians per second
b	0.025 to 1.257 radians per second
c	- 55.4 to +70 seconds
d	2.65 to 125.7 seconds
s	.0710 radians per second per second

4.3 Discussion of Solutions

For convenience in the operation of the differential analyzer, all solutions were run with a fixed filter resonant frequency "a" and a fixed input signal sweep-rate s.[1] The differential analyzer solutions were run to observe the effect of varying the four remaining parameters: (1) the center time of the input pulse c, (2) the length of the input pulse d, (3) the bandwidth of the filter b, and (4) the number of single-tuned circuits. These parameters are displayed on the time-frequency diagram of Fig. 4.4.

Some of the solutions as observed at the output of the differential analyzer are discussed here. Additional solutions and discussion are found in Appendix F. A comparison between the solutions measured on the differential analyzer and the solutions calculated analytically is given in Section 5.

[1] See Appendix E for a discussion of this and for details of the differential analyzer operating proceedure.

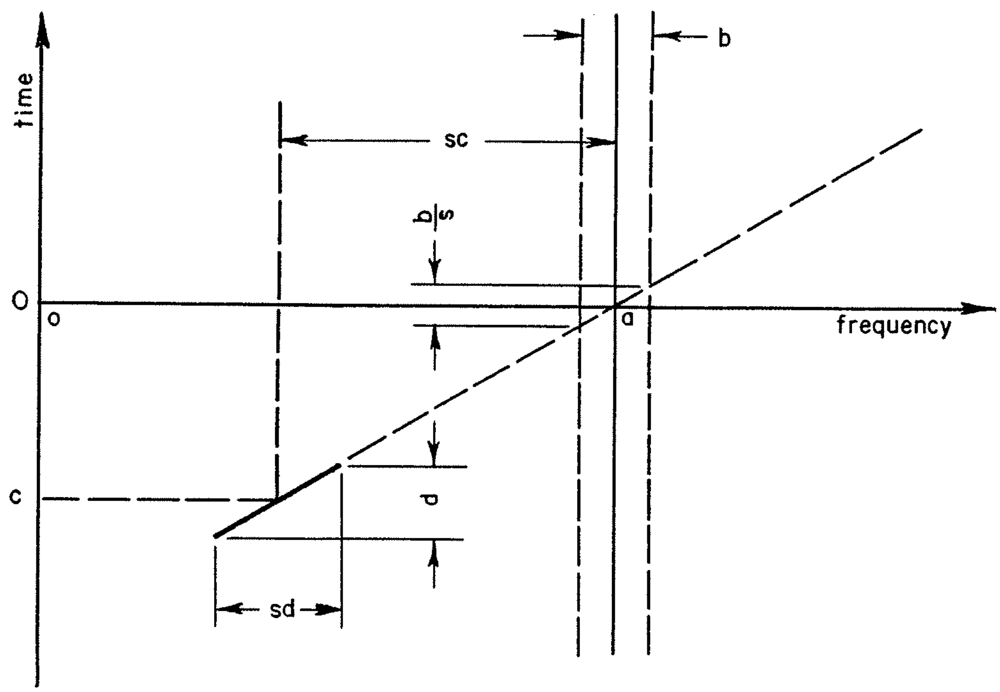


Fig. 4.4 Time-Frequency Diagram Showing Problem Parameters

4.3.1 Qualitative Results

A. Varying Filter Parameters The effect of varying the filter parameters b and n is shown by Figs. 4.5 to 4.7. Figure 4.5 shows the response of a two circuit filter to an extremely long input signal as the filter bandwidth is varied. The four output traces illustrate an important effect. For a fixed sweep-rate there is a filter bandwidth which leads to the minimum output pulse width in seconds, $\frac{Wb}{s}$. From the data shown, the optimum bandwidth appears to be about 0.34 radians per second.

Figure 4.6 shows the effect of varying the bandwidth of a one circuit filter; the input signal is a pulse 10.6 seconds long with a center frequency equal to the filter center frequency. The two important effects observable here are: (1) the reduction in the relative amplitude A_o as the filter bandwidth is

FIG. 4.5
RESPONSES FOR VARIOUS BANDWIDTHS
2 CIRCUITS
$d = \infty$
$s = .0710$

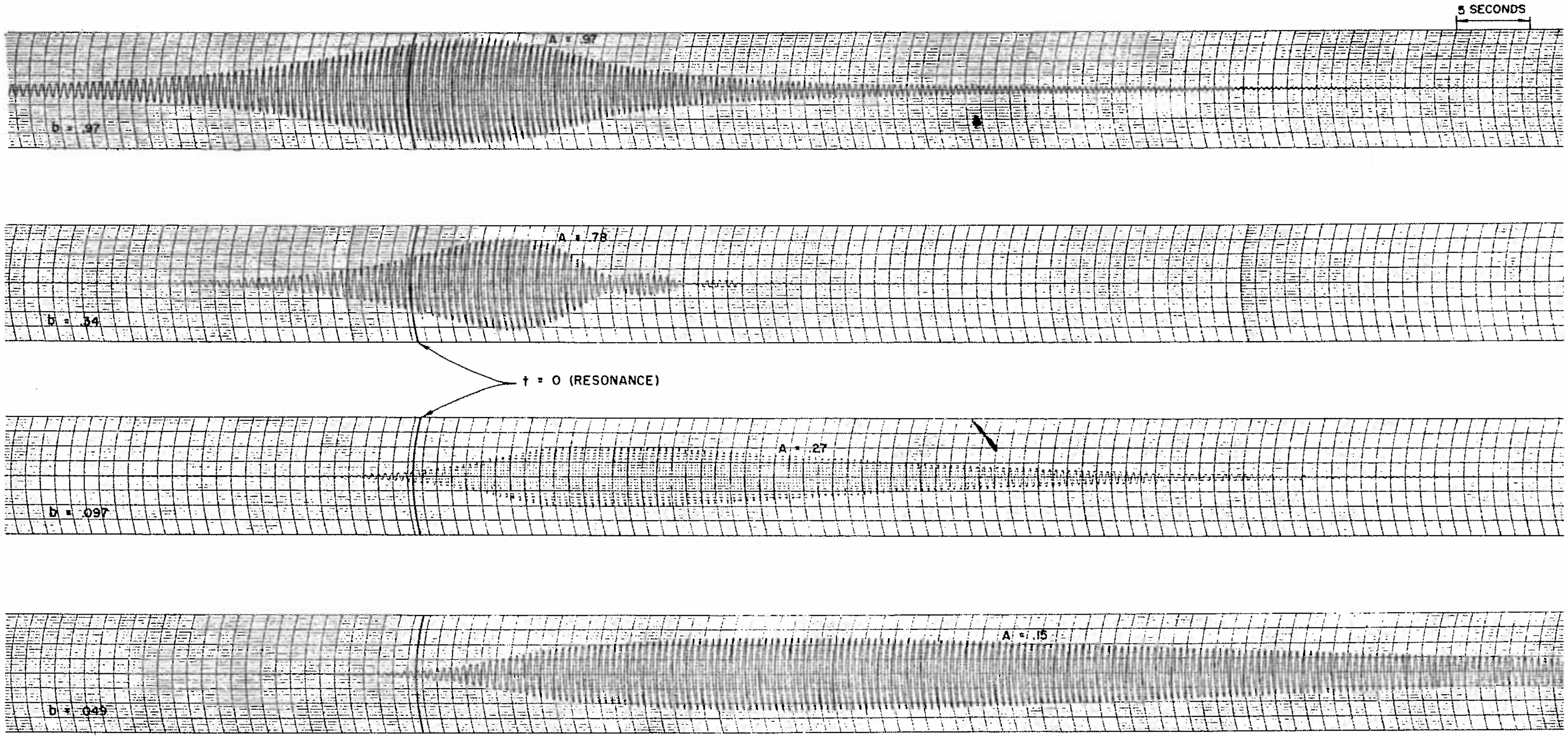
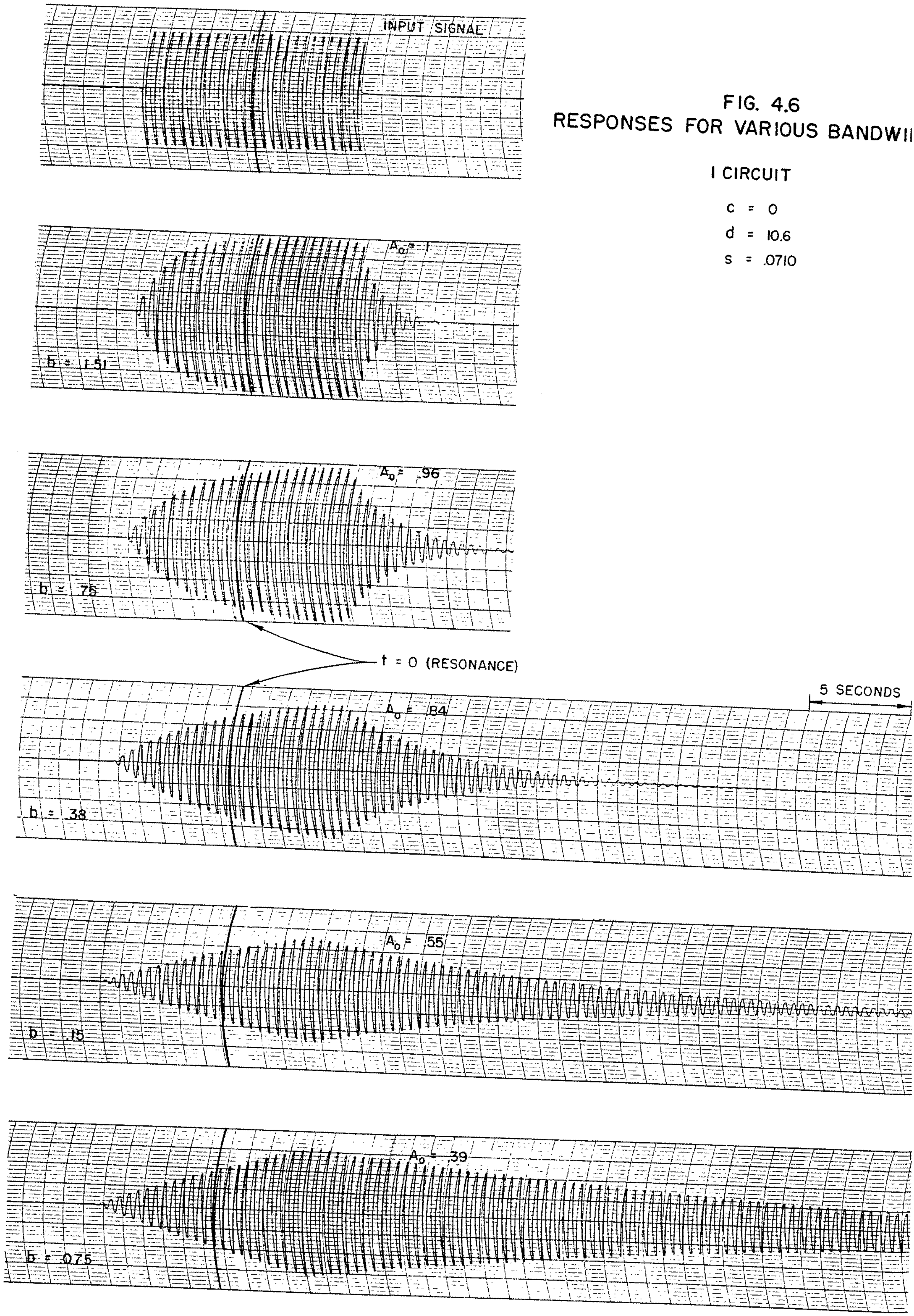


FIG. 4.6
RESPONSES FOR VARIOUS BANDWI

I CIRCUIT

c = 0
d = 10.6
s = .0710



M-970 B-P3-4 RKL 5-1-52

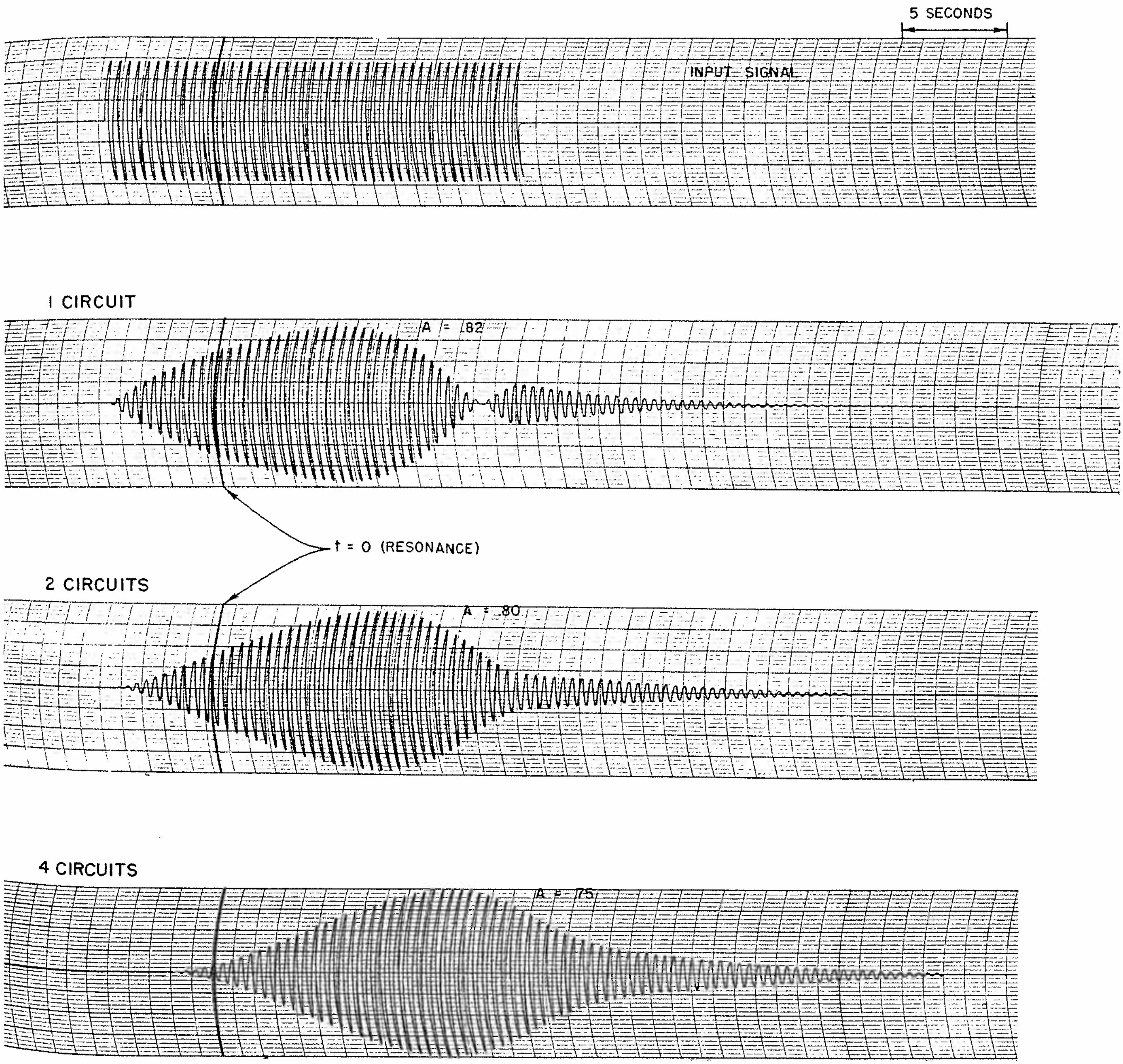
FIG. 4.7
RESPONSES FOR 1, 2 AND 4 CIRCUITS

b = .31

c = 5

d = 20

s = .0710



reduced (A_o drops from 1 to .39 as the bandwidth b is reduced from 1.51 to .07) and (2) the "spreading" of the output pulse as the filter bandwidth is reduced. The spreading of the output signal for large bandwidths observed in the previous figure does not occur here since for large bandwidth the input pulse width determines the output pulse width directly.

Figure 4.7 shows the response of 1, 2 and 4 circuit filters, all having the same bandwidth, to an input pulse having a center frequency somewhat above the resonant frequency of the filters. There is an increase in the delay of the output pulse relative to the input pulse as the number of circuits is increased, but otherwise increasing the number of circuits has little effect on the relative amplitude and output pulse width. Note the envelope of the output pulse tends towards a Gaussian shape as the number of circuits increases.

B. Varying Signal Parameters The effect of varying the signal duration d and center frequency (a - sc) is shown in Figs. 4.8 and 4.9 respectively. As one would expect, there is a reduction in the relative amplitude of the output if either (a) the input pulse duration is reduced, or (b) the center frequency of the input pulse differs appreciably from the filter center frequency. Figure 4.8 shows the effect on the output of varying d. Figure 4.9 illustrates the change in filter response as c is varied. Each output pulse is the result of a separate differential analyzer solution. The various output tapes were cut and pasted together in the proper sequence to form the figure.

4.3.2 Quantitative Results Figures 4.5 to 4.9 indicate qualitatively the effect on the filter output of varying the various signal and filter parameters. A quantitative measure of the filter response is furnished by the factors introduced in Section 2.3: the output relative amplitude A_o, the output pulse width W, and the apparent bandwidth B. More than four hundred solutions of the

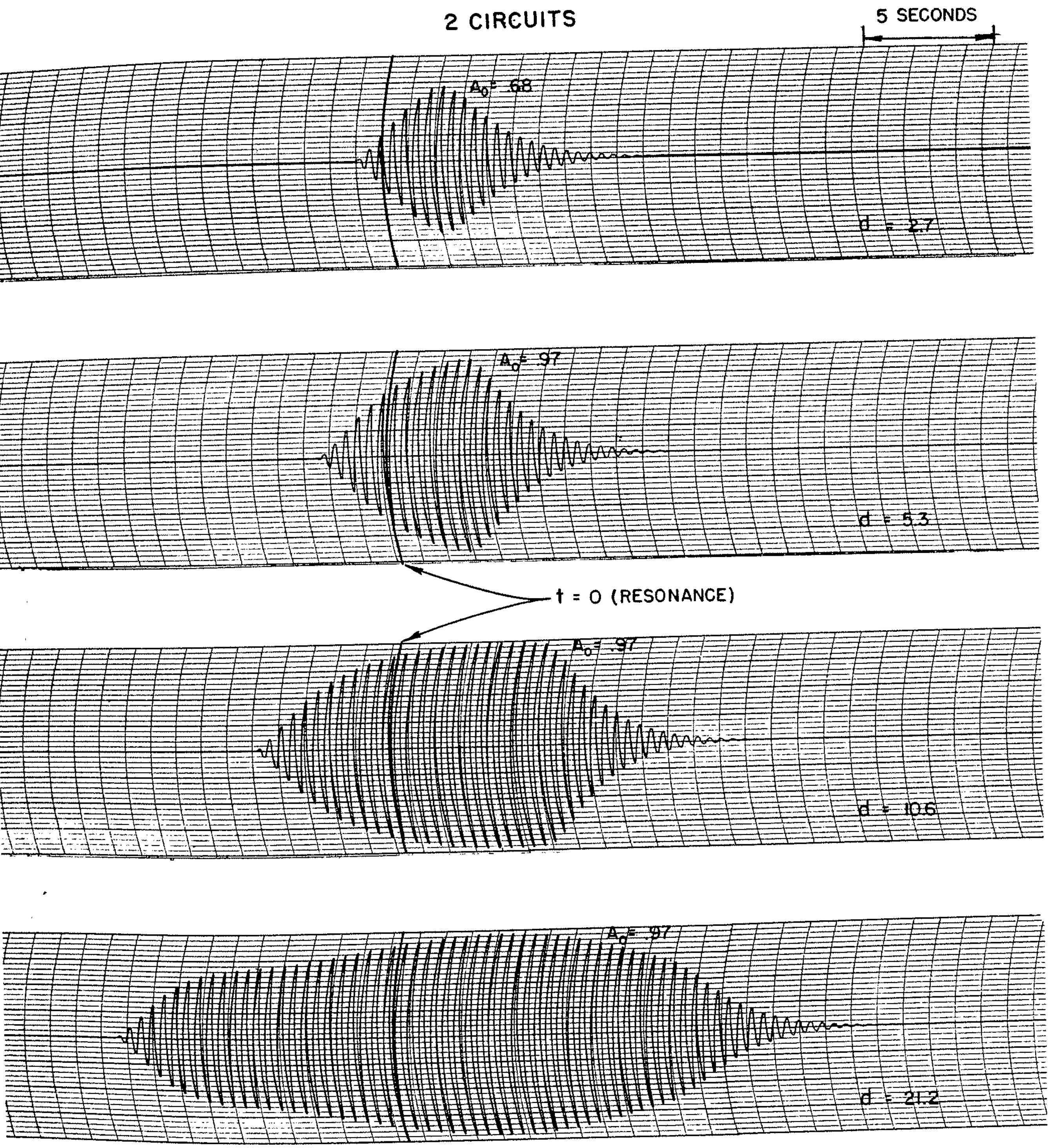
FIG. 4.8
RESPONSES FOR VARIOUS INPUT PULSE WIDTHS

b = .97

c = 0

s = .0710

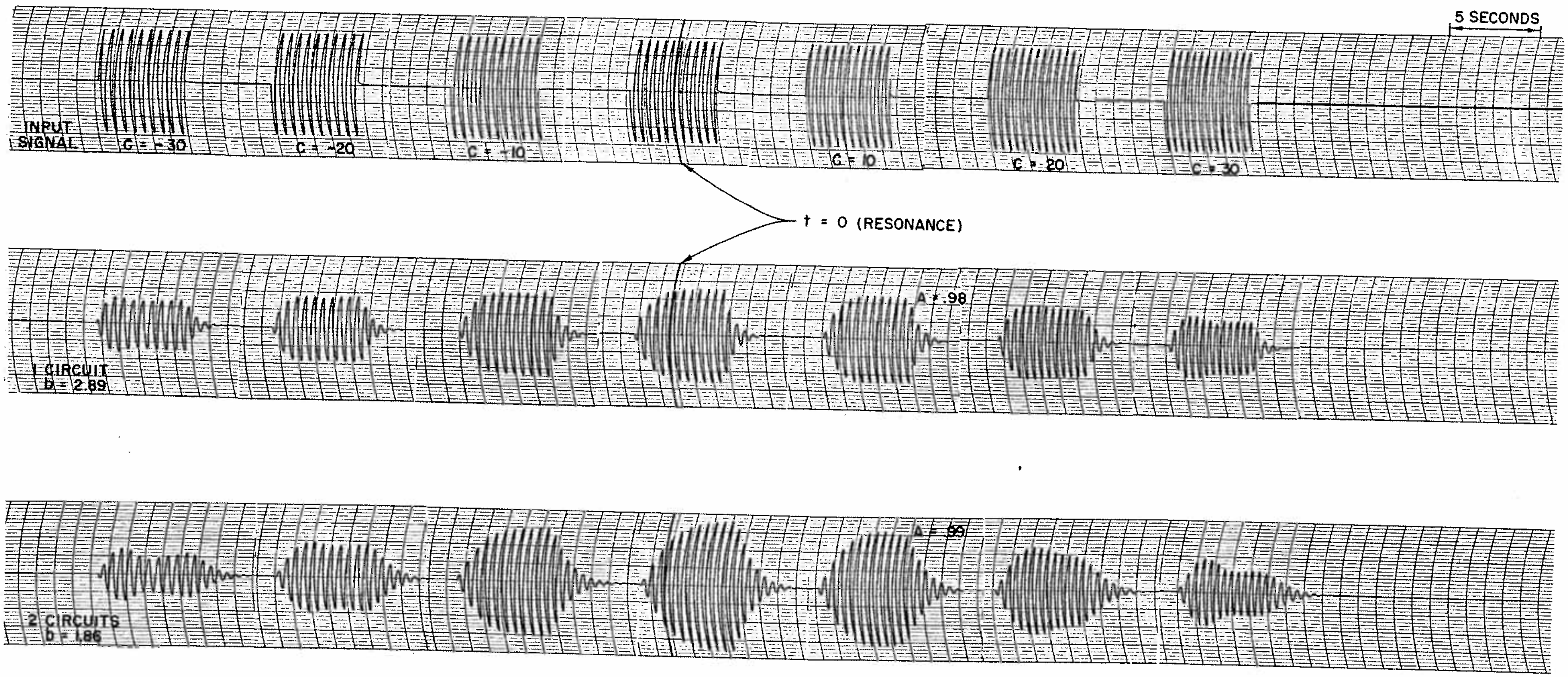
2 CIRCUITS



M-970 B-P3-7 RKL 5-5-52

FIG. 4.9
RESPONSES OF 1 AND 2 CIRCUITS FOR VARIOUS PULSE POSITIONS

$d = 5$
$s = .0710$



type shown in Figs. 4.5 to 4.9 were run on the differential analyzer to determine the dependence of these factors on the circuit and signal parameters. Some typical curves of A_o, W and B are discussed below. Additional curves and discussion are included in Appendix F.

Relative Amplitude A The relative amplitude is the peak amplitude of the output pulse normalized to the steady-state amplitude of the output for a cw input signal at the bandcenter frequency. A_o is the value of A which corresponds to bandcenter pulses ($c = 0$). Fig. 4.10 shows the variation of the relative amplitude versus sweep-rate for a cw receiver input. Curves for one, two and four circuit filters are plotted. For every case, the response is unity for very low sweep-rates. For very high sweep-rates, the responses drop off rapidly. The response of the single circuit filter is considerably greater than that of the other filters. For all filters measured the response has dropped 3 decibels at $\frac{a}{b^2} \approx 1$.

The relative amplitude as a function of the input pulse length and the sweep-rate are shown in Fig. 4.11. The cw response curve is replotted in this figure for comparison. If the pulses are short, relative to the reciprocal of the bandwidth in cycles per second, ($bd < 2\pi$), the relative amplitude response is less than the response for the cw case for very low sweep-rates. At high sweep-rates the response to pulses is approximately bounded by the cw response curve, but in every case there is a range of sweep-rates over which the response to a short pulse exceeds the response to a cw signal. For very high sweep-rates one would expect the response curves for all pulse lengths to approach the cw response curve.

Output Pulse Width W W is the width of the output pulse in time (measured between 3 decibel points) relative to the time it takes the input

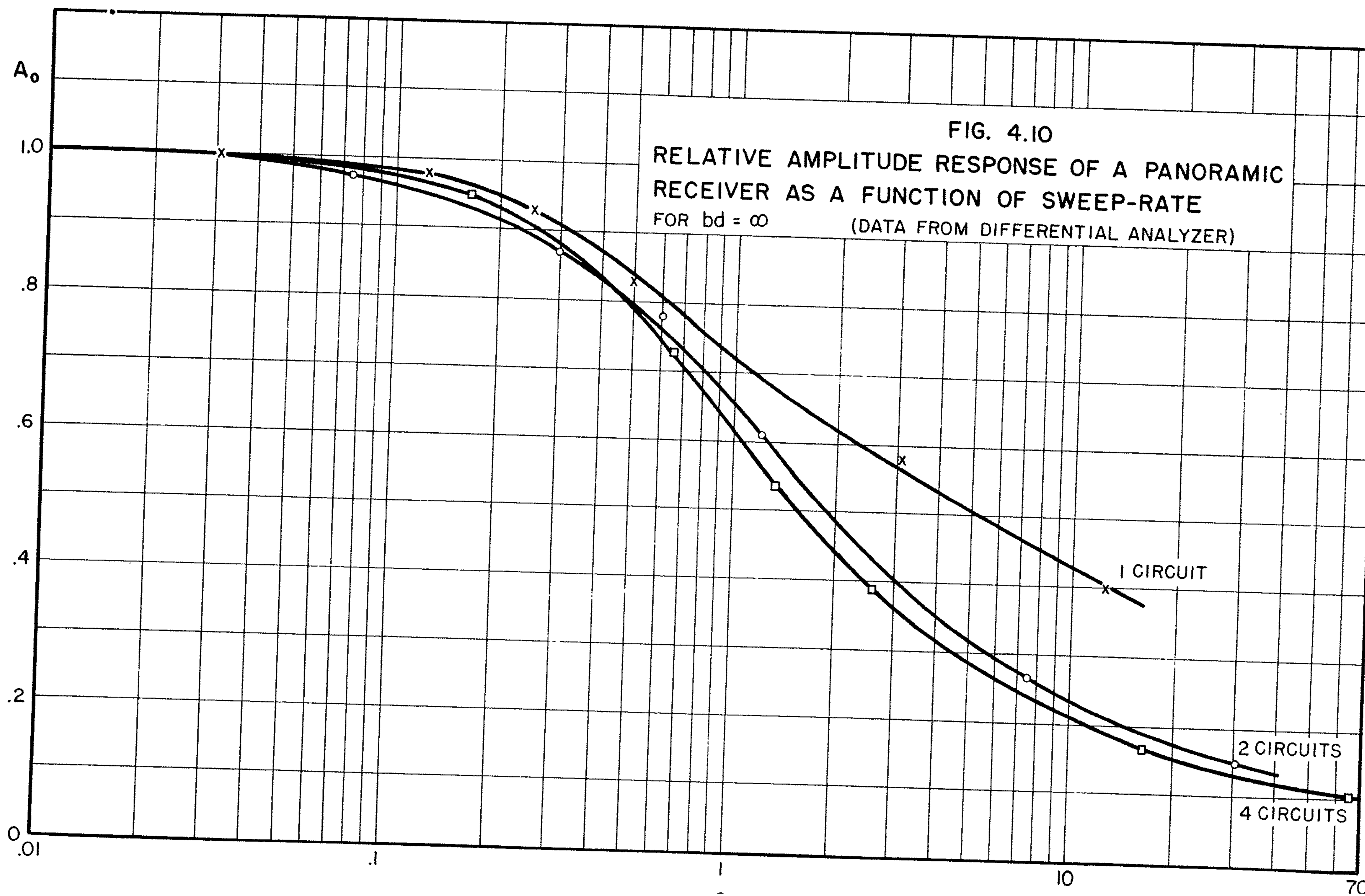


FIG. 4.10
RELATIVE AMPLITUDE RESPONSE OF A PANORAMIC RECEIVER AS A FUNCTION OF SWEEP-RATE
FOR bd = ∞ (DATA FROM DIFFERENTIAL ANALYZER)

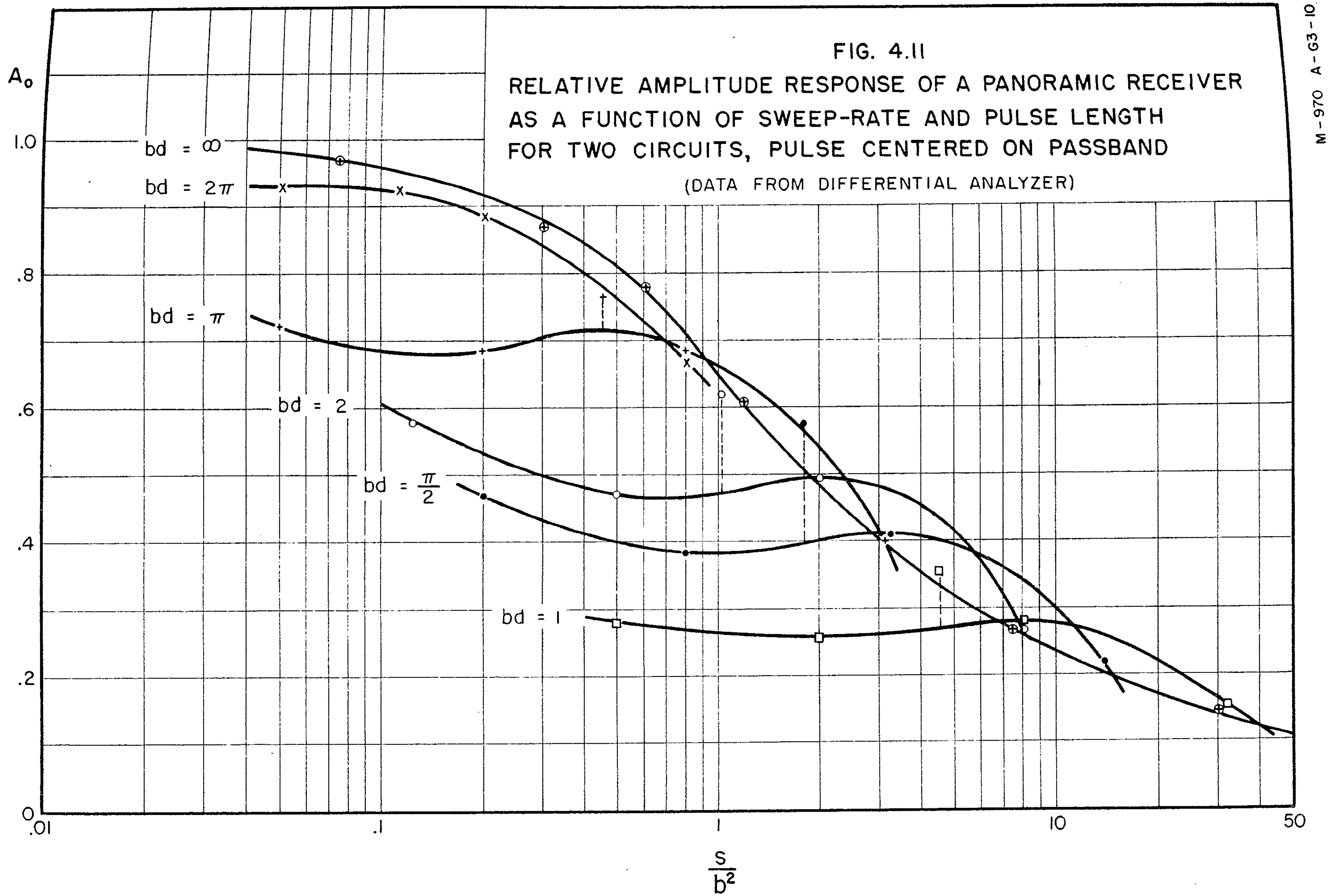


FIG. 4.11
RELATIVE AMPLITUDE RESPONSE OF A PANORAMIC RECEIVER AS A FUNCTION OF SWEEP-RATE AND PULSE LENGTH FOR TWO CIRCUITS, PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)

signal to sweep over the filter passband. Data have been obtained giving the dependence of W on the signal and circuit parameters. The curves of Figs. 4.: and 4.13 are typical of the results obtained. There is an increase of W as tl sweep-rate is raised. For large sweep-rates the width of the output pulse t_o approaches a constant determined by the filter impulse response. Then W = $\frac{s}{}$ becomes linear in s. The length of the input signal d and the number of circu n have little influence on W for high sweep-rates. An exception occurs for th case of one circuit; W is about one-half the value for the two and four circui cases. The beat phenomena which are observed in the output for the one circui case have the effect of appreciably reducing the output pulse length as measur between 3 db points.[1] It should be pointed out that this beating phenomenon c give rise to some other anomalous results.[2]

Apparent Bandwidth B As indicated qualitatively in Fig. 4.9, the relative amplitude of the output pulse is a function of the difference between the center frequency of the input pulse and the filter bandcenter, sc. The pea response drops off as the magnitude of sc is increased. A typical series of plots of relative amplitude A versus $\frac{sc}{b}$ is shown in Fig. 4.14. The center valu of the curves shown is A_o. The apparent bandwidth, B, is defined as the distan between the points on these curves at which A = $\frac{A_o}{\sqrt{2}}$. Each curve of Fig. 4.14 determines one value of B.

Plots of B as a function of sweep-rate with the number of circuits and the length of the input pulse as parameters are given in Fig. 4.15. For very low sweep-rates B has a horizontal asymptote. For very large sweep-rates B has

[1] See Fig. F.4, Appendix F
[2] See Appendix F

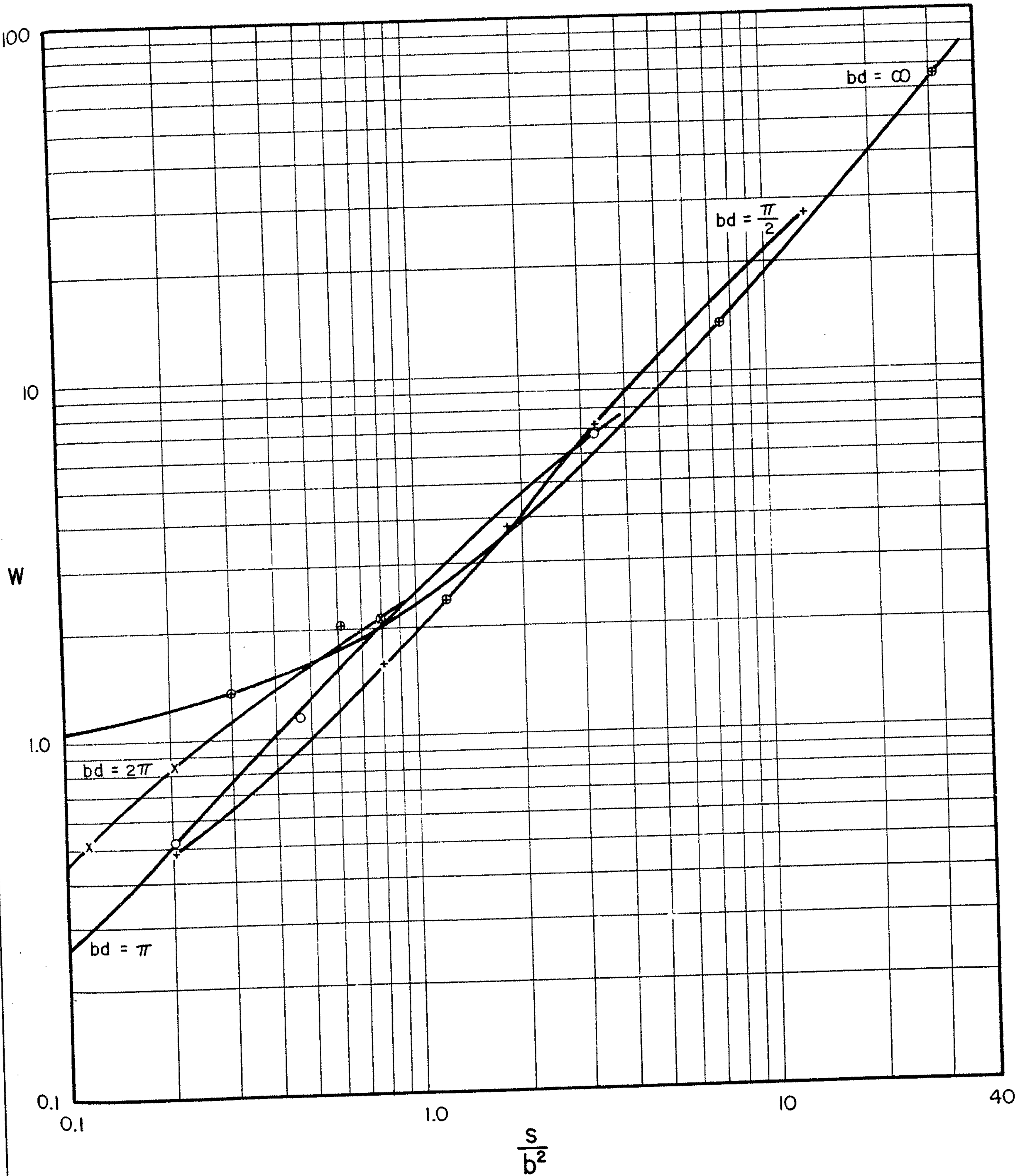


FIG. 4.12

FREQUENCY SWEPT BY PANORAMIC RECEIVER DURING OUTPUT PULSE AS A FUNCTION OF SWEEP-RATE. (2 CIRCUITS, PULSE CENTERED ON PASSBAND)

(DATA FROM DIFFERENTIAL ANALYZER)

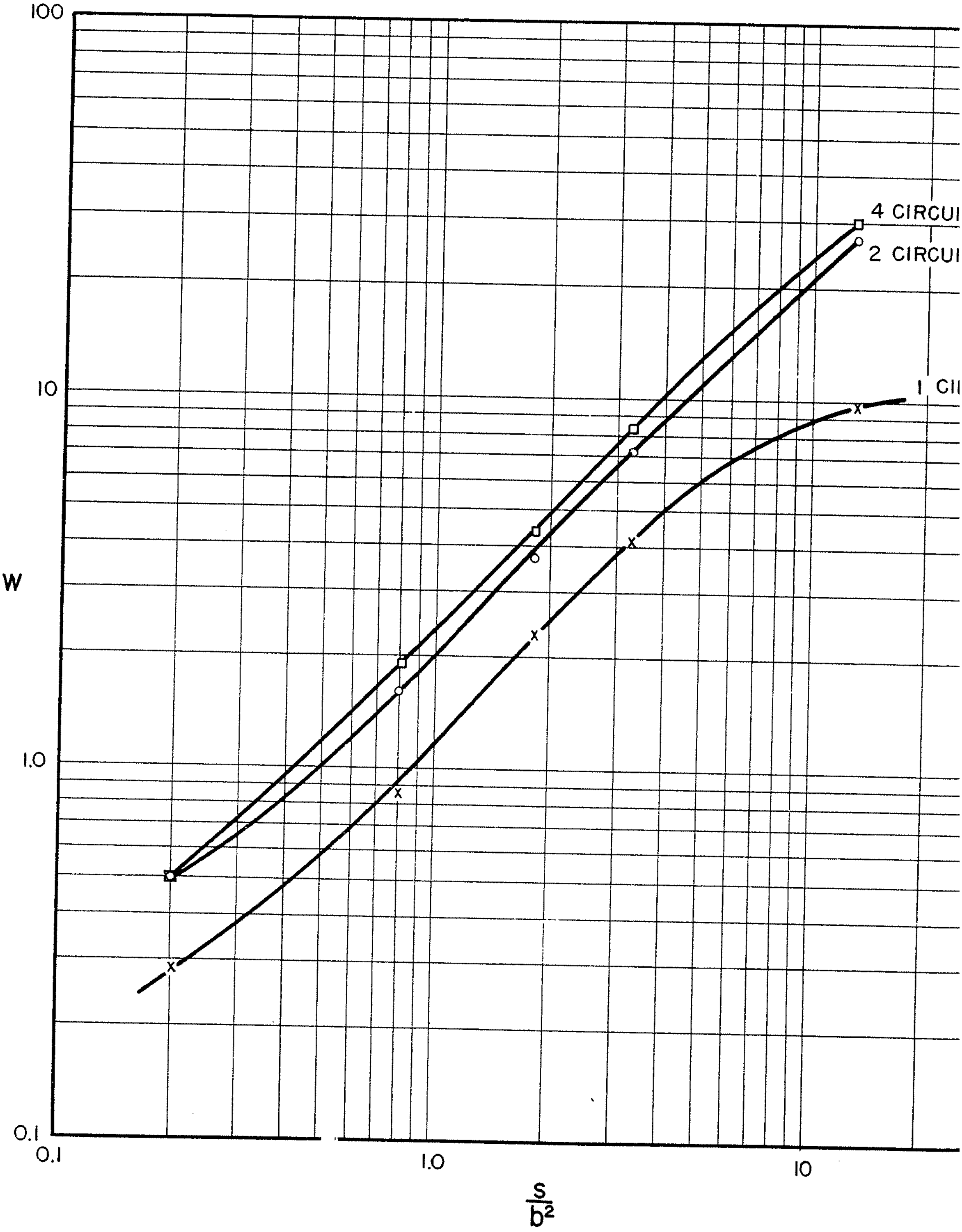


FIG. 4.13

FREQUENCY SWEPT BY PANORAMIC RECEIVER DURING OUTPUT PULS
AS A FUNCTION OF SWEEP-RATE. PULSE CENTERED ON PASSBAND
FOR $bd = \frac{\pi}{2}$ (DATA FROM DIFFERENTIAL ANALYZER)

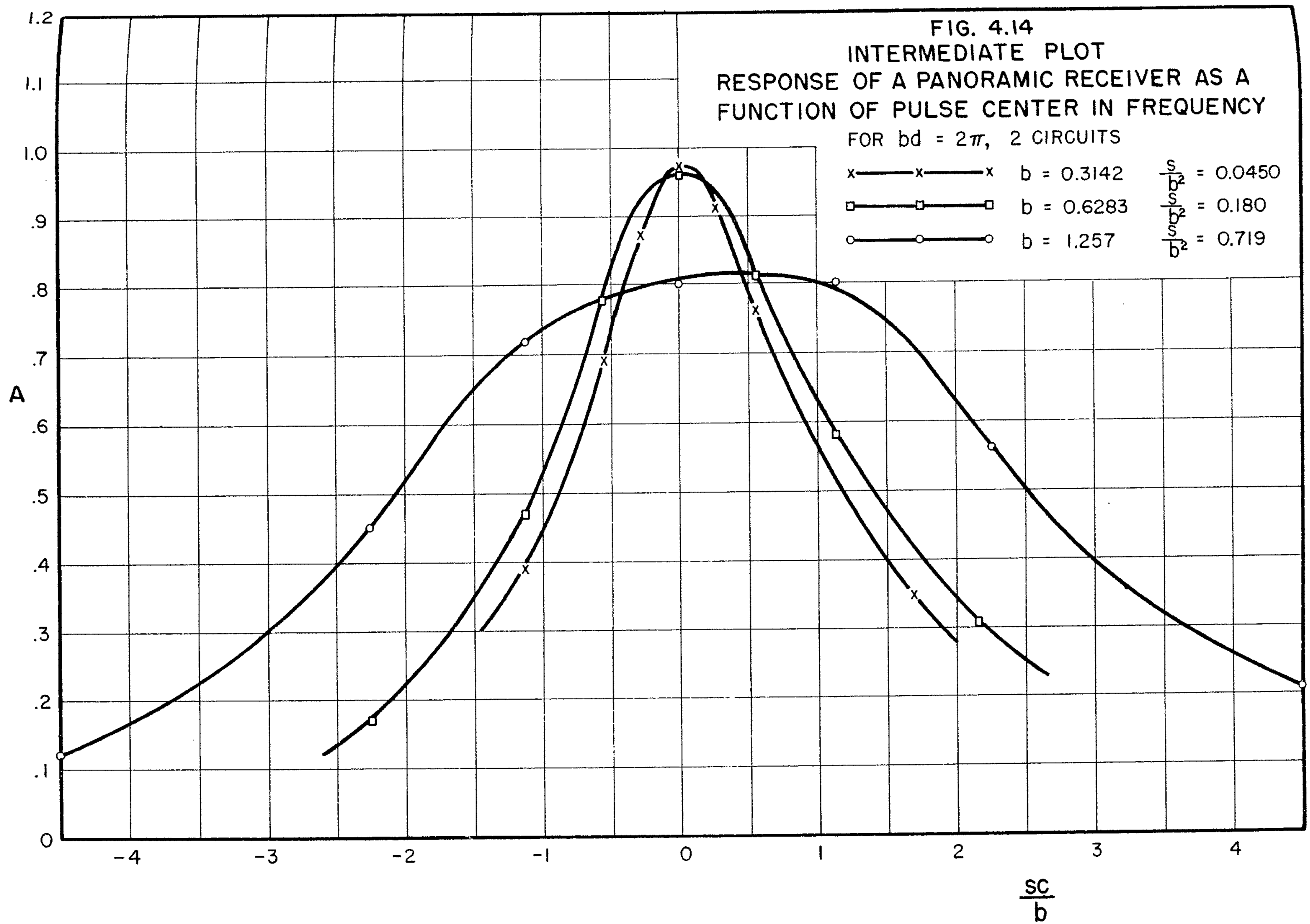


FIG. 4.14
INTERMEDIATE PLOT
RESPONSE OF A PANORAMIC RECEIVER AS A
FUNCTION OF PULSE CENTER IN FREQUENCY
FOR bd = 2π, 2 CIRCUITS
b = 0.3142 s/b² = 0.0450
b = 0.6283 s/b² = 0.180
b = 1.257 s/b² = 0.719
A
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-4
-3
-2
-1
0
1
2
3
4
sc/b

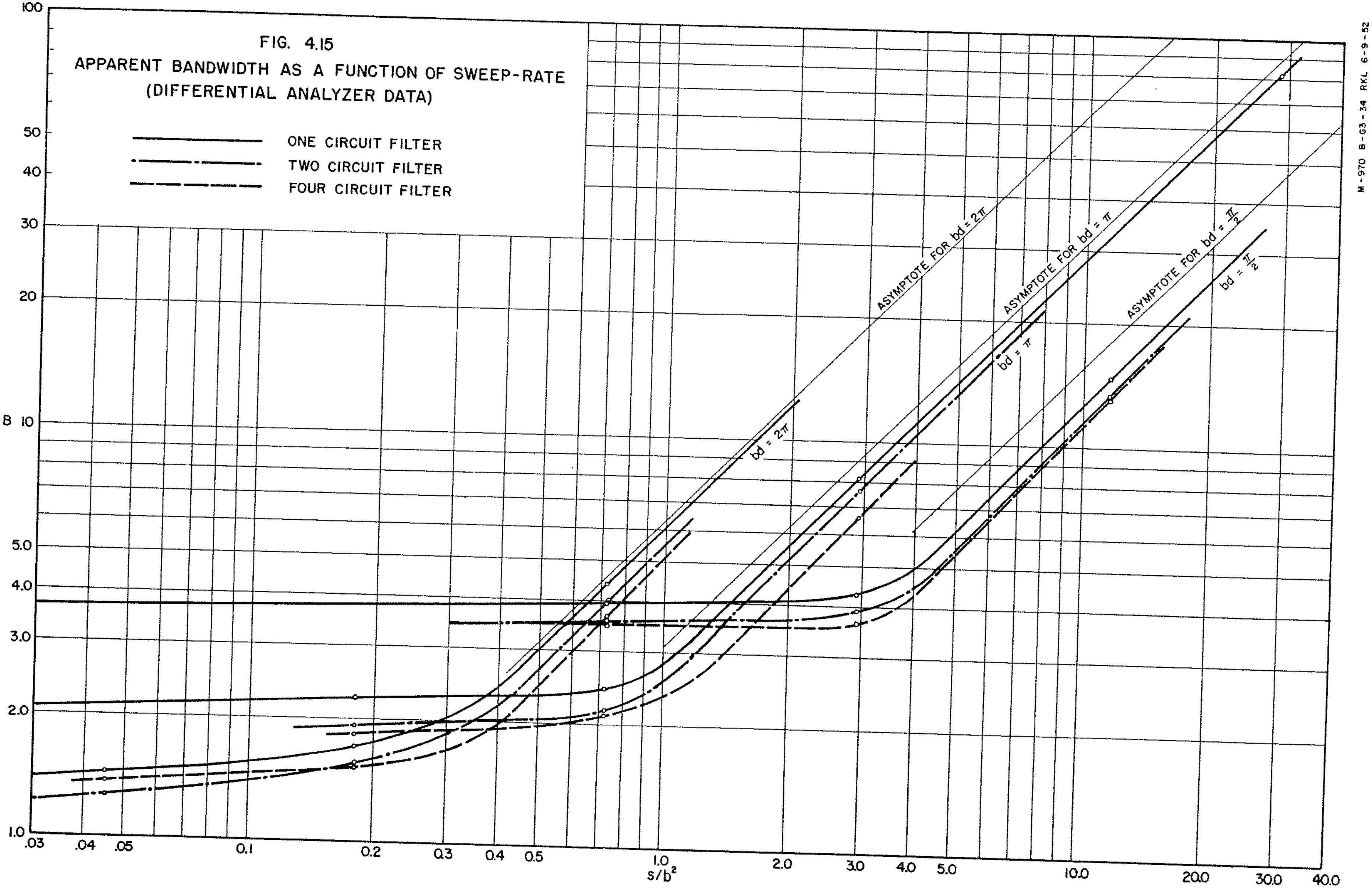


FIG. 4.15
APPARENT BANDWIDTH AS A FUNCTION OF SWEEP-RATE
(DIFFERENTIAL ANALYZER DATA)

an asymptote given by $(\frac{s}{b^2} \cdot bd)$.[1] As a result the curves of Fig. 4.15 are reasonably reliable even though only three points were obtained for most curves. Since each point on this plot represents about seven differential analyzer solutions, about two hundred solutions were required to obtain the data plotted. The number of circuits is seen to be of little importance, but the length of the input pulse has a major effect. For low sweep-rates narrowing the pulse increases the apparent bandwidth while for high sweep-rates the reverse is true.

[1] See Section 3.3

5. COMPARISON OF SOLUTIONS

5.1 Single-tuned Circuit by Two Methods

The response of single-tuned circuits to pulses with square envelopes has been discussed in earlier sections of this report.[1] Both the theoretical and differential analyzer curves for certain cases were presented. The results of the two investigations are compared here. Figure 5.1 illustrates curves obtained by these two techniques for very broad pulses. Figures 5.2 and 5.3 are similar except that the pulses started at frequencies near the center of the passband of the filter. The bandwidths for the three figures are different. A part of the discrepancy of Fig. 5.3 probably is due to the slight difference in starting time of the two curves. These three figures and Figs. 5.4 and 5.5 indicate a very good agreement between the numerical calculations and the differential analyzer solutions.

5.2 The Gaussian Case and Differential Analyzer Solutions

The relative amplitude A_c, output pulse width W, and apparent bandwidth B have been calculated both for the Gaussian case (Section 3.2) and from the differential analyzer solutions for one, two and four circuit synchronous single-tuned filters (Section 4.3). Eight typical families of curves comparing these data are given in Figs. 5.4 to 5.11.

The relative amplitude for a cw input signal is shown in Fig. 5.4. The curves for two circuits, four circuits, and the Gaussian case differ very little, the curve for the one circuit filter is higher at fast sweep-rates, but not by more than a factor of two. The relative amplitude curves for pulses (Figs. 5.5

[1]Section 3.1 and Section 4

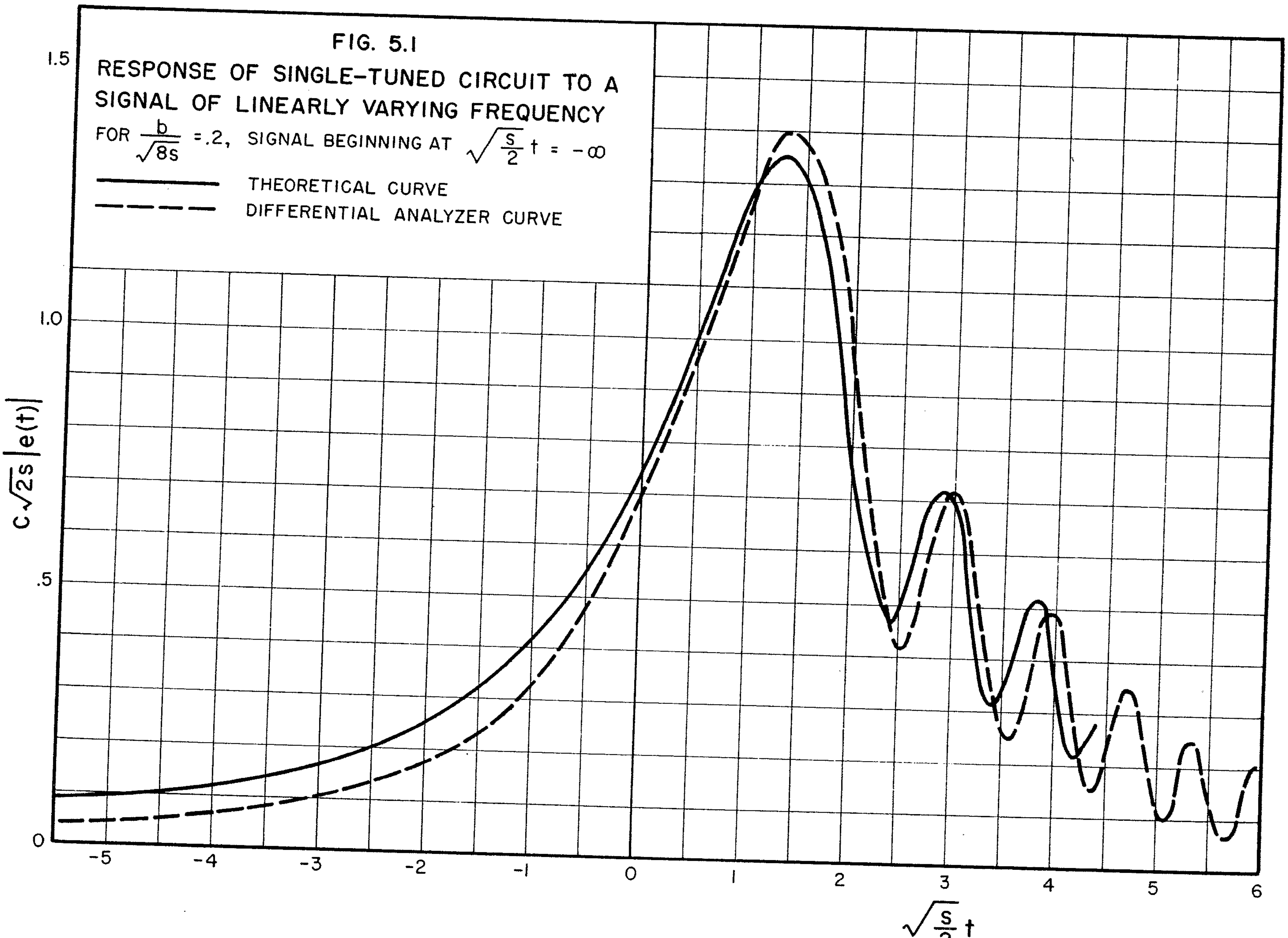


FIG. 5.1

RESPONSE OF SINGLE-TUNED CIRCUIT TO A SIGNAL OF LINEARLY VARYING FREQUENCY

FOR $\frac{b}{\sqrt{8s}} = .2$, SIGNAL BEGINNING AT $\sqrt{\frac{s}{2}}\,t = -\infty$

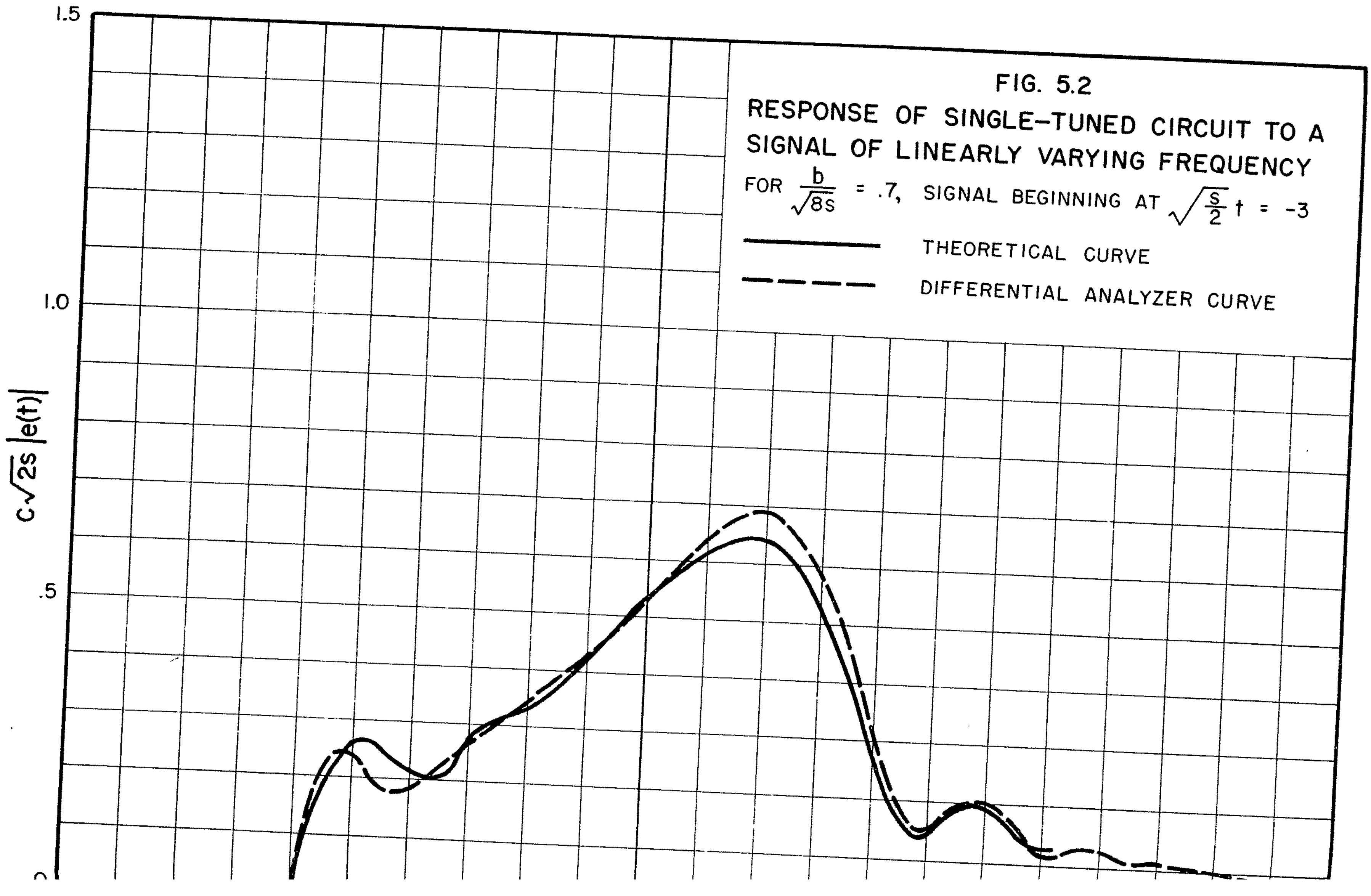


FIG. 5.2
RESPONSE OF SINGLE-TUNED CIRCUIT TO A SIGNAL OF LINEARLY VARYING FREQUENCY
FOR $\frac{b}{\sqrt{8s}} = .7$, SIGNAL BEGINNING AT $\sqrt{\frac{s}{2}}\,t = -3$

——— THEORETICAL CURVE
– – – DIFFERENTIAL ANALYZER CURVE

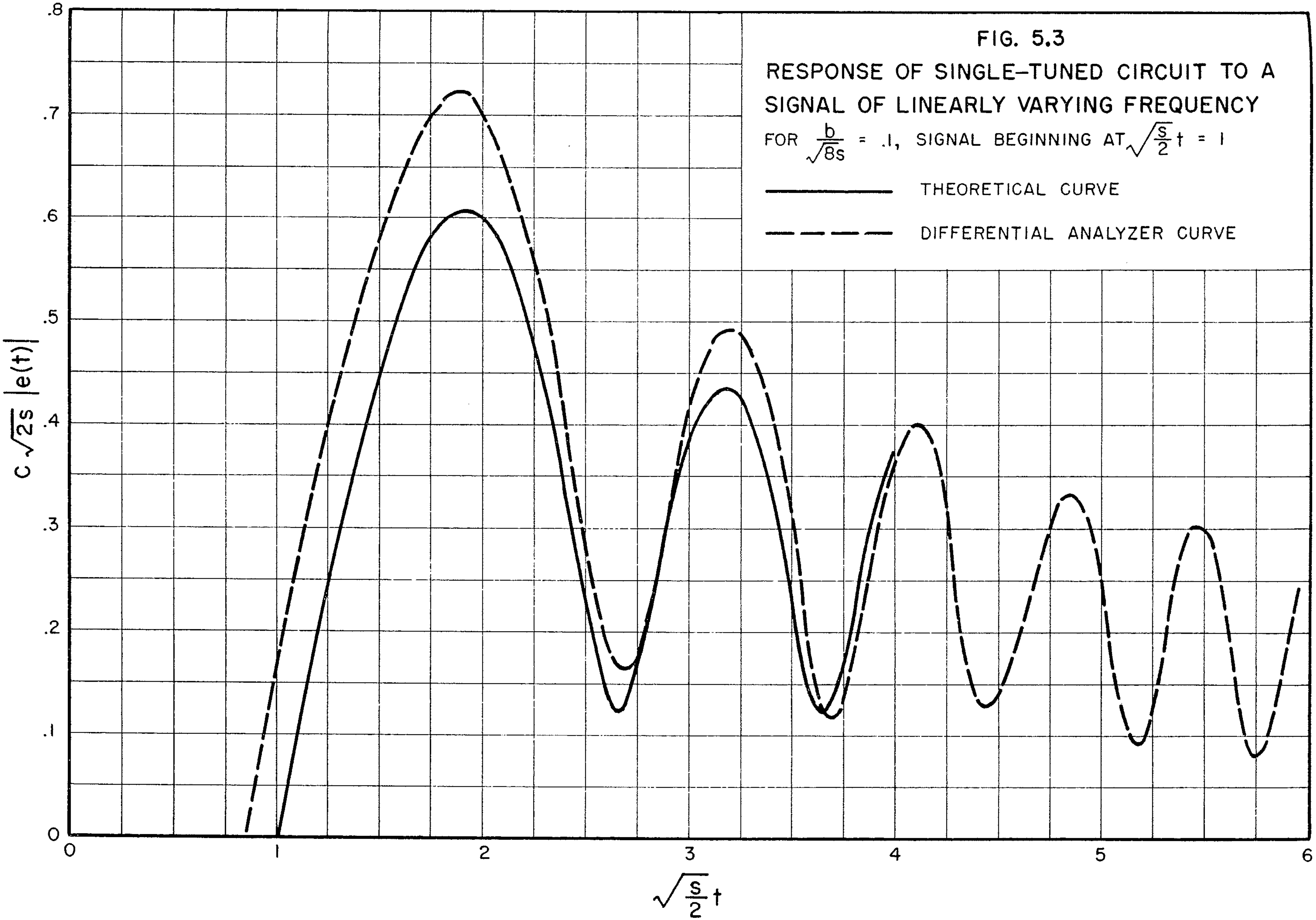


FIG. 5.3

RESPONSE OF SINGLE-TUNED CIRCUIT TO A SIGNAL OF LINEARLY VARYING FREQUENCY FOR $\frac{b}{\sqrt{8s}} = .1$, SIGNAL BEGINNING AT $\sqrt{\frac{s}{2}}\,t = 1$

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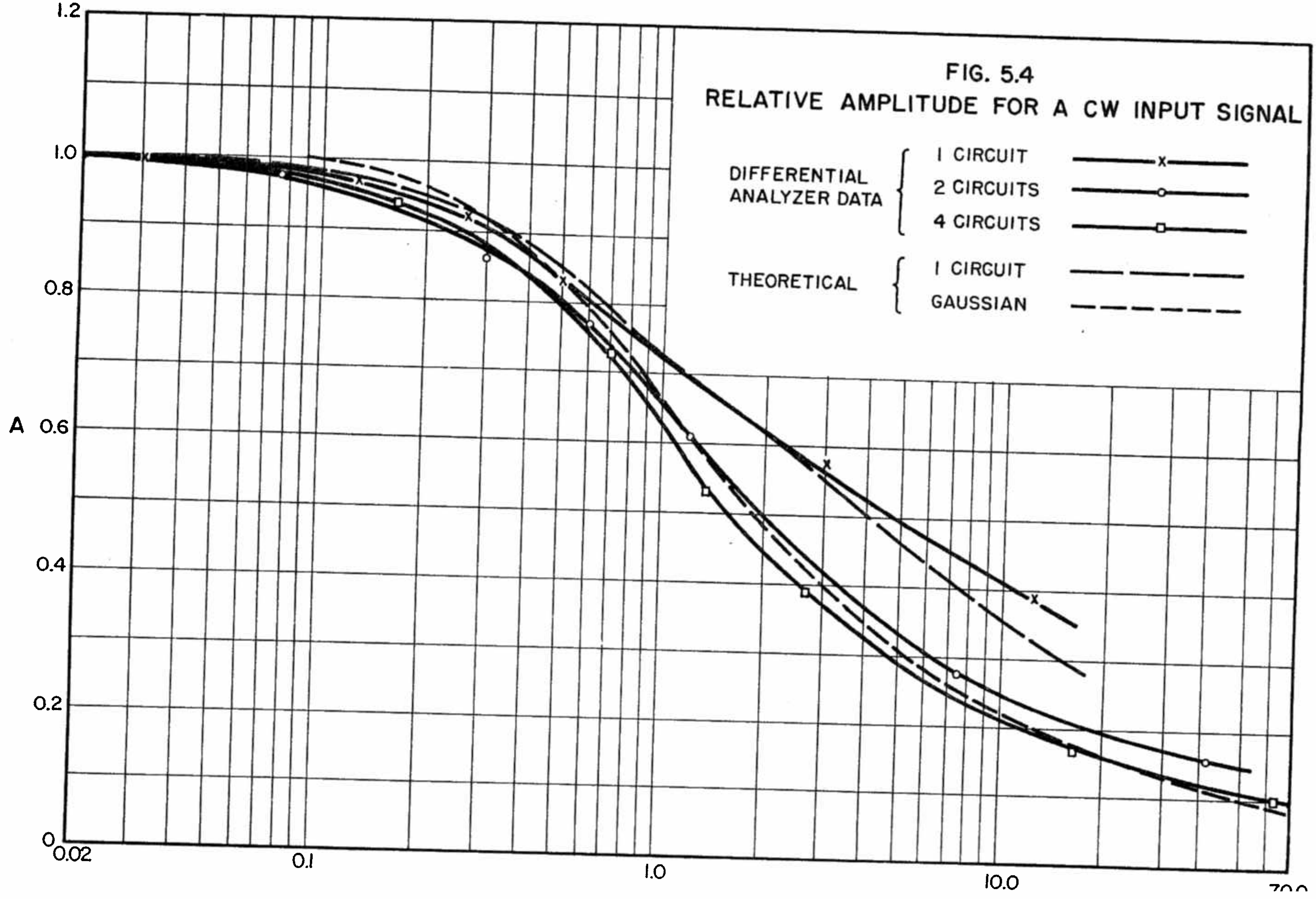
FIG. 5.4
RELATIVE AMPLITUDE FOR A CW INPUT SIGNAL
DIFFERENTIAL ANALYZER DATA
1 CIRCUIT
2 CIRCUITS
4 CIRCUITS
THEORETICAL
1 CIRCUIT
GAUSSIAN
A
1.2
1.0
0.8
0.6
0.4
0.2
0
0.02
0.1
1.0
10.0

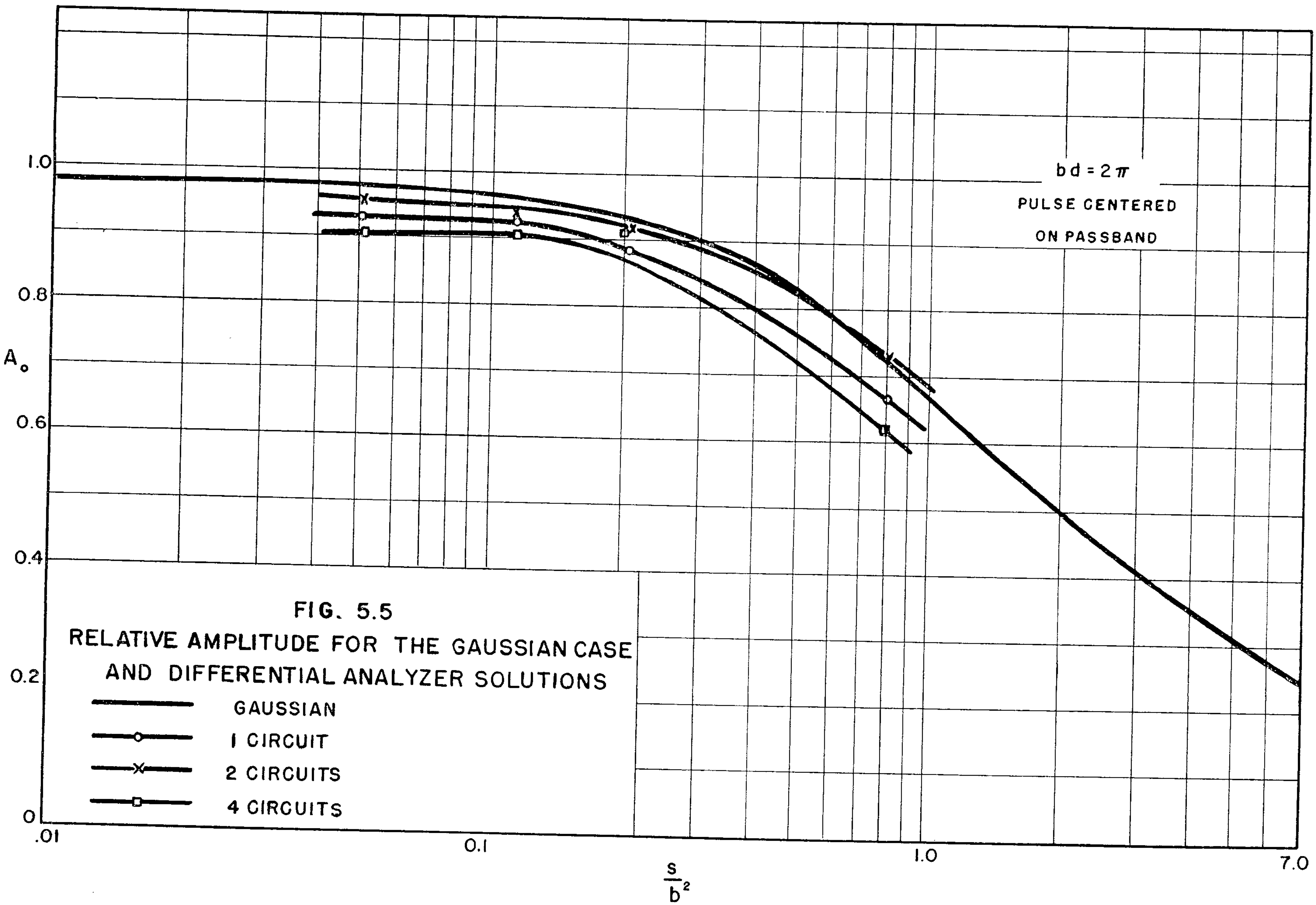

FIG. 5.5
RELATIVE AMPLITUDE FOR THE GAUSSIAN CASE AND DIFFERENTIAL ANALYZER SOLUTIONS

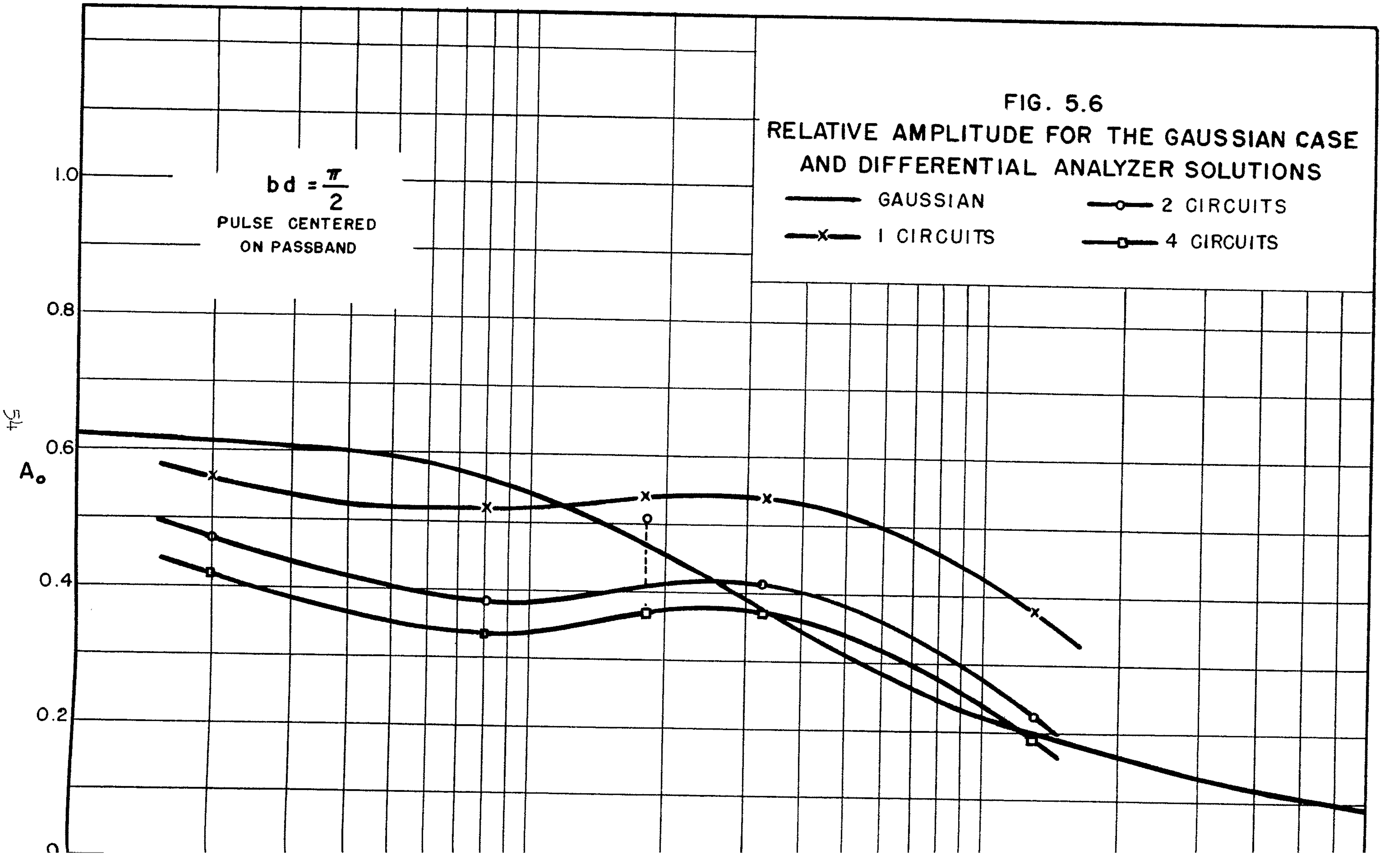

FIG. 5.6
RELATIVE AMPLITUDE FOR THE GAUSSIAN CASE AND DIFFERENTIAL ANALYZER SOLUTIONS

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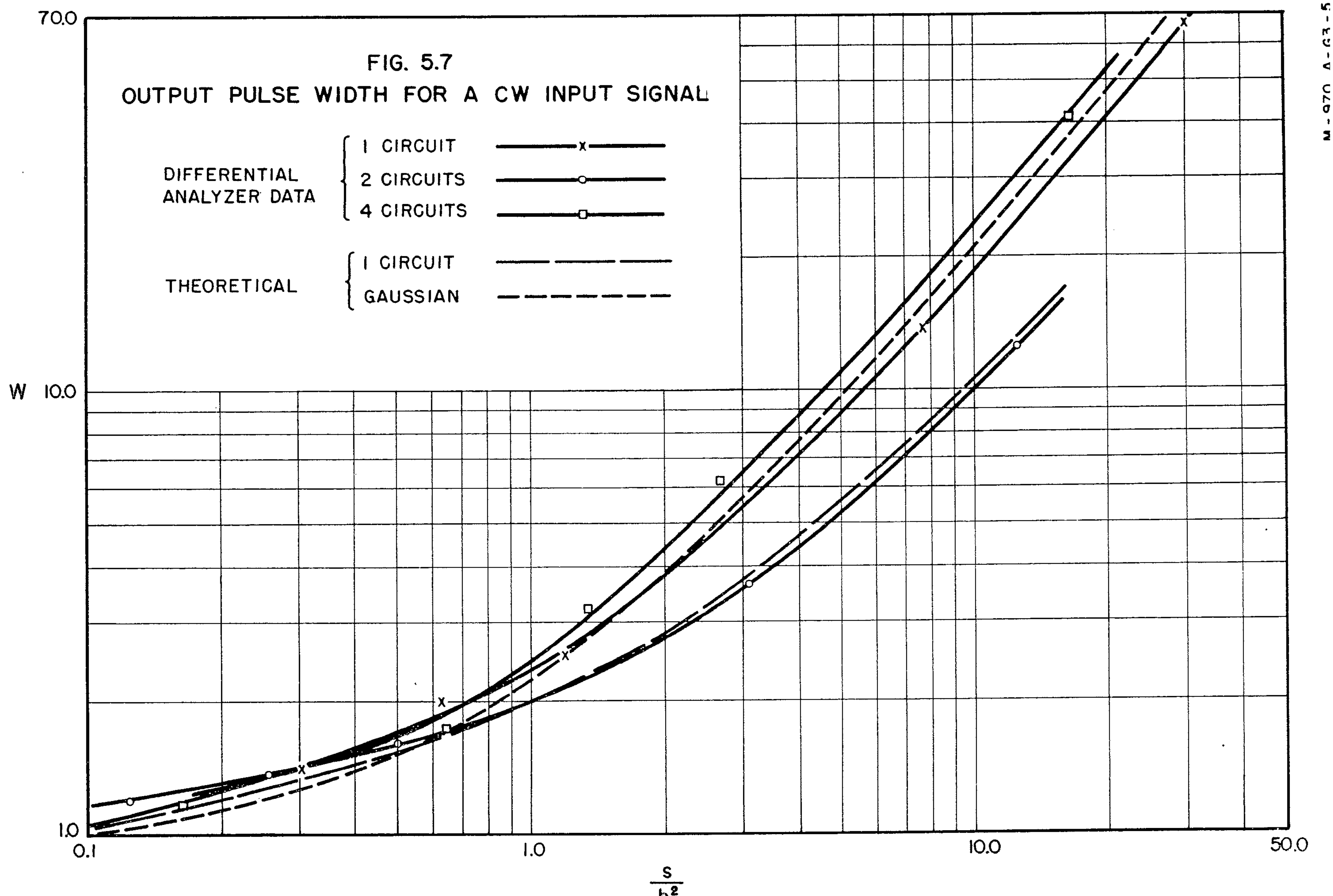


FIG. 5.7
OUTPUT PULSE WIDTH FOR A CW INPUT SIGNAL
DIFFERENTIAL ANALYZER DATA
1 CIRCUIT
2 CIRCUITS
4 CIRCUITS
THEORETICAL
1 CIRCUIT
GAUSSIAN
W
70.0
10.0
1.0
0.1
1.0
10.0
50.0
s/b²
M-970 A-G3-51 RKL 6-

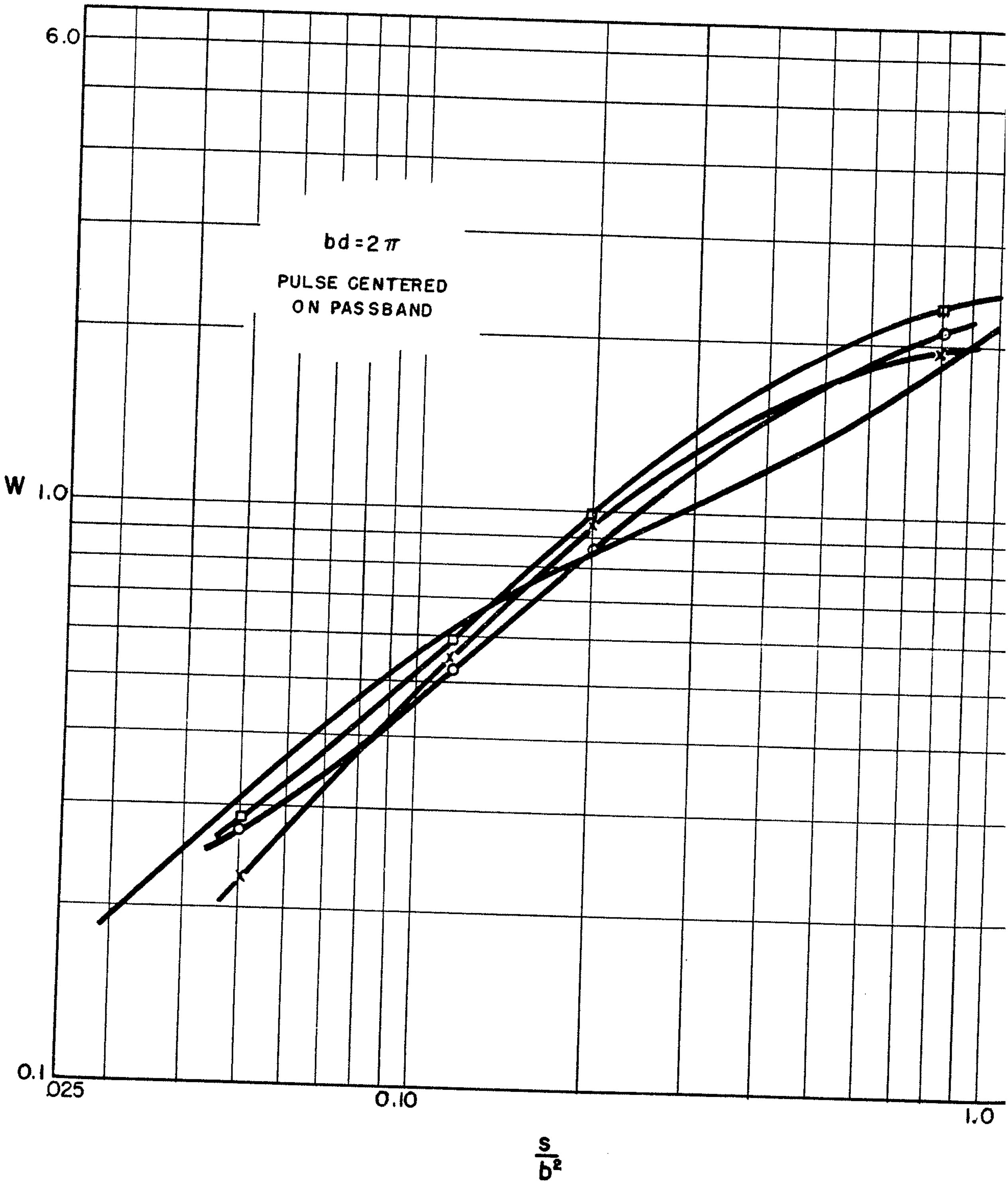


FIG. 5.8

OUTPUT PULSE WIDTH FOR THE GAUSSIAN CASE AND DIFFERENTIAL ANALYZER SOLUTIONS

——— GAUSSIAN
—x— 1 CIRCUIT
—o— 2 CIRCUITS
—□— 4 CIRCUITS

M-970 A-63-45 PLW 6

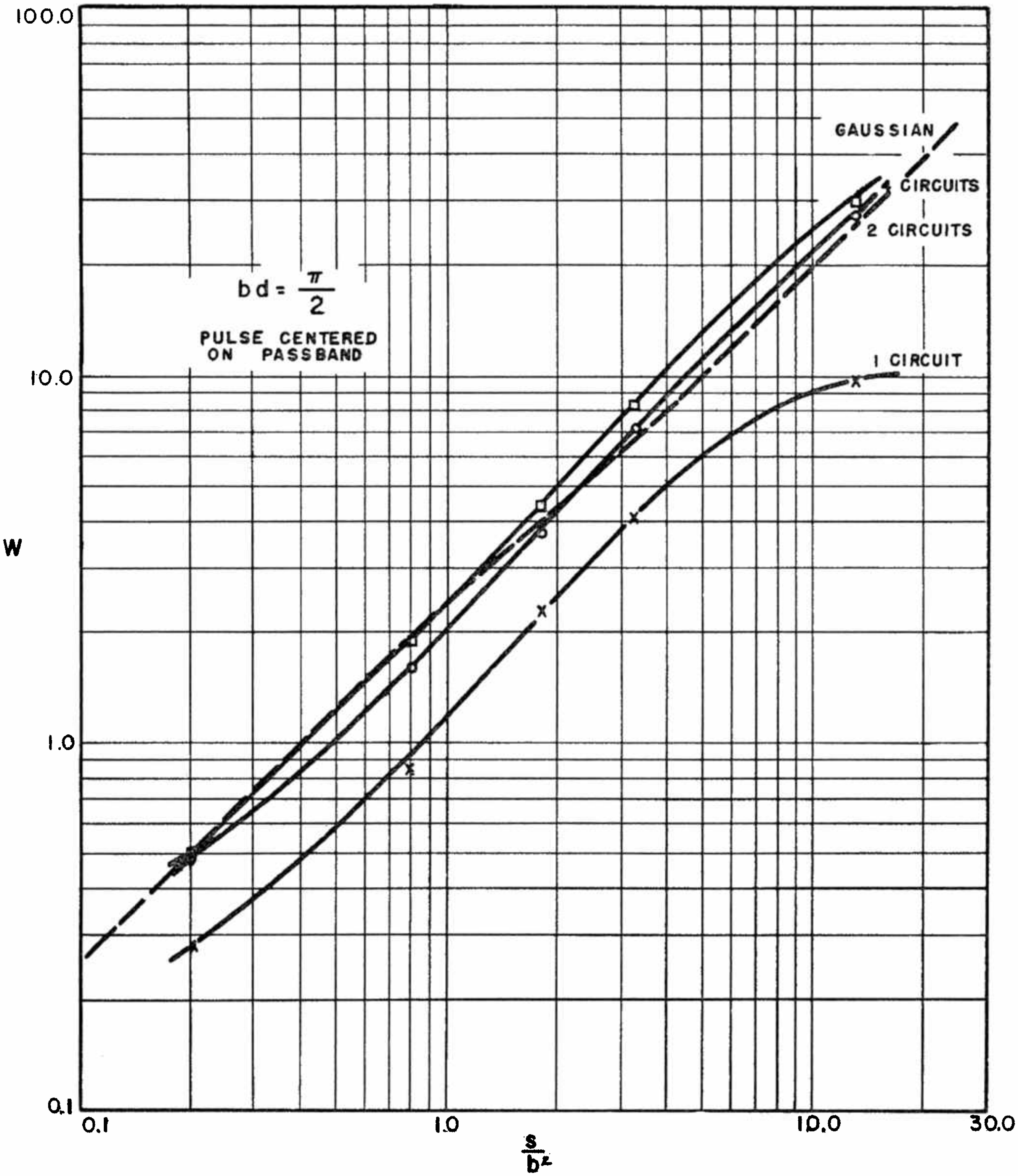


FIG. 5.9

OUTPUT PULSE WIDTH AS A FUNCTION OF SWEEP-RATE FOR GAUSSIAN CASE AND DIFFERENTIAL ANALYZER DATA

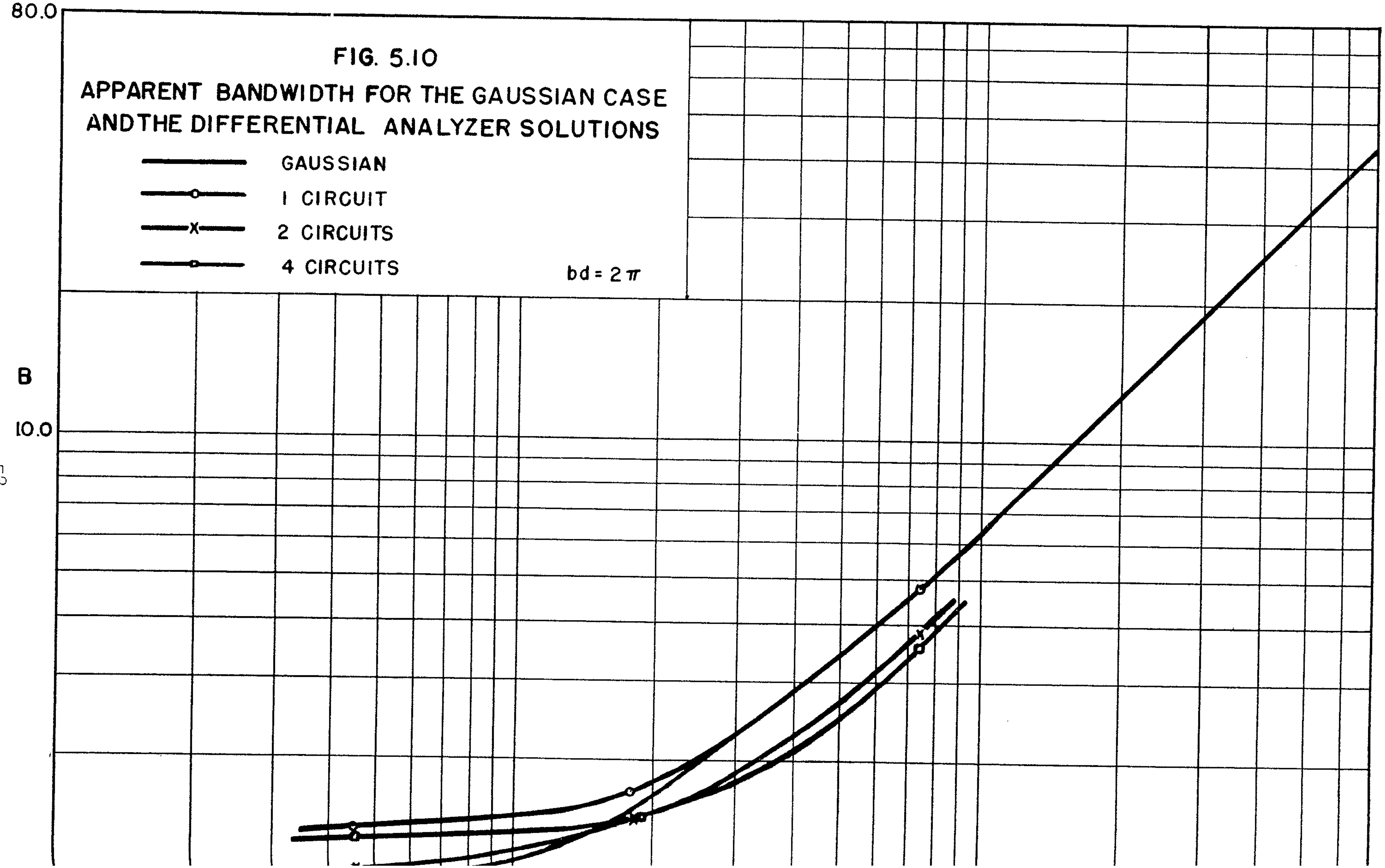
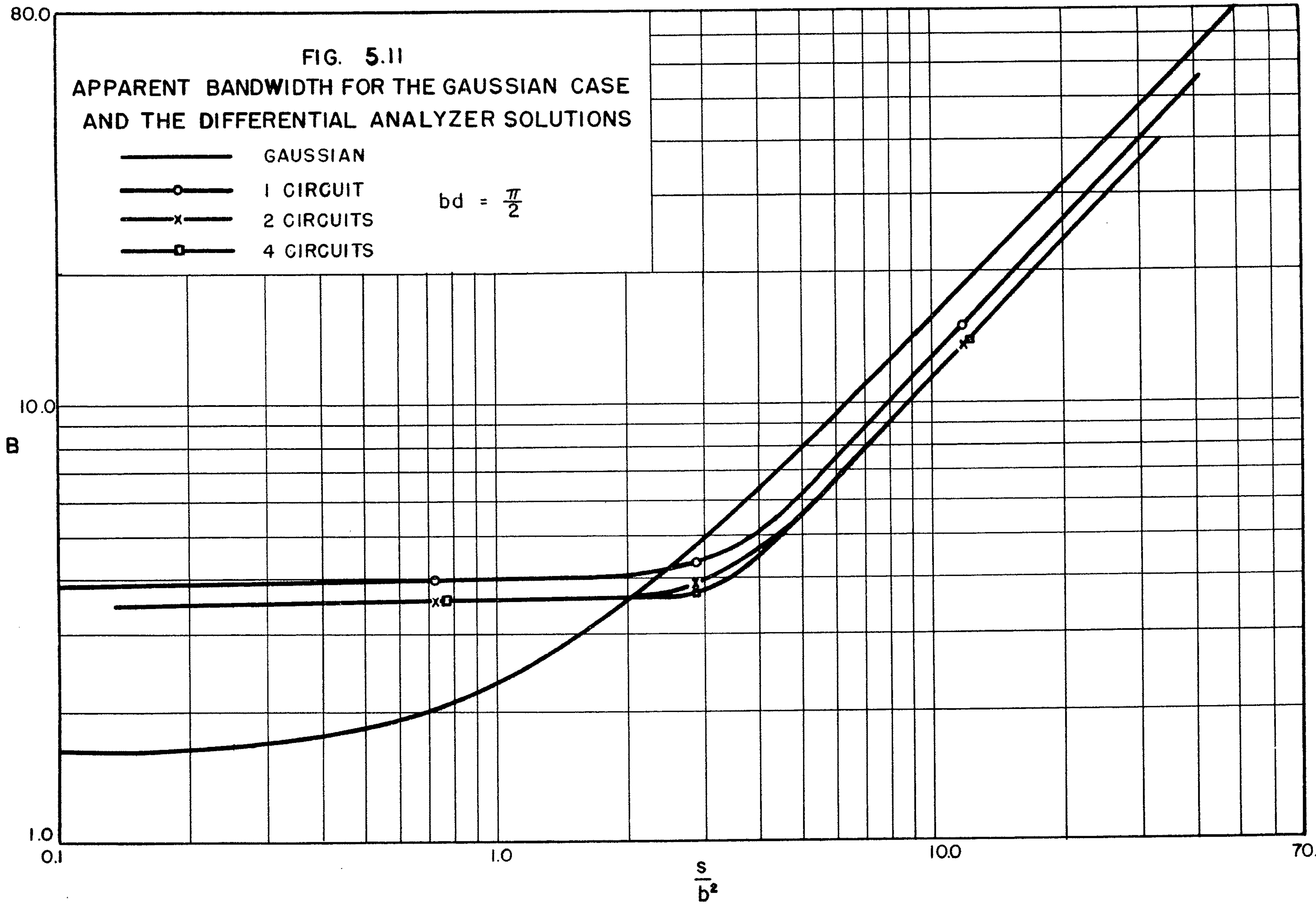


FIG. 5.10

APPARENT BANDWIDTH FOR THE GAUSSIAN CASE AND THE DIFFERENTIAL ANALYZER SOLUTIONS

FIG. 5.11

APPARENT BANDWIDTH FOR THE GAUSSIAN CASE AND THE DIFFERENTIAL ANALYZER SOLUTIONS



and 5.6) are more irregular. The curves still have the same general features, however, and they differ by no more than a factor of two.

Figure 5.7 shows the output pulse width for a constant amplitude signal. Again the curves for two and four circuits and the Gaussian case are very close. The curves for the one circuit filter are lower, but not by more than a factor of three. The output pulse width curves for a pulse input (Figs. 5.8 and 5.9) are all of the same general shape and are within a factor of three of each other, with the curve for a one circuit filter diverging the most.

The curves for B in Figs. 5.10 and 5.11 are within forty per cent of each other for fast sweep-rates and within a factor of two and one-half for slow sweep-rates. The curves for a particular value of bd have the same asymptote for fast sweep-rates (see Section 3.3) and the asymptotes are parallel for slow sweep-rates. The limiting values of B for slow sweep-rates can be obtained theoretically for the single-tuned circuit as well as for the Gaussian case, and these are compared in Table 5.1.

TABLE 5.1 APPARENT BANDWIDTH FOR ZERO SWEEP-RATE

bd	$\frac{\pi}{2}$	π	2π
1 circuit	3.61	1.89	1.17
Gaussian Filter	1.62	1.18	1.05

Generally the one circuit filter gives somewhat greater relative amplitude, a little narrower output pulse, and a slightly larger apparent bandwidth than any of the other filters considered.

Certain features of the receiver response in the differential analyzer solutions differ considerably from the solution of the Gaussian case. For example, the time of maximum response can hardly be expected to agree, since the Gaussian filter is assumed to have no phase delay. Also, the output pulse shape for the Gaussian case is always Gaussian, while a wide variety of shapes appear in the differential analyzer solutions, as observed in Section 4.3.

The solution of the Gaussian case gives an understanding of the nature of the response of a panoramic receiver. Moreover, the Gaussian case is <u>quantitatively</u> consistant enough with the other cases studied to be used in many design problems involving peak amplitude, output pulse width, apparent bandwidth, and resolution.

6. SUMMARY

The response of a panoramic receiver to cw signals and pulses has been investigated in this report. The receivers studied were idealized superheterodynes; several types of i-f amplifier filters were considered. In many cases the analysis applies to a trf panoramic receiver.

Theoretical studies were made of the following cases:

a) Single-tuned i-f filter with cw and square envelope pulses as as input signals, and

b) Gaussian shaped i-f passband with cw and Gaussian envelope pulses as input signals.

Differential analyzer solutions were made for i-f filters of one, two, and four single-tuned circuits with cw and square envelope pulses as input signals.

The character of the response is found to depend in a minor way upon the type of i-f filter used. General formulas presented in Section 3.3 hold regardless of the type of filter or the type of input pulse. Thus one anticipates that the important features of the response are little altered with other types of bandpass filters.

Only the envelope of the response is studied in this report. The pertinent response factors investigated were the relative amplitude, the output pulse width, and the apparent bandwidth. These factors largely determine the character of the response. They are functions of the receiver sweep-rate, the i-f bandwidth, and the signal pulse width.

The Gaussian case has been presented in considerable detail in this study. Analytically, the response for this case is very satisfying. The answer is given in closed form, and it is simple when expressed in terms of the important factors: relative amplitude, output pulse width, and effective bandwidth. More significant is the fact that the Gaussian case is fairly representative and can be used with suitable caution in the design of panoramic equipment.

The formulas and the many curves in this report will enable the engineer to optimize certain design features of a panoramic receiver subject to the application requirements.

ACKNOWLEDGEMENT

The authors wish to thank Mr. R. Bradley, Mr. R. Lyons, and Mrs. C. Martin for assistance in the preparation of the text. The authors are also indebted to members of the engineering staff at Sperry Gyroscope Company for their work on the Gaussian filter.

APPENDIX A[1]

Derivation of Response to a Single-Tuned Circuit

Let

$$i(t) = \begin{cases} \exp\left\{jat + \frac{jst^2}{2}\right\}, & \text{for } c - \frac{d}{2} \leq t \leq c + \frac{d}{2} \\ 0, & \text{for } t < c - \frac{d}{2} \text{ and } t > c + \frac{d}{2} \end{cases} \quad (A.1)$$

be the input signal current to a passive lumped-constant network and let $\tilde{i}$ designate its Fourier transform. Similarly, let e(t) be the voltage produced at a given output and $\tilde{e}$ its Fourier transform. Assume the circuit is quiescent prior to the application of the signal. Then,

$$\tilde{e} = z(p)\tilde{i} = \frac{z(p)}{\sqrt{2\pi}} \int_{c-\frac{d}{2}}^{c+\frac{d}{2}} \exp\left[jaT + \frac{jsT^2}{2}\right] \exp(-pT)\, dT \quad (A.2)$$

where z(p) is the appropriate transfer impedance. For a passive network z(p) is a rational function and can be expanded in partial fractions as follows.

$$z(p) = \frac{g(p)}{h(p)} = \sum_{k=1}^{n} \frac{g(p_k)}{h'(p_k)(p-p_k)} \quad (A.3)$$

where p_k designates a root of $h(p) = 0$, and multiple roots are excluded.[2] Then equation A.2 becomes

$$\tilde{e} = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} \frac{g(p_k)}{h'(p_k)(p-p_k)} \int_{c-\frac{d}{2}}^{c+\frac{d}{2}} \exp\left[jaT + \frac{jsT^2}{2}\right] \exp(-pT)\, dT. \quad (A.4)$$

[1] The proof given here follows that of Gunnar Hok, Ref. 1.

[2] The extension to multiple roots is not difficult.

Taking the inverse transform and interchanging the order of integration,

$$e(t) = \sum_{k=1}^{n} \frac{g(p_k)}{h'(p_k)} \int_{c-\frac{d}{2}}^{c+\frac{d}{2}} \exp\left[ja\tau + \frac{js\tau^2}{2}\right] \int_{-j\infty}^{j\infty} \frac{\exp p(t-\tau)}{2\pi j\,(p-p_k)}\, dp\,d\tau. \tag{A.5}$$

By the inversion theorem of operational calculus, if the real part of $p_k < 0$,

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{\exp p(t-\tau)}{(p-p_k)}\,dp = \begin{cases} 0 \text{ for } t < \tau \\ \exp p_k(t-\tau) \text{ for } t > \tau, \end{cases} \tag{A.6}$$

and therefore the expression for the voltage becomes,

$$e(t) = \sum_{k=1}^{n} \frac{g(p_k)}{h'(p_k)} \int_{c-\frac{d}{2}}^{t} \exp\left\{p_k t + (ja - p_k)\tau + \frac{js\tau^2}{2}\right\} d\tau \tag{A.7}$$

for $c - \frac{d}{2} \leq t \leq c + \frac{d}{2}$. The upper limit of the integration t is replaced by $c + \frac{d}{2}$ for $t > c + \frac{d}{2}$.

Now define,

$$\gamma_k(\tau) = \frac{1}{\sqrt{s}}(ja - p_k) + j\sqrt{s}\,\tau. \tag{A.8}$$

Then the exponent of the integrand in Eq A.7 can be written

$$p_k t + (ja - p_k)\tau + \frac{js\tau^2}{2} = -j\,\frac{\gamma_k^2(\tau)}{2} + j\,\frac{\gamma_k^2(t)}{2} + j\left[at + \frac{st^2}{2}\right],$$

and the voltage expression assumes the form

$$e(t) = \sum_{k=1}^{n} \frac{g(p_k)}{h'(p_k)} \sqrt{\frac{2}{s}} \exp\left[jat + \frac{jst^2}{2}\right] G(\chi_k) \qquad (A.9)$$

where $G(\chi_k)^1 = -j \exp\left[j \frac{\chi_k^2(t)}{2}\right] \int_{\chi_k(c - \frac{d}{2})}^{\chi_k(t)} \exp - j \frac{\chi_k^2}{2} \, d\chi_k .$ (A.10)

The error function of a complex variable defined by $\frac{1}{\sqrt{2\pi}} \int_0^{x+iy} e^{-\frac{z^2}{2}} dz$ has been tabulated and can be used to evaluate $G(\chi_k)$.[2]

For the case of the single tuned circuit shown in Fig. 3.1, the general formula for the voltage reduces as follows:

$$z(p) = \frac{1}{\frac{1}{R} + \frac{1}{pL} + pC} = \frac{RLp}{p^2 \cdot RLC + pL + R} .$$

Then,

$$g(p) = RLp,$$

$$h(p) = RLCp^2 + Lp + R,$$

$$h'(p) = 2RLCp + L, \text{ and}$$

$$\frac{g(p)}{h'(p)} = \frac{RLp}{2RLCp + L} = \frac{1}{2C} \cdot \frac{1}{1 + \frac{1}{2RCp}} .$$

The roots of $h(p) = 0$ are

---

[1] $G(\chi_k) = 0$ for $t < c - \frac{d}{2}$, and for $t > c + \frac{d}{2}$ the upper limit of the integral is to be replaced by $\chi_k(c + \frac{d}{2})$.

[2] "Tables of Integrals Associated with the Error Function of a Complex Variable," Hastings, C. and Mercum, J., RAD-284, Douglas Aircraft, August, 1948.

$$\left.\begin{matrix} p_1 \\ \\ p_2 \end{matrix}\right\} = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left[\frac{1}{2RC}\right]^2} = -\frac{1}{2RC} \pm ja = -\frac{b}{2} \pm ja .$$

Then, if $\frac{2a}{b} >> 1$,

$$\frac{g(p_1)}{h'(p_1)} \simeq \frac{g(p_2)}{h'(p_2)} \simeq \frac{1}{2C} .$$

By Eq A.9 the voltage is $e(t) \simeq \frac{1}{C\sqrt{2s}} \exp\left(jat + \frac{jst^2}{2}\right) \cdot \left[G(\gamma_1) + G(\gamma_2)\right]$. (A.11)

Also, it is readily shown that

$\lim_{a \to \infty} G(\gamma_2) = 0$, so that for high frequency the voltage response is given approximately by

$$e(t) \simeq \frac{1}{C\sqrt{2s}} \exp\left(jat + \frac{jst^2}{2}\right) G(\gamma_1) \qquad (A.12)$$

where $\gamma_1 = \frac{b}{2\sqrt{s}} + j\sqrt{s}\, t$. (A.13)

APPENDIX B

Derivation of the Response of a Gaussian Filter

In this appendix the formulae are derived for the response of a filter with a Gaussian shaped transfer function to a signal which is changing linearly in frequency and has either a constant amplitude or a Gaussian shaped envelope.

Assume the filter transfer function is

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\omega-a)^2}{b^2}\right], \qquad \text{(B.1)}$$

and the input signal is (the real part of)

$$f(t) = \exp\left[j\left(\frac{st^2}{2} + at\right) - \frac{(t-c)^2}{d^2}\right]. \qquad \text{(B.2)}$$

The procedure is to find the Fourier transform $F(\omega)$ of $f(t)$, multiply it by $H(\omega)$, and transform back to the t-plane. The filter response is the real part of the resulting function $g(t)$. The calculation is simplified by using the convolution formula:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) \cdot F(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda)\, h(t-\lambda)\, d\lambda \qquad \text{(B.3)}$$

where $h(t)$ is the Fourier transforms of $H(\omega)$.[1,2]

Two preliminary remarks will make the derivation go smoothly. In the first place, the envelope of the real part of a complex function of time is just

[1] Titchmarsh, E. C., "Introduction to the Theory of Fourier Integrals", Oxford University Press, 1937, p. 51.

[2] The use of complex functions for the signal $f(t)$ and the impulse response $h(t)$ is justified if the response of the filter is negligible at zero frequency.

the absolute value of the function. This can be seen as follows: Let $Z(t)$ be any complex function of t. It can be written, $Z(t) = |Z(t)| \exp(j\theta(t))$, where $\theta(t)$ is the argument of $Z(t)$. The real part of $Z(t)$ is then $|Z(t)| \cos\theta(t)$, and the envelope of this is $|Z(t)|$.

Secondly, in computing the Fourier transforms, use will be made of the following formula:

$$\int_{-\infty}^{\infty} \exp(-ut^2 + vt)\, dt = \sqrt{\frac{\pi}{u}} \exp\frac{v^2}{4u} . \tag{B.4}$$

The integration is along the real axis in the t-plane, and u and v are complex numbers, with the real part of u positive. This formula can be derived as follows:

$$\int_{-\infty}^{\infty} \exp\left[-ut^2 + vt\right] dt = \exp\left[\frac{v^2}{4u}\right] \int_{-\infty}^{\infty} \exp\left[-u\left[t - \frac{v}{2u}\right]^2\right] dt.$$

Letting $Z = u(t - \frac{v}{2u})$,

$$\int_{-\infty}^{\infty} \exp(-ut^2 + vt)\, dt = \frac{1}{\sqrt{u}} \exp\frac{v^2}{4u} \int_{-\infty}^{\infty} \exp(-Z^2)\, dZ = \sqrt{\frac{\pi}{u}} \exp\frac{v^2}{4u} .$$

Note that the path of integration in the Z-plane is not along the real axis, but along a line which may be oblique to the real axis. From the requirement that the real part of u be positive, it can be shown that the path of integration in the Z-plane makes no more than a 45° angle with the real axis. With this restriction the integral $\int_{-\infty}^{\infty} \exp(-Z^2)\, dZ$ is independent of the angle of the path and thus equal to $\sqrt{\pi}$, which is given by integration along the real axis.

Now the calculation of $g(t)$ can be carried out. Before use is made of (B.3), $h(t)$ must be calculated.

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) \exp(j\omega t)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[j\omega t - \frac{(\omega - a)^2}{b^2} \right] d\omega$$

$$= \frac{1}{2\pi} \exp\left[\frac{-a^2}{b^2}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\omega^2}{b^2} + (jt + \frac{2a}{b^2})\,\omega \right] d\omega$$

which yields, after application of (B.4) and simplification,

$$h(t) = \frac{b}{2\sqrt{\pi}} \exp\left[jat - \frac{b^2t^2}{4} \right]. \qquad (B.5)$$

The expressions for f(t) and h(t) can now be substituted in (B.3):

$$g(t) = \int_{-\infty}^{\infty} f(\lambda)\, h(t-\lambda)\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{js\lambda^2}{2} + ja\lambda - \frac{(\lambda-c)^2}{d^2} \right] \cdot \frac{b}{\sqrt{2}} \exp\left[\frac{-b^2}{4}(t-\lambda)^2 + ja(t-\lambda) \right] d\lambda$$

$$= \frac{b}{2\sqrt{\pi}} \exp\left[-\frac{c^2}{d^2} - \frac{b^2t^2}{4} + jat \right] \int_{-\infty}^{\infty} \exp\left[-\lambda^2 \left(\frac{1}{d^2} + \frac{b^2}{4} - \frac{js}{2} \right) + \lambda \left(\frac{2c}{d^2} + \frac{b^2}{2} \right) \right] d\lambda$$

and using (B.4) again;

$$g(t) = \frac{b}{2\cdot\sqrt{\frac{1}{d^2} + \frac{b^2}{4} - \frac{js}{2}}} \exp\left\{ -\frac{c^2}{d^2} - \frac{b^2t^2}{4} + jat + \frac{\left[\frac{2c}{d^2} + \frac{b^2t}{2}\right]^2}{4\left[\frac{1}{d^2} + \frac{b^2}{4} - \frac{js}{2}\right]} \right\}. \qquad (B.6)$$

As has been remarked, the required answer is the absolute value of g(t), which can be obtained by taking the absolute value of the first factor and keeping only the real part of the exponent.

$$\left|g(t)\right| = \frac{b}{2\left[\left(\frac{1}{d^2} + \frac{b^2}{4}\right)^2 + \frac{s^2}{4}\right]^{\frac{1}{4}}} \exp\left\{ -\frac{c^2}{d^2} - \frac{b^2t^2}{4} + \frac{\left[\frac{2c}{d^2} + \frac{b^2t}{2}\right]^2 \left[\frac{1}{d^2} + \frac{b^2}{4}\right]}{4\left\{\left[\frac{1}{d^2} + \frac{b^2}{4}\right]^2 + \frac{s^2}{4}\right\}} \right\} \qquad (B.7)$$

The exponent of (B.7) can be put into the following form:

$$-\frac{b^2\left[\frac{4}{d^2}+b^2+s^2d^2\right]}{d^2\left[\left(\frac{4}{d^2}+b^2\right)^2+4s^2\right]}\cdot\left[t-\frac{c\left[\frac{4}{d^2}+b^2\right]}{\left[\frac{4}{d^2}+b^2+s^2d^2\right]}\right]^2-\frac{s^2c^2}{\left[\frac{4}{d^2}+b^2+s^2d^2\right]}\quad . \qquad (B.8)$$

Referring to the definitons of A, W, and B in section 2.3, and recalling that in the Gaussian case the width of a curve is taken to the $e^{-\frac{1}{4}}$ points, it is clear that

$$A_o = \frac{b}{\left[\left(\frac{4}{d^2}+b^2\right)^2+4s^2\right]^{\frac{1}{4}}}\ , \qquad (B.9)$$

$$B = \frac{1}{b}\sqrt{\frac{4}{d^2}+b^2+s^2d^2},\ \text{and} \qquad (B.10)$$

$$W = \frac{sd}{b^2}\sqrt{\frac{\left(\frac{4}{d^2}+b^2\right)^2+4s^2}{\frac{4}{d^2}+b^2+s^2d^2}} = \frac{sd}{b}\ \frac{1}{A_o^2B}\quad . \qquad (B.11)$$

The time of maximum response is given by

$$t_m = c\left[\frac{\frac{4}{d^2}+b^2}{\frac{4}{d^2}+b^2+s^2d^2}\right]\ . \qquad (B.12)$$

Now $|g(t)|$ can be written

$$|g(t)| = A_o\ \exp\left\{-\frac{1}{W^2}\cdot\left[\frac{s(t-t_m)}{b}\right]^2-\frac{1}{B^2}\left[\frac{sc}{b}\right]^2\right\}\ . \qquad (B.13)$$

For a cw input the signal is $\exp j\left[\frac{s\,t^2}{2}+at\right]$, and the output can be obtained from (B.7) by taking the limit as d approaches infinity.

$$\lim_{d\to\infty}|g(t)| = \frac{b}{\left[b^4+4s^2\right]^{\frac{1}{4}}}\ \exp\left\{-\frac{b^2\,s^2\,t^2}{b^4+4s^2}\right\}\ . \qquad (B.7')$$

In the notation of (B.9) to (B.13),

$$A_o = \frac{b}{\left[b^4 + 4s^2\right]^{\frac{1}{4}}} \tag{B.9'}$$

$$W = \frac{1}{b^2}\left(b^4 + 4s^2\right) = \frac{1}{A_o^2}\text{, and} \tag{B.11'}$$

$$\lim_{d \to \infty} \left|g(t)\right| = A_o \exp\left\{-\frac{1}{W^2}\left[\frac{st}{b}\right]^2\right\}. \tag{B.13'}$$

APPENDIX C

Curves for the Gaussian Case

The curves in this appendix together with those of Section 3.2 make up a complete set of curves of A_o, W, and B for the Gaussian case. They are listed in Table C.1.

TABLE C.1 Curves for the Gaussian Case

I. Normalization with Respect to Bandwidth

A. Sweep-Rate as Abscissa, Pulse Width as Parameter

[1]3.4 A_o
3.5 W
3.6 B

B. Pulse Width as Abscissa, Sweep-Rate as Parameter

C.1 A_o
C.2 W
C.3 B

II. Normalization with Respect to Pulse Width

A. Sweep-Rate as Abscissa, Bandwidth as Parameter

C.4 A_o
C.5 bdW
C.6 bdB

B. Bandwidth as Abscissa, Sweep-Rate as Parameter

C.7 A_o
C.8 bdW
C.9 bdB

III. Normalization with Respect to Sweep-Rate

A. Bandwidth as Abscissa, Pulse Width as Parameter

C.10 A_o

C.11 $\frac{b}{\sqrt{s}}$ W

C.12 $\frac{b}{\sqrt{s}}$ B

[1]These numbers are figure numbers.

B. Pulse Width as Abscissa, Bandwidth as Parameter

C.13 A_o

C.14 $\frac{b}{\sqrt{s}}$ W

C.15 $\frac{b}{\sqrt{s}}$ B

The output pulse width W and apparent bandwidth B are normalized with respect to bandwidth by definiton. Where they are presented with a different normalization, they are multiplied by the factor which will correct their normalization. Suppose one wishes to know how apparent bandwidth varies as a function of pulse width for a fixed sweep-rate. Normalization with respect to sweep-rate should be chosen; then the apparent bandwidth in radians per second is $\frac{bB}{2\pi}$, or $\frac{\sqrt{s}}{2\pi} \cdot \frac{bB}{\sqrt{s}}$. Since $\frac{\sqrt{s}}{2\pi}$ is fixed, apparent bandwidth in radians per second is proportional to $\frac{bB}{\sqrt{s}}$, and its dependence upon pulse width is shown in Fig. C.15.

A brief discussion of the dependence of A_o, W, and B on sweep-rate and pulse width was given in Section 3.2. Several features are brought out by the curves in this appendix. For any given sweep-rate, there is a pulse width which leads to a minimum apparent bandwidth. It is given by $sd^2 = 2$. This can be seen in Figs. C.3 and C.15. For a fixed sweep-rate and for pulses such that $sd^2 > 2$, there is a bandwidth which gives a minimum output pulse width (see Fig. C.11). For long pulses ($sd^2 > 10$), the minimum occurs very nearly where $b^2 = 2s$.

In many of the graphs there is an unattainable region. For example, in Fig. C.7, the relative amplitude is seen to be bounded by the curves for $sd^2 = 0$. Thus, if $bd < 1$, A_o must be less than 0.45 regardless of the sweep-rate. In Fig. C.3, the curve for $\frac{s}{b^2} = 0$ is the envelope of the family —

no curve is found below it. In Fig. 3.6, the envelope is found to be $B = \sqrt{1 + \frac{4s}{b^2}}$. B is never less than this, and it has this value only when $sd^2 = 2$. The envelope is also shown in Fig. C.14; it corresponds to the minima in Fig. C.11.

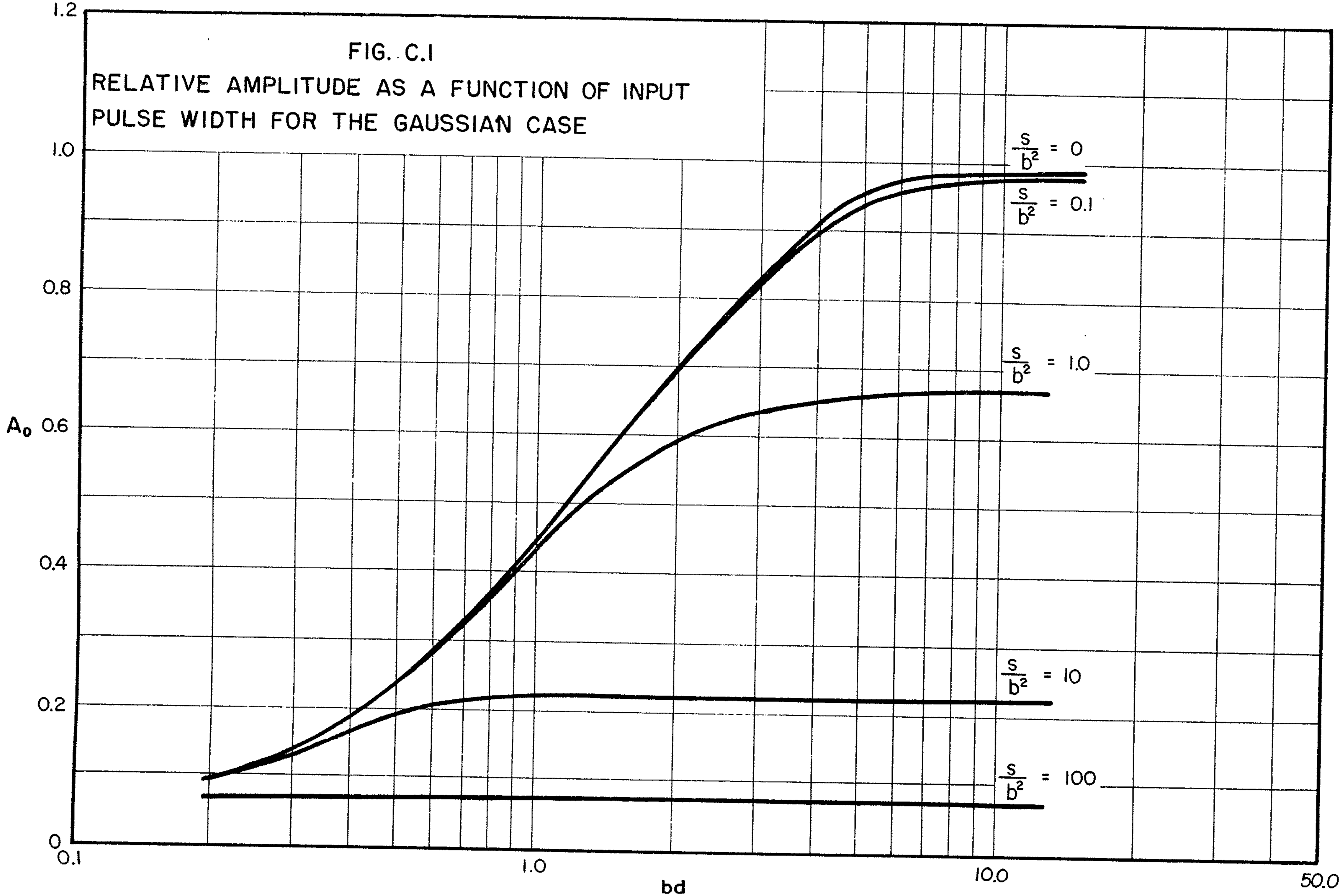


FIG. C.1
RELATIVE AMPLITUDE AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE

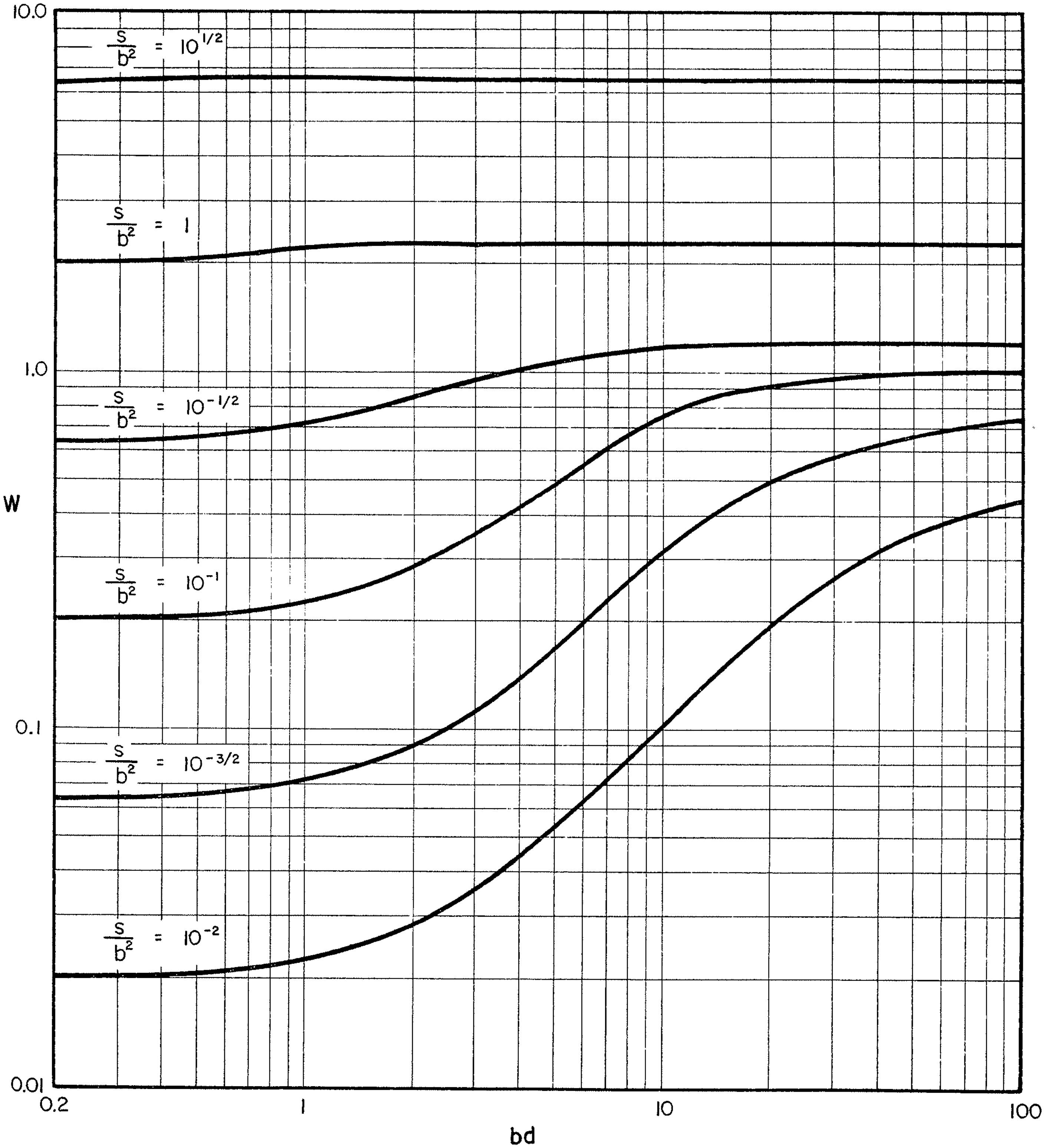


FIG. C.2

OUTPUT PULSE WIDTH AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE

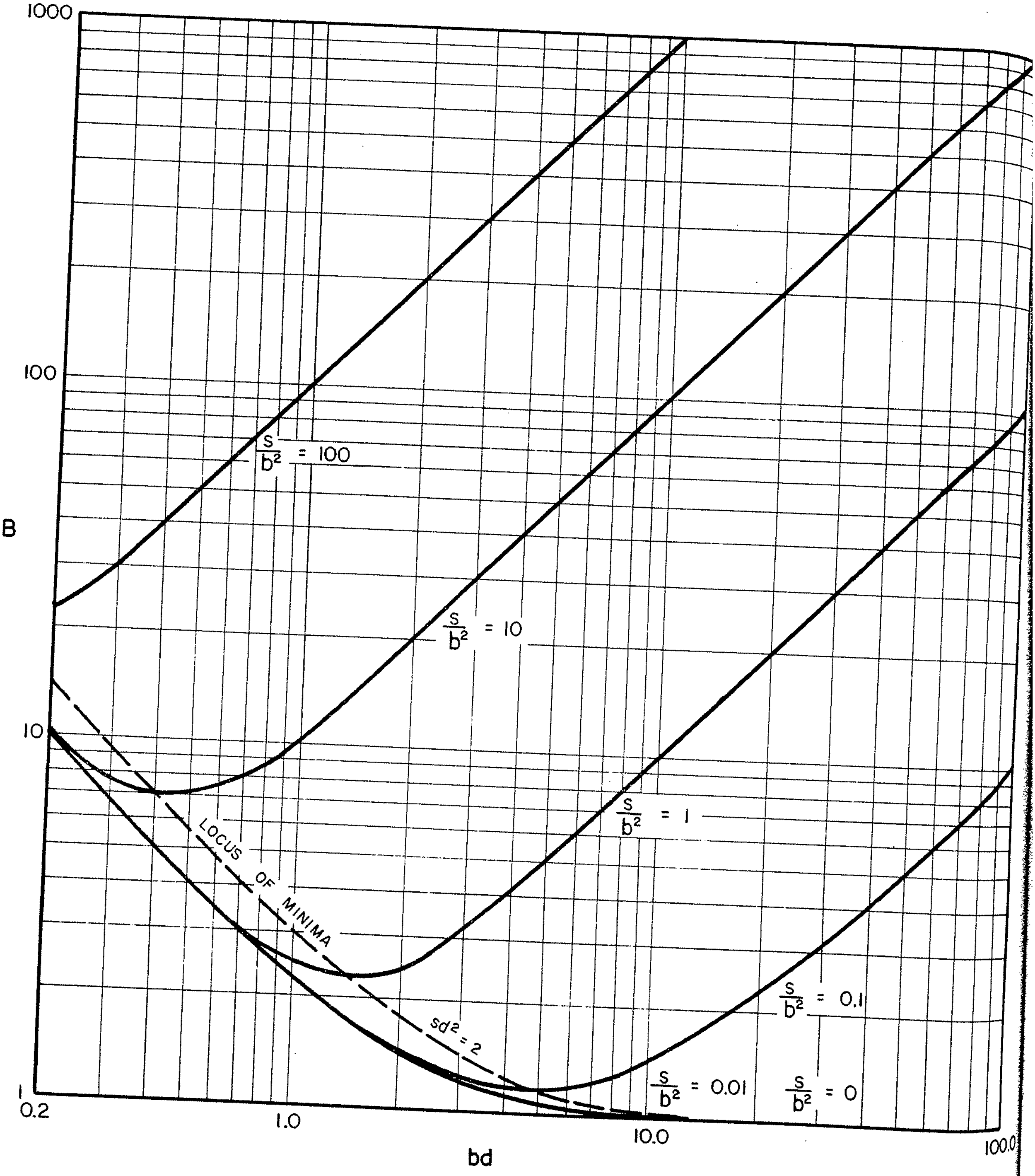


FIG. C.3
APPARENT BANDWIDTH AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE

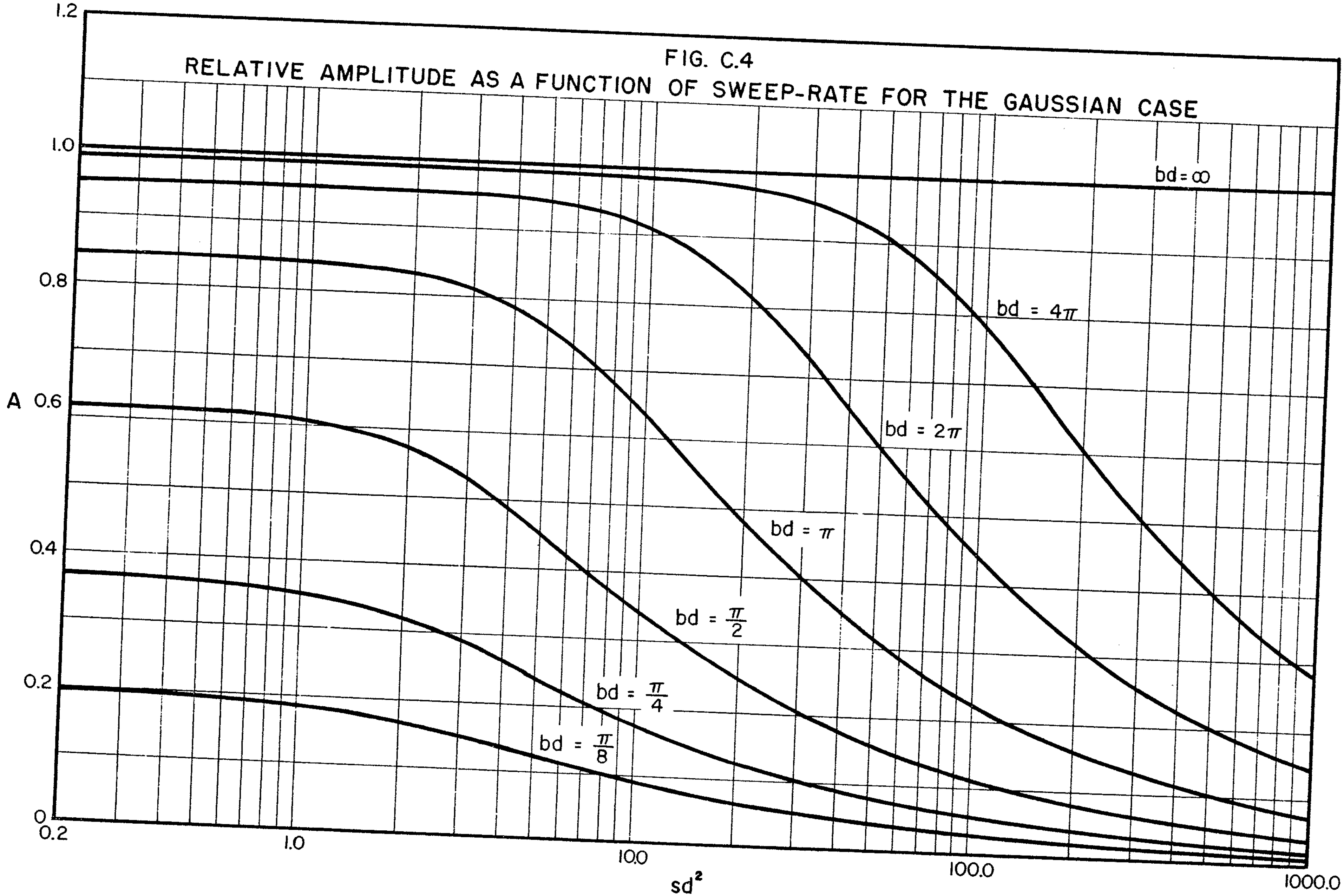


FIG. C.4
RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE FOR THE GAUSSIAN CASE

M-970 A-G3-25 RKI 5-28

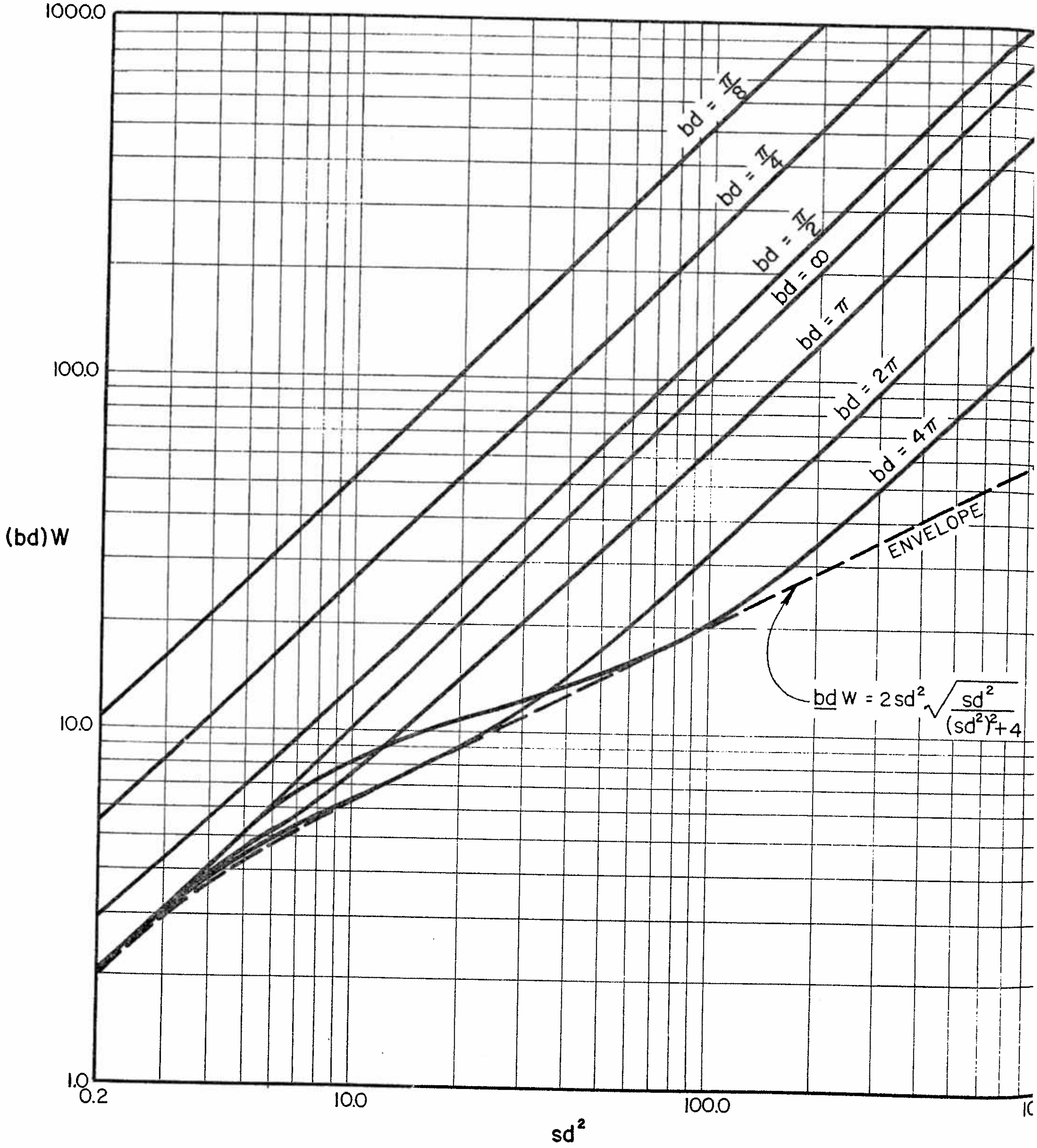


FIG. C.5
OUTPUT PULSE WIDTH AS A FUNCTION OF
SWEEP-RATE FOR THE GAUSSIAN CASE

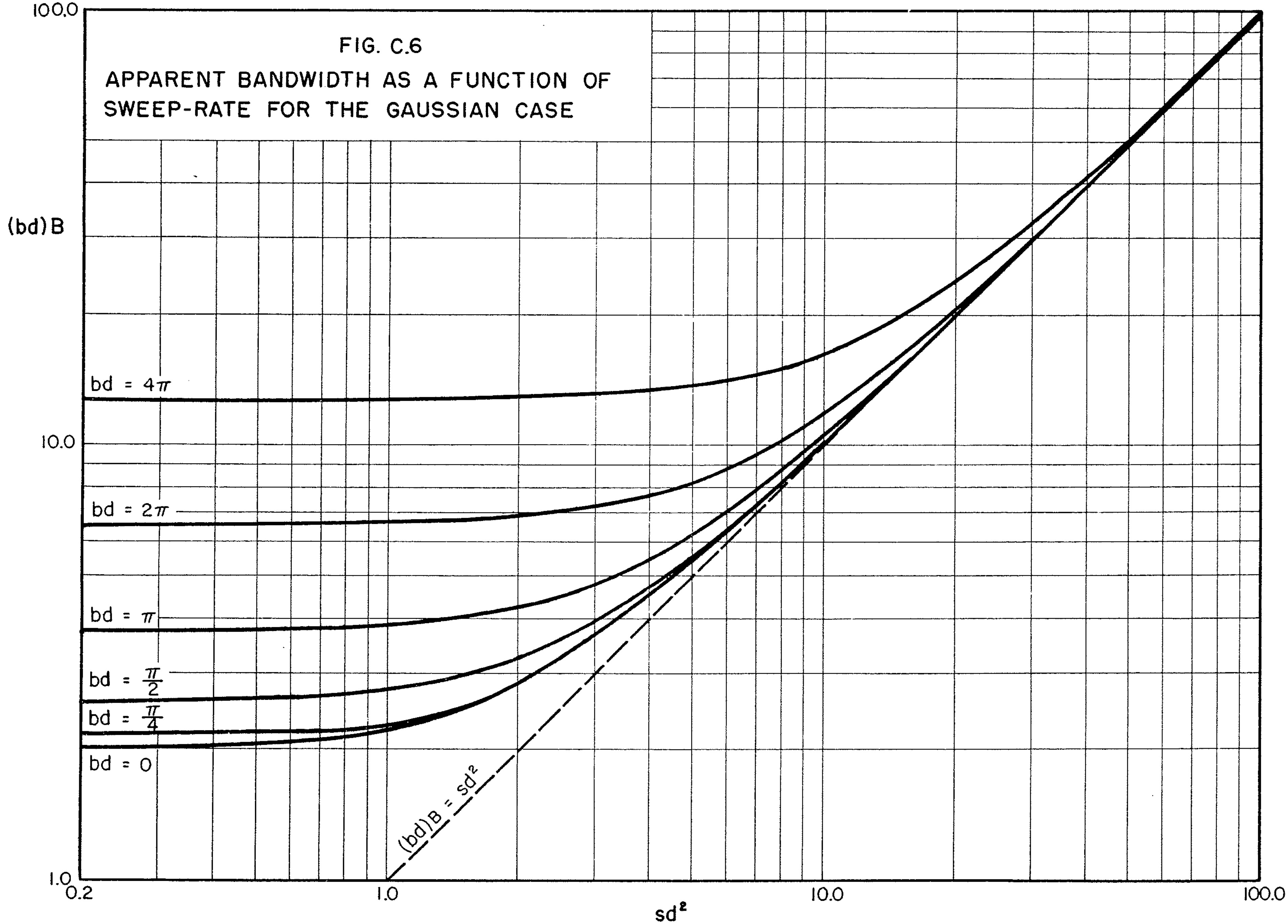
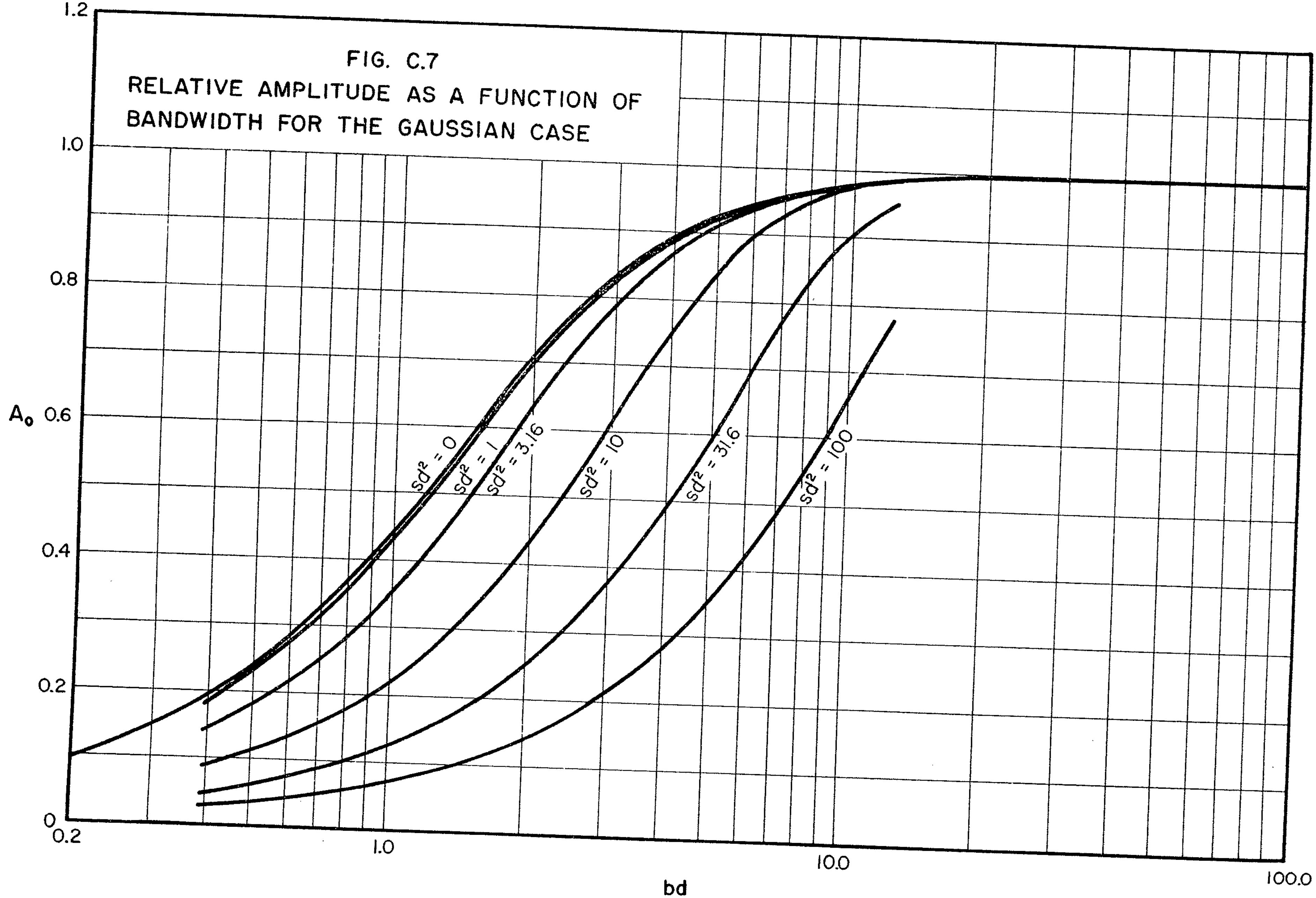


FIG. C.6
APPARENT BANDWIDTH AS A FUNCTION OF SWEEP-RATE FOR THE GAUSSIAN CASE

FIG. C.7
RELATIVE AMPLITUDE AS A FUNCTION OF BANDWIDTH FOR THE GAUSSIAN CASE



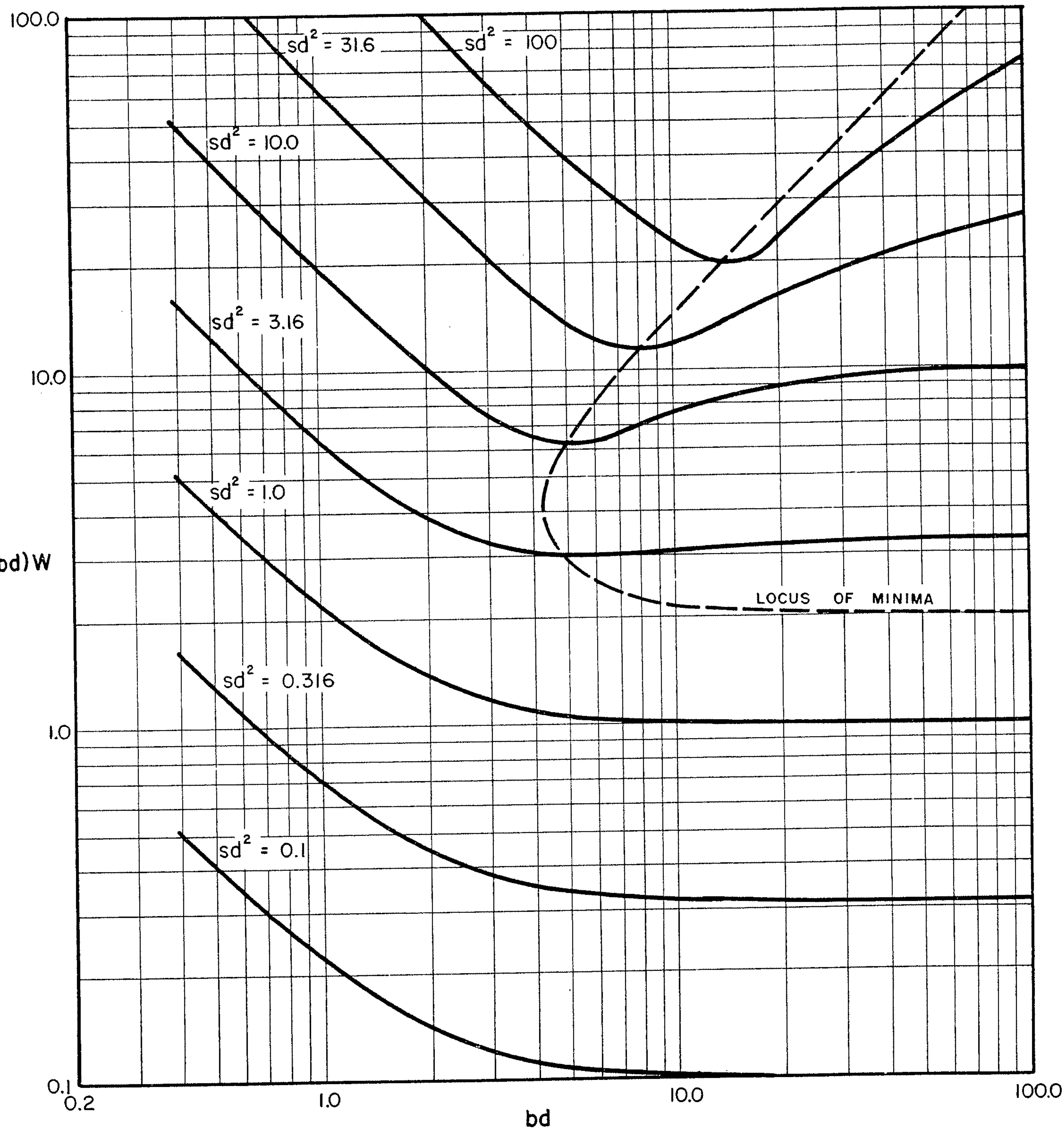


FIG. C.8

OUTPUT PULSE WIDTH AS A FUNCTION OF BANDWIDTH FOR THE GAUSSIAN CASE

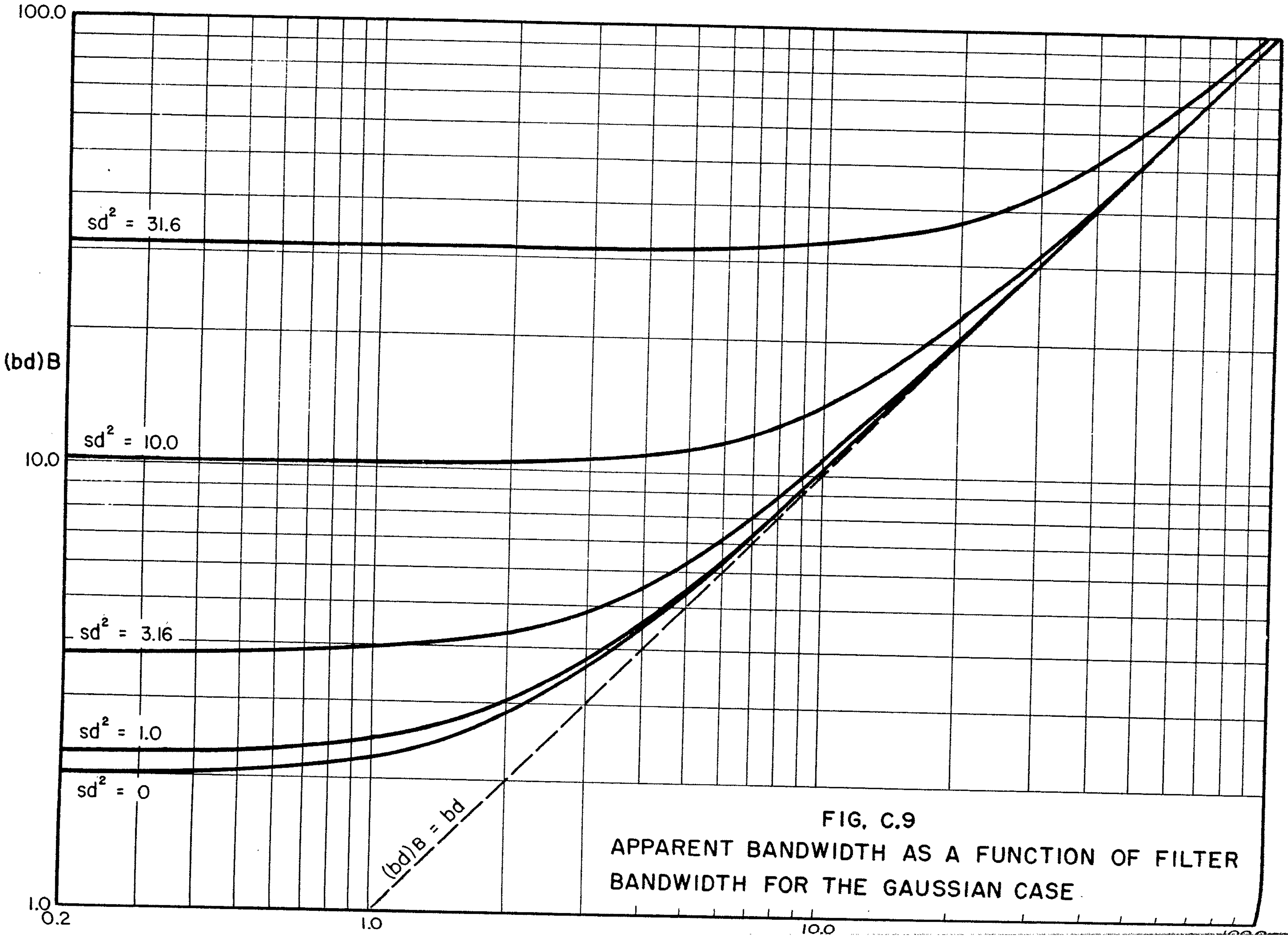


FIG. C.9
APPARENT BANDWIDTH AS A FUNCTION OF FILTER BANDWIDTH FOR THE GAUSSIAN CASE

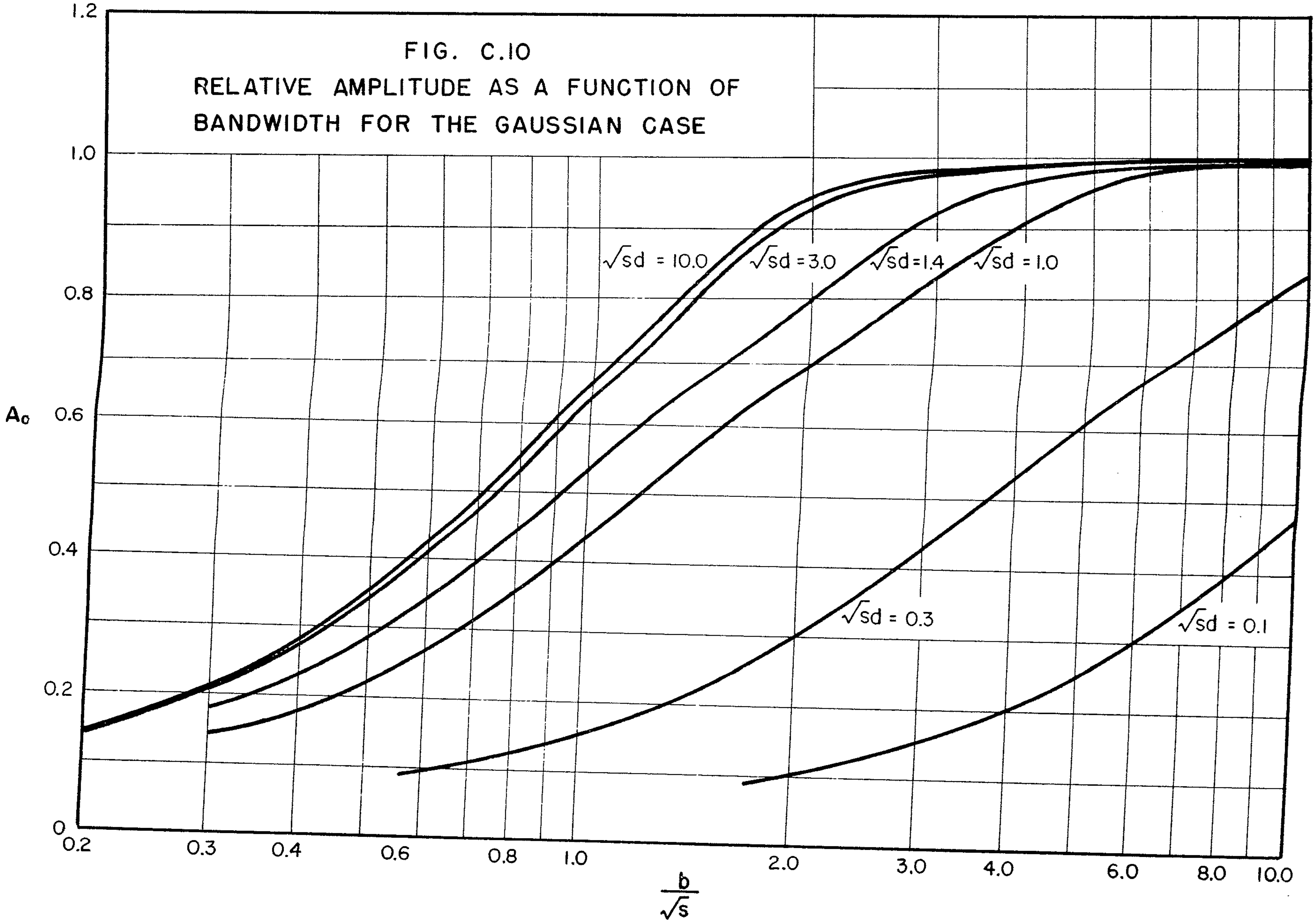


FIG. C.10
RELATIVE AMPLITUDE AS A FUNCTION OF BANDWIDTH FOR THE GAUSSIAN CASE

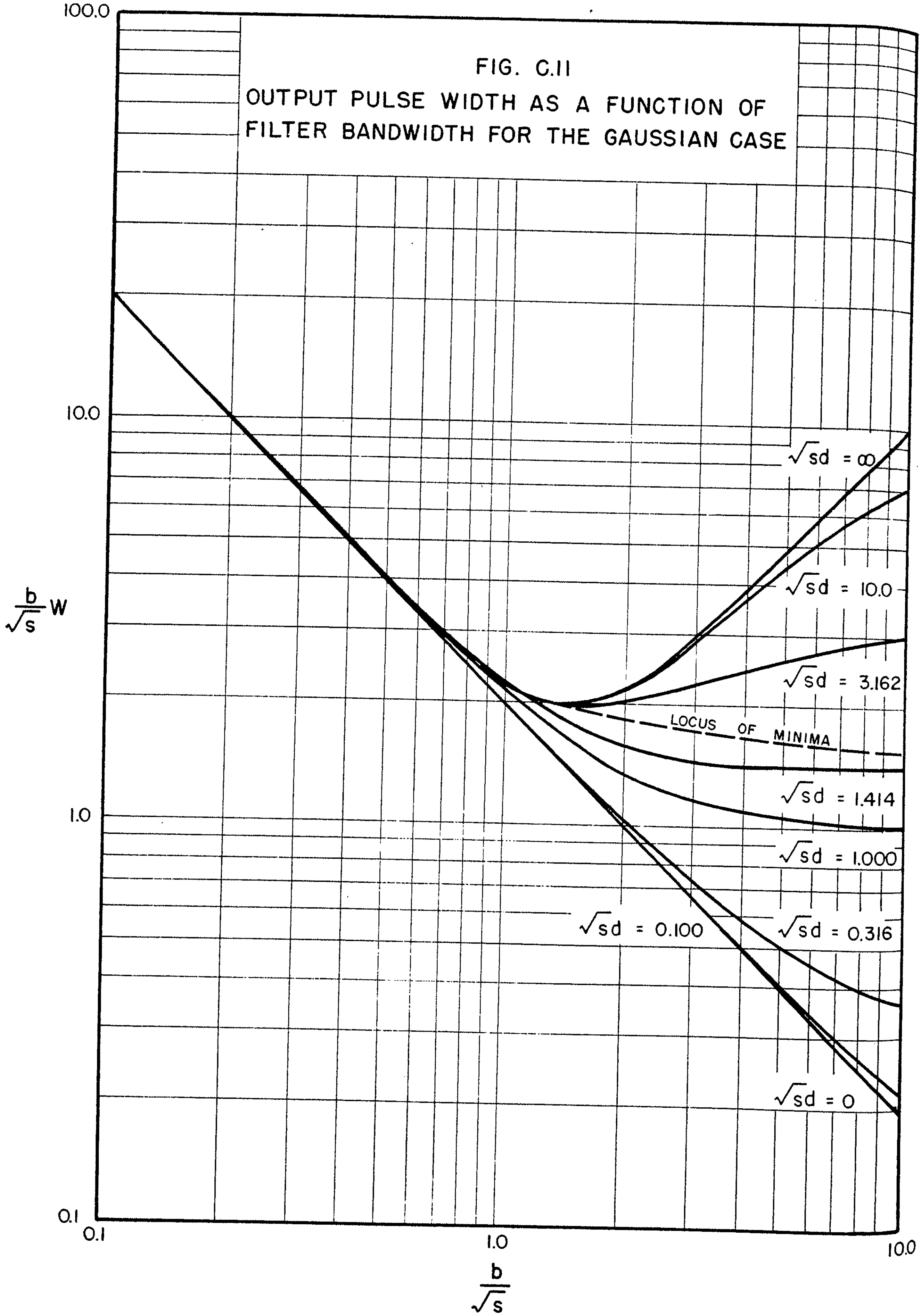


FIG. C.11
OUTPUT PULSE WIDTH AS A FUNCTION OF FILTER BANDWIDTH FOR THE GAUSSIAN CASE

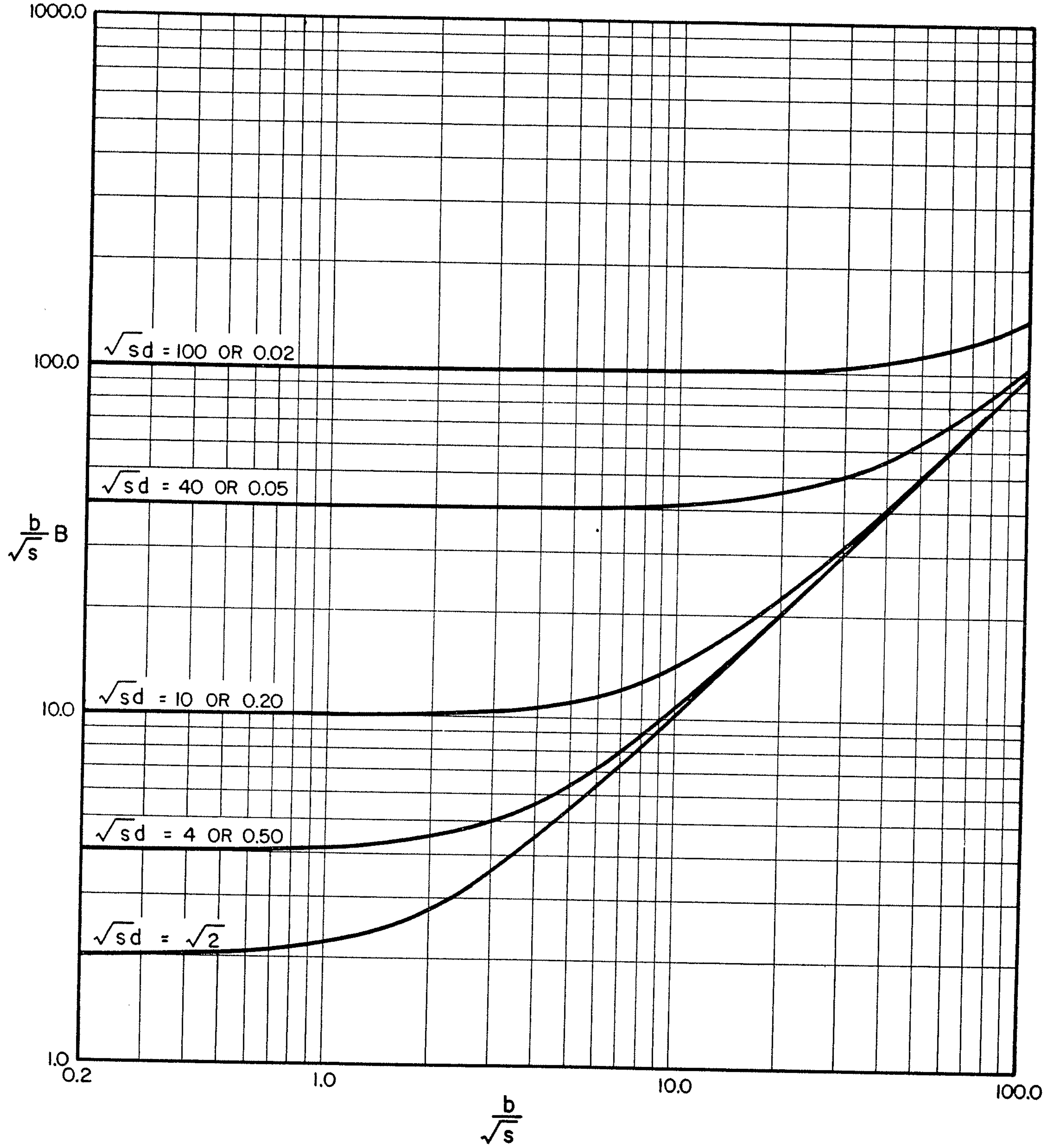


FIG. C.12

APPARENT BANDWIDTH AS A FUNCTION OF FILTER BANDWIDTH FOR THE GAUSSIAN CASE

FIG. C.13

RELATIVE AMPLITUDE AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE

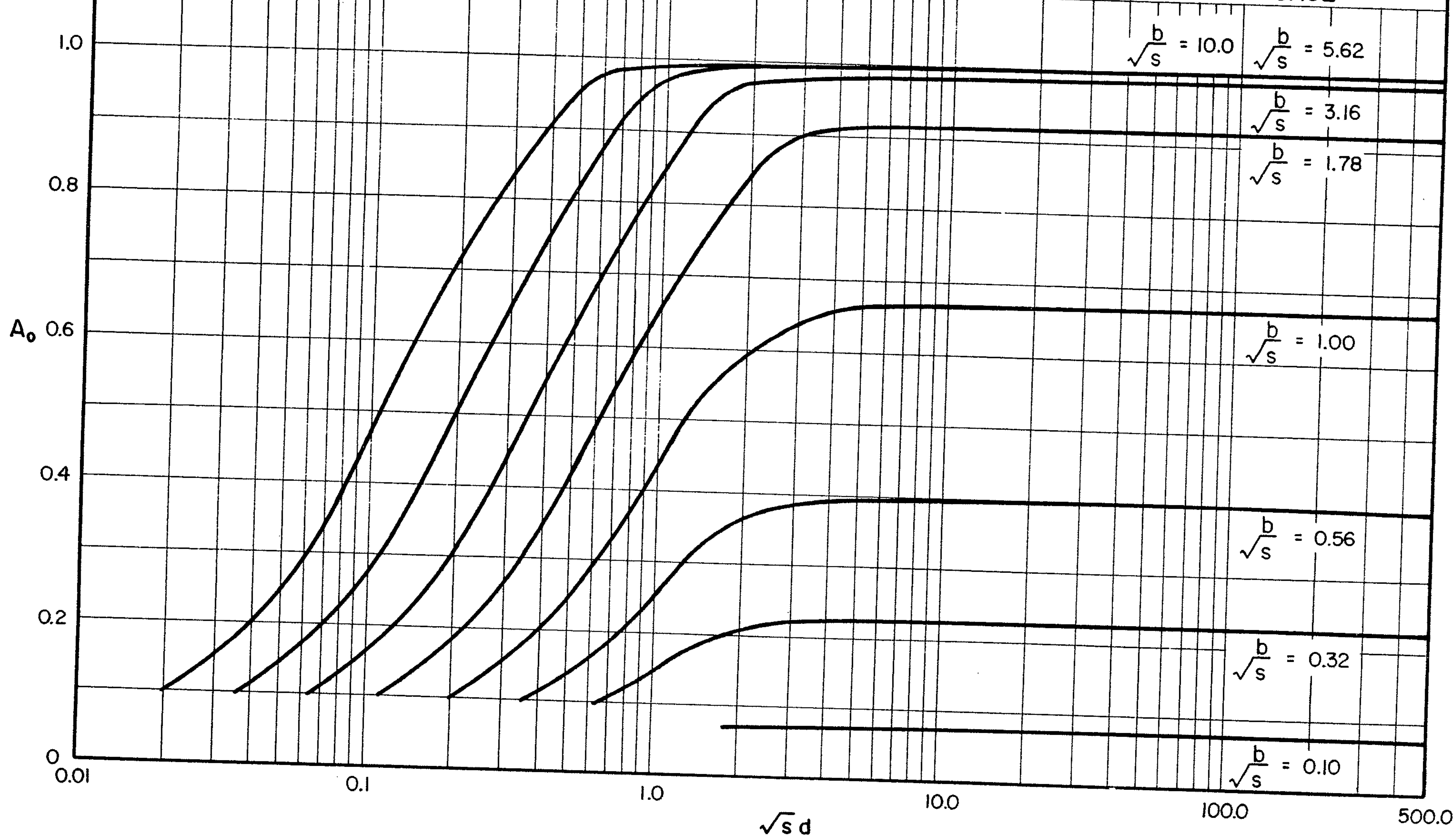
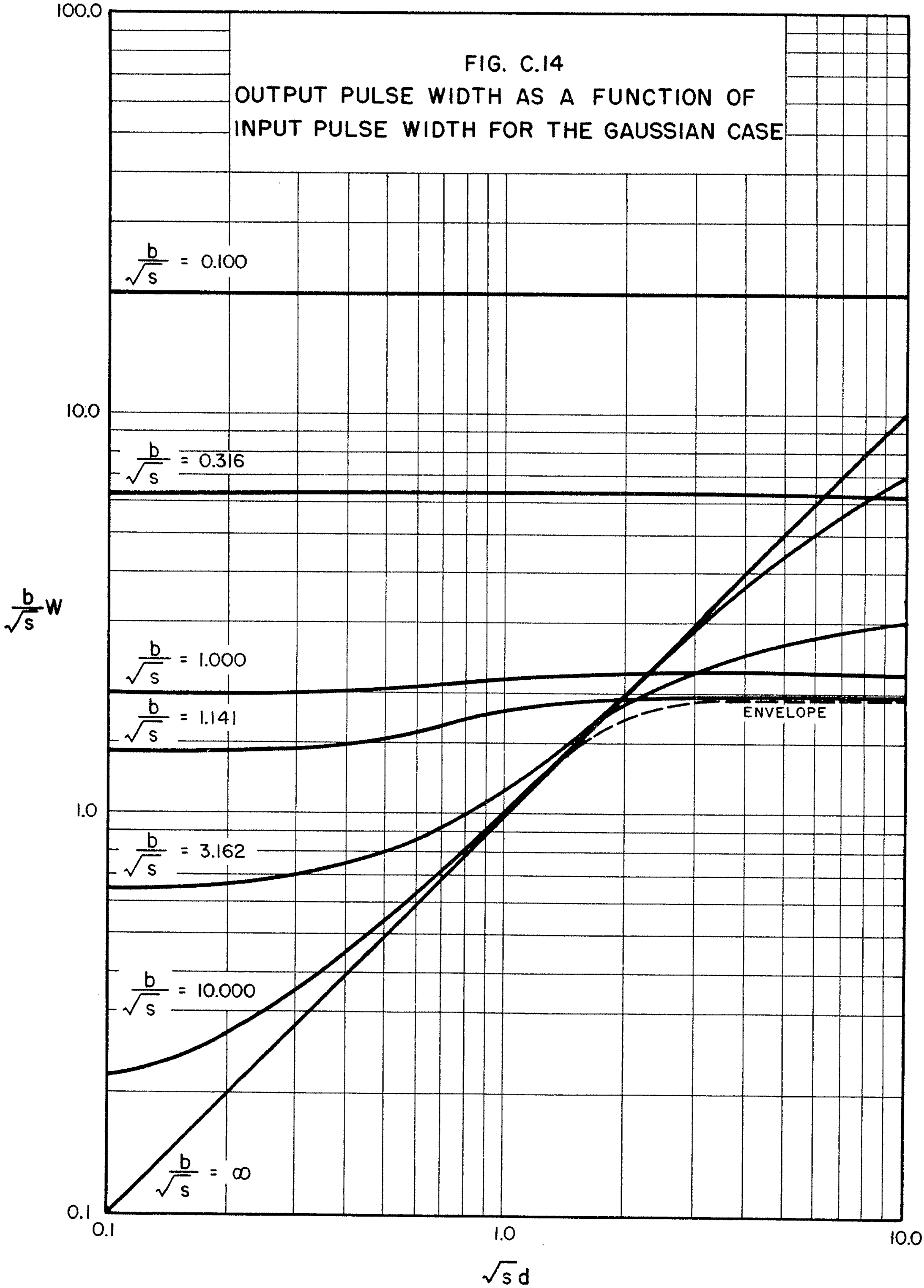


FIG. C.14
OUTPUT PULSE WIDTH AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE



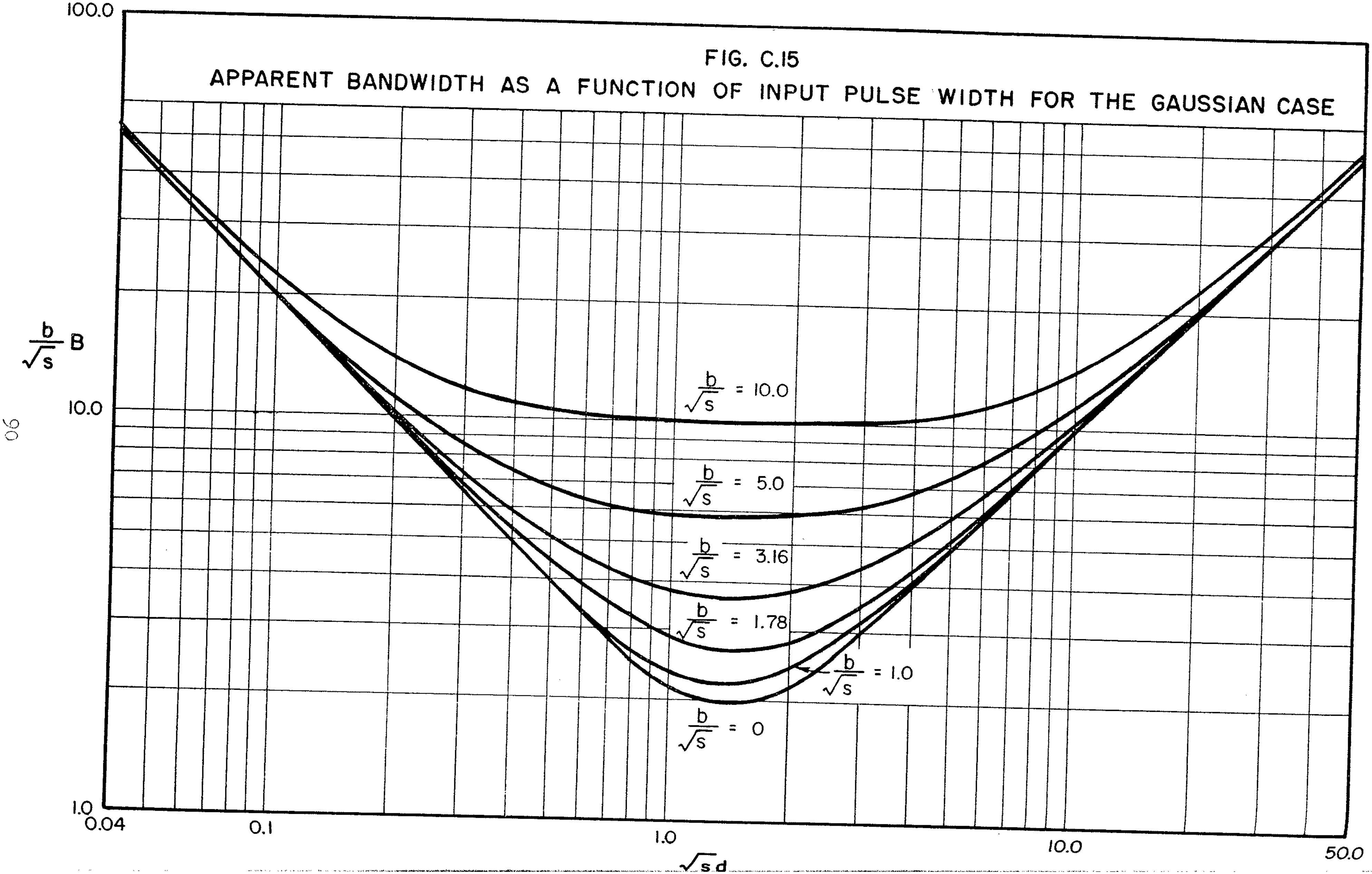


FIG. C.15
APPARENT BANDWIDTH AS A FUNCTION OF INPUT PULSE WIDTH FOR THE GAUSSIAN CASE
$\frac{b}{\sqrt{s}} B$
100.0
10.0
1.0
0.04
0.1
1.0
10.0
50.0
$\sqrt{s}\, d$
$\frac{b}{\sqrt{s}} = 10.0$
$\frac{b}{\sqrt{s}} = 5.0$
$\frac{b}{\sqrt{s}} = 3.16$
$\frac{b}{\sqrt{s}} = 1.78$
$\frac{b}{\sqrt{s}} = 1.0$
$\frac{b}{\sqrt{s}} = 0$

APPENDIX D

Derivation of the General Relations Among Factors Describing Response

The success of the derivation of Eq 3.16 and Eq 3.17 depends on a different definition of the width of a curve. Suppose f(x) is a function which approaches zero when $|x|$ becomes large. Then the width w of the graph of f(x) is defined here as

$$w = \frac{\int_{-\infty}^{\infty} |f(x)|^2 \, dx}{\max |f(x)|^2} \quad . \tag{D.1}$$

For example, the width of a pulse is defined as its total energy divided by its maximum power, and the bandwidth of a filter with transfer function $H(\omega)$ is

$$b = \frac{\int_{-\infty}^{\infty} H(\omega) \, \overline{H(\omega)} \, d\omega}{\max \; H(\omega) \, \overline{H(\omega)}} \quad . \tag{D.2}$$

This is known as the "noise bandwidth."[1]

Suppose a scanning signal of constant amplitude,

$$f(t) = f_o \exp\left[jat + \frac{jst^2}{2} \right] \tag{D.3}$$

is applied to a filter with transfer function $H(\omega)$. Then the output signal is given by

$$g(t) = \int_{-\infty}^{\infty} F(\omega) \, H(\omega) \, e^{j\omega t} \, d\omega$$

where $F(\omega)$ is the Fourier transform of f(t). By Parseval's Theorem,[2] the energy

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[1]Wallman, H. and Valley, G.E., Ref. 9, p. 169
Lawson, J.L. and Uhlenbeck, G.E., Ref. 10, p. 101

[2]Titchmarsh, An Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937, p. 50

in the output pulse is,

$$\int_{-\infty}^{\infty} \left[g(t)\right]^2 dt = 2\pi \int_{-\infty}^{\infty} F(\omega)\, H(\omega)\, \overline{F(\omega)}\, \overline{H(\omega)}\, d\omega. \qquad (D.4)$$

But,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = f_o \frac{(1+j)}{\sqrt{2s}}\, e^{-\frac{j(a-\omega)^2}{2s}} \qquad (D.5)$$

so that,

$$F(\omega)\, \overline{F(\omega)} = \frac{f_o^{\,2}}{s},$$ and from (D.4),

$$\int_{-\infty}^{\infty} \left[g(t)\right]^2 dt = \frac{2\pi f_o^{\,2}}{s} \int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega\,. \qquad (D.6)$$

Now all that remains is to interpret these integrals using the definition given above for the width of a curve.

As was pointed out above, the energy of a pulse is the product of pulse width and maximum power, so

$$\int_{-\infty}^{\infty} \left[g(t)\right]^2 dt = t_o \cdot (\text{maximum power of output pulse}) \qquad (D.7)$$

where t_o is the width of the output pulse. The maximum of $2\pi H(\omega)\, \overline{H(\omega)}$ is the maximum power gain of the filter, so by Eq D.2,

$$\int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega = \frac{b}{2\pi} \cdot (\text{maximum power gain of filter}). \qquad (D.8)$$

Substituting in Eq D.6,

$$t_o \cdot (\text{maximum power of output pulse}) = \frac{b f_o^{\,2}}{s} \cdot (\text{maximum power gain of filter}) \qquad (D.9)$$

f_o^2 is the power of the input signal, and by the definition of relative amplitude A_o,

$$A_o^2 = \frac{\text{(maximum power of output pulse)}}{\text{(input signal power)}\cdot\text{(maximum power gain of filter)}} \quad . \tag{D.10}$$

Substituting this in Eq D.9 gives

$$A_o^2 \cdot \frac{st_o}{b} = A_o^2 \cdot W = 1 \tag{D.11}$$

which is Eq 3.16.

The difficult part of proving Eq 3.17 is defining "apparent bandwidth." Suppose the input signal $f(t)$ to the filter is a pulse of arbitrary shape and frequency, and suppose its transform is $F(\omega)$. If the pulse is shifted in frequency by α radians, the transform of the resulting pulse $f_\alpha(t)$ is $F(\omega + \alpha)$. The response of the filter to each of the family of pulses resulting from shifting $f(t)$ in frequency is a function of α. In particular, the energy of the output pulse is a function of α which approaches zero for large $|\alpha|$. The apparent bandwidth B is defined here as the width of the graph of energy of the output pulse as a function of α, divided by the filter bandwidth (see Fig. D.1).

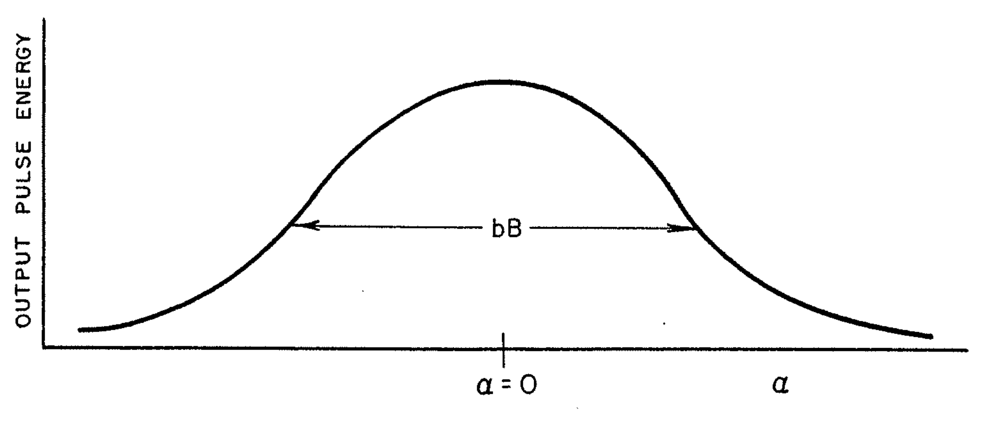

Figure D.1

There is no loss of generality in assuming that the output pulse energy is a maximum for $\alpha = 0$. The output pulse energy as a function of α is, by Parseval's Theorem,

$$2\pi\int_{-\infty}^{\infty} F(\omega+\alpha)\, H(\omega)\, \overline{F(\omega+\alpha)}\, \overline{H(\omega)}\, d\omega \,, \tag{D.12}$$

and by Eq D.1,

$$B = \frac{1}{b}\,\frac{2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega+\alpha)\, H(\omega)\, \overline{F(\omega+\alpha)}\, \overline{H(\omega)}\, d\omega\, d\alpha}{2\pi\int_{-\infty}^{\infty} F(\omega)\, H(\omega)\, \overline{F(\omega)}\, \overline{H(\omega)}\, d\omega} \,. \tag{D.13}$$

Changing the order of integration in the numerator,

$$B = \frac{1}{b}\,\frac{2\pi\int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\int_{-\infty}^{\infty} F(\omega+\alpha)\, \overline{F(\omega+\alpha)}\, d\alpha\, d\omega}{2\pi\int_{-\infty}^{\infty} F(\omega)\, H(\omega)\, \overline{F(\omega)}\, \overline{H(\omega)}\, d\omega} \,. \tag{D.14}$$

But,

$$\int_{-\infty}^{\infty} F(\omega+\alpha)\, \overline{F(\omega+\alpha)}\, d\alpha = \int_{-\infty}^{\infty} \overline{F(\alpha)}\, F(\alpha)\, d\alpha \,,$$

so that

$$B = \frac{1}{b}\,\frac{\int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega\; 2\pi\int_{-\infty}^{\infty} F(\alpha)\, \overline{F(\alpha)}\, d\alpha}{2\pi\int_{-\infty}^{\infty} F(\omega)\, H(\omega)\, \overline{F(\omega)}\, \overline{H(\omega)}\, d\omega} \,, \tag{D.15}$$

and all that remains is to interpret the three integrals.

By (D.2),

$$2\pi\int_{-\infty}^{\infty} H(\omega)\, H(\omega)\, d\omega = b\cdot(\text{max power gain of filter}), \tag{D.16}$$

and by Parseval's Theorem again,

$$\int_{-\infty}^{\infty} F(\alpha)\,\overline{F(\alpha)}\,d\alpha = d\cdot(\text{max power of input pulse}). \qquad (D.17)$$

$$2\pi\int_{-\infty}^{\infty} F(\omega)\,H(\omega)\,\overline{F(\omega)}\,\overline{H(\omega)}\,d\omega = t_o(\text{maximum power of output pulse for "centered" input pulse}), \qquad (D.18)$$

where d and t_o are the widths of the input and output pulses respectively. Substituting in Eq D.15 gives

$$B = \frac{d\cdot(\text{max power of input pulse})\cdot b\cdot(\text{max power gain of filter})}{b\cdot t_o\cdot(\text{max power of output pulse for "centered" input pulse})} . \qquad (D.19)$$

By the definition of relative amplitude A_o,

$$A_o^{\,2} = \frac{(\text{max power of output pulse for "centered" input pulse})}{(\text{max power of input pulse})(\text{max power gain of filter})} . \qquad (D.20)$$

Substituting (D.20) in (D.19) gives

$$A_o^{\,2}\, B\, \frac{t_o}{d} = 1 \qquad (D.21)$$

which is Eq 3.17.

As an application of (D.21), a proof is given here that the straight lines $B^* = bd\cdot\frac{s}{b^2}$ are asymptotes for the apparent bandwidth as the sweep-rate becomes large. Since the curves are plotted on log-log graph paper, the condition for B* to be an asymptote for B is

$$\lim_{s\to\infty} \frac{B^*}{B} = 1 . \qquad (D.22)$$

By Eq D.21,

$$B = \frac{d}{t_o\, A_o^{\,2}} .$$

Substituting this for B and bd $\cdot \frac{s}{b^2}$ for B*, the condition (D.22) becomes

$$\lim_{s\to\infty} \frac{sd}{b} \cdot \frac{t_o A_o^2}{d} = \lim_{s\to\infty} \frac{s t_o A_o^2}{b} = 1. \qquad (D.23)$$

Let M(t) denote the envelope of the input signal. It is convenient to choose the time origin so that M(t) is a maximum when t = 0; this is achieved by substituting t + c for t. The frequency of the input signal when t = 0 is then sc + a, and the input signal is

$$f(t) = M(t)\, e^{jst^2 + j(sc + a)t} \qquad (D.24)$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega + \alpha)\, e^{j\omega t}\, d\omega = \frac{e^{-j\alpha t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega + \alpha)\, e^{j(\omega + \alpha)t}\, d(\omega + \alpha)$$

$$= e^{-j\alpha t} f(t),$$

a translation in frequency in general is equivalent to multiplying f(t) by $e^{-j\alpha t}$. This can be achieved by changing the parameter c in (D.24). Thus, the "centered pulse" is the input signal which has the value for the parameter c which maximizes the output power.

The output signal is, by the convolution formula[1],

$$g(t,c) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} M(\tau)\, \exp\left\{\frac{js\tau^2}{2} + j(sc + a)\tau\right\} h(t - \tau)\, d\tau , \qquad (D.25)$$

where h(t) is the transform of H(ω), the filter transfer function.

[1]Titchmarsh, loc.cit., p. 51

As in Eq D.20,

$$A_o^2 = \frac{\text{(max power of output pulse for centered input pulse)}}{\text{(max power of input pulse)}\cdot\text{(max power gain of filter)}}$$

$$= \frac{\max \left| g(t,c) \right|^2}{\max \left| f(t) \right|^2 \cdot \max 2\pi\, H(\omega)\, \overline{H(\omega)}}\,, \qquad \text{(D.26)}$$

and

$$\lim_{s\to\infty} \frac{s t_o A_o^2}{b} = \frac{\lim_{s\to\infty} \left[t_o \cdot \max \left| \sqrt{s}\; g(t,c) \right|^2 \right]}{b \cdot \max \left[M(t) \right]^2 \cdot \max 2\pi\, H(\omega)\, \overline{H(\omega)}}$$

$$= \frac{\max \lim_{s\to\infty} t_o \cdot \left| \sqrt{s}\; g(t,c) \right|^2}{\max \left[M(t) \right]^2 \cdot \max 2\pi\, H(\omega)\, \overline{H(\omega)}}\,. \qquad \text{(D.27)}$$

Referring to Eq D.25,

$$\lim_{s\to\infty} \left| \sqrt{s}\; g(t,c) \right|^2 \doteq \left| \lim_{s\to\infty} \sqrt{s} \int_{-\infty}^{\infty} M(\tau)\, \exp\left\{ \frac{js\tau^2}{2} + j(a + sc)\tau \right\} h(t-\tau)\, d\tau \right|^2 .$$

Now,[1]

$$\int_{-\infty}^{\infty} \exp\left\{ \frac{js\tau^2}{2} \right\} d\tau = (1 + j)\sqrt{\frac{\pi}{s}}\,, \qquad \text{(D.28)}$$

and from this it can be shown that

$$\lim_{s\to\infty} \frac{1}{(1 + j)\sqrt{\frac{s}{\pi}}} \int_{-\infty}^{\infty} \exp\left\{ \frac{js(\tau + c)^2}{2} \right\} F(\tau)\, d\tau = F(-c). \qquad \text{(D.29)}$$

[1] Dwight, H.B., Tables of Integrals and Other Mathematical Data, Eq. 859.5, Macmillan, New York, 1947

It follows that

$$\lim_{s \to \infty} \sqrt{s}\ g(t) = (1 + j)\sqrt{\pi}\ \exp\left\{\frac{-jsc^2}{2}\right\} M(-c)\ h(t - c), \qquad (D.30)$$

and the response, in the limit, is the same as the response to an impulse. This observation makes possible the calculation of the limiting value of t_o. By (D.1) and using Parseval's Theorem,

$$\lim_{s \to \infty} t_o = \frac{\int_{-\infty}^{\infty} |h(t)|^2\, dt}{\max |h(t)|^2} = \frac{\int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega}{\max |h(t)|^2}, \text{ and}$$

$$\lim_{s \to \infty} t_o = \frac{b \max H(\omega)\, \overline{H(\omega)}}{\max |h(t)|^2}. \qquad (D.31)$$

Substituting (D.30) and (D.31) into (D.27) gives

$$\lim_{s \to \infty} \frac{st_o A_o^2}{b} = \frac{|1 + j|^2\, \pi \max |M(-c)|^2 \max |h(t - c)|^2\, b \max H(\omega)\, \overline{H(\omega)}}{b \max |M(t)|^2 \max 2\pi H(\omega)\, \overline{H(\omega)} \max |h(t)|^2}, \text{ or}$$

$$\lim_{s \to \infty} \frac{st_o A_o^2}{b} = 1 \qquad (D.32)$$

which is the required condition, Eq D.23.

APPENDIX E

Differential Analyzer Procedure

E.1 Analyzer Setup

The mathematical problem is to study solutions of the equations

$$
\begin{aligned}
C\frac{de_1}{dt} + \frac{e_1}{R} + \frac{1}{L}\int e_1\,dt &= i(t), \\
C\frac{de_2}{dt} + \frac{e_2}{R} + \frac{1}{L}\int e_2\,dt &= -g_m e_1, \\
C\frac{de_3}{dt} + \frac{e_3}{R} + \frac{1}{L}\int e_3\,dt &= -g_m e_2, \\
C\frac{de_4}{dt} + \frac{e_4}{R} + \frac{1}{L}\int e_4\,dt &= -g_m e_3,
\end{aligned} \tag{E.1}
$$

where,

$$
\begin{aligned}
i(t) &= 0 && \text{for } t < c - \frac{d}{2}, \\
i(t) &= \cos\left(at + \frac{st^2}{2}\right) && \text{for } c - \frac{d}{2} \le t \le c + \frac{d}{2}, \text{ and} \\
i(t) &= 0 && \text{for } t > c + \frac{d}{2}.
\end{aligned} \tag{E.2}
$$

The system of equations E.1 is of a particularly simple type in that the solution of each differential equation becomes the forcing function for the succeeding equation; otherwise all four differential equations are identical.

The electronic differential analyzer is an analogue machine employing voltages as the dependent variables and time as the independent variable. It consists of a number of high-gain amplifiers which, through suitable feedback connections, can be made to perform the operations of addition and integration of these variables. The interconnection of analyzer units to solve any one of equations E.1 is given in Fig. E.1. In this figure the ground connection is common to all elements, although it is omitted in the drawing. All voltages are measured with respect to ground. The equation solved by this setup is

$$\frac{de_j}{dt} = -\frac{P_r}{250} \cdot e_j - 200 \int e_j \, dt + \frac{P_{ij}}{2000} \cdot i(j) \qquad (E.3)$$

which is identical with any of Eq E.1 provided,

$$\frac{1}{RC} = b_o = \frac{P_r}{250}$$

and

$$\frac{1}{LC} = a^2 = 200. \qquad (E.4)$$

b_o is the circuit bandwidth in radians per second and "a" is the circuit center-frequency, which for these solutions was held constant at $10\sqrt{2}$ radians per second. Fig. E.1a represents the contents of any one of the blocks labelled "circuit" in Fig. 4.3.

A sinusoidal signal with frequency proportional to time was generated by solving the differential equation,

$$\frac{d^2f}{dt^2} + (10 + st)^2 f = 0 \qquad (E.5)$$

with the initial conditions

$$\left.\frac{df}{dt}\right|_o = 1.5 \,,$$

$$f_o = 0 \quad .$$

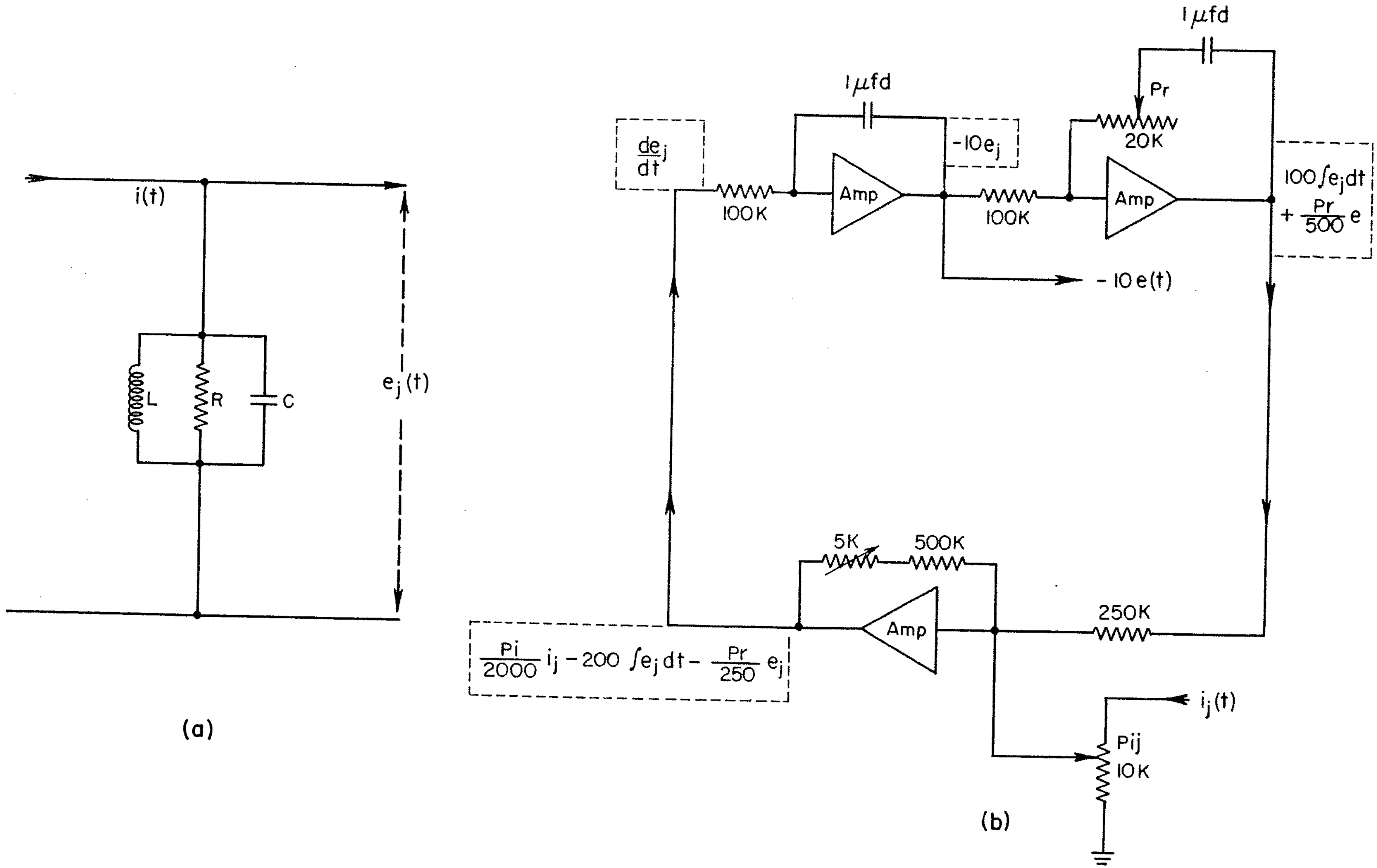

FIG. E.1
SINGLE-TUNED FILTER AND ITS ANALOGUE

For small s, one would expect the solution of this equation to be

$$\frac{df}{dt} \approx 1.5 \cos\left[10t + \frac{st^2}{2}\right]. \qquad (E.6)$$

When Eq E.5 is solved on the differential analyzer, it is observed that the amplitude of the first derivative increases as $0 \leq st \leq 10$. This increase can be reduced to negligible proportions by modifying the equation solved to

$$\frac{d^2f}{dt^2} + \epsilon\frac{df}{dt} + (10 + st)^2 f = 0. \qquad (E.5')$$

The differential analyzer circuit for the solution of Eq E.5 is shown in Fig. E.2. The variable coefficient of Eq E.5' is generated by the two ganged potentiometers M-1 and M-2 driven by a synchronous motor at $\frac{3}{14}$ RPM. The circuit shown solves the equation

$$\frac{d^2f}{dt^2} + \epsilon\frac{df}{dt} + \left[10 + \frac{10\,\theta}{\pi}\right]\cdot f = 0. \qquad (E.7)$$

Since $\theta = \frac{\pi t}{140}$, $\frac{10\,\theta}{\pi} = \frac{t}{14}$; and Eq E.7 is the same as Eq E.5' with $s = \frac{1}{14}$ radian per second per second. The amplitude of the first derivative, which is used as the output signal, is observed to remain constant to within five per cent when $\epsilon = 3 \times 10^{-5}$. Thus the signal used is

$$i(t) = \frac{df}{dt} = 15 \cos\left[10t + \frac{t^2}{28}\right] \text{ volts} \qquad (E.8)$$

for

$$0 \leq t \leq 140 \text{ seconds.}$$

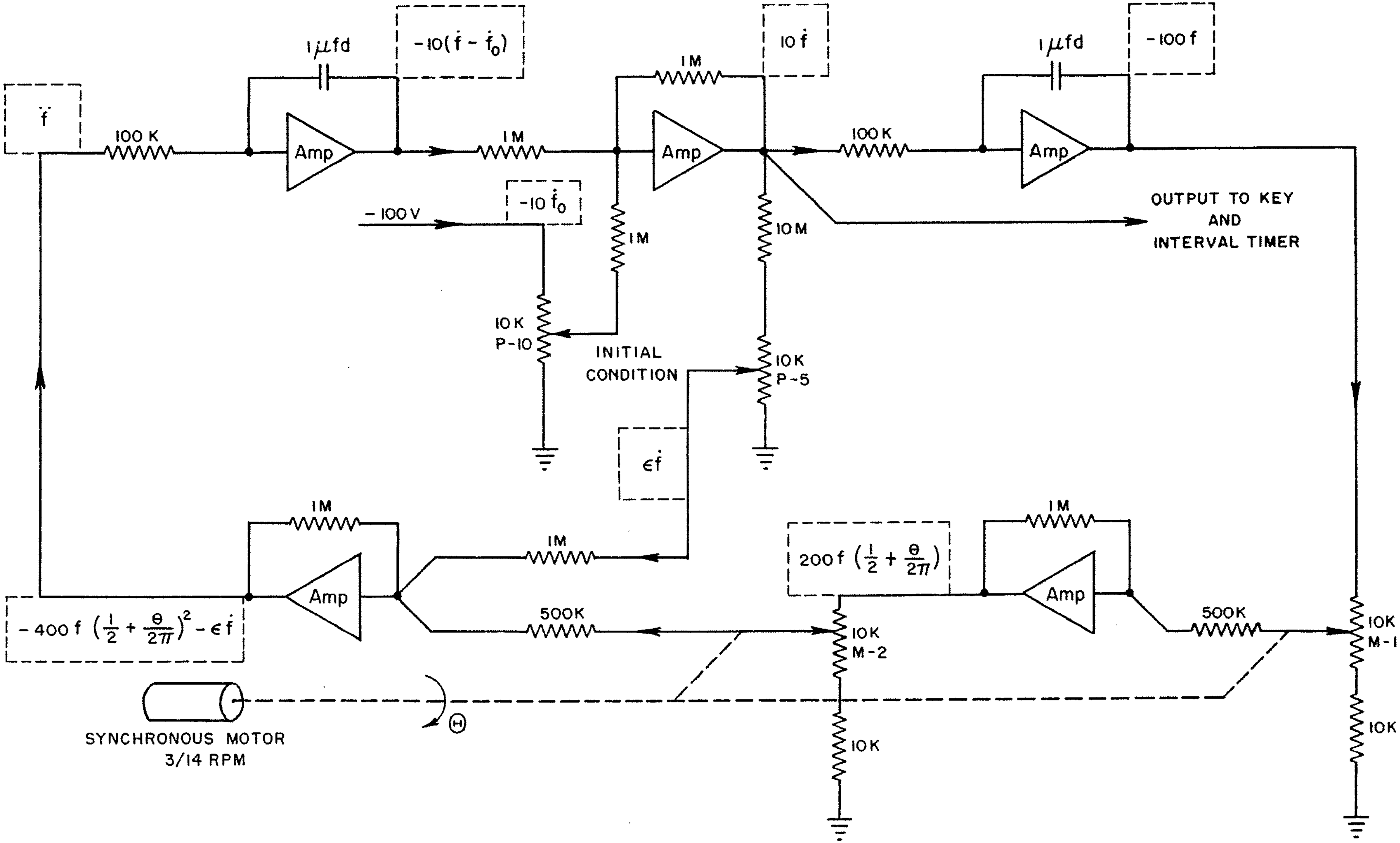

FIG. E.2
DIFFERENTIAL ANALYZER SETUP OF INPUT FUNCTION GENERATOR

The interconnection of the input function generator and the various circuits to solve equation E.1 is given in Fig. 4.3. The block labelled key and interval timer consists of a switch which is closed manually at $c - \frac{d}{2}$ seconds and an electronic interval timer which opens the circuit d seconds later. The four channel recorder is made up of two dual Brush recorders.

E.2 Differential Analyzer Solutions

E.2.1 Run Proceedures For all differential analyzer solutions the input signal sweep-rate s, the filter center frequency a, and the input signal amplitude are held constant. In each run four quantities have to be specified: (1) the type of filter (number of single-tuned circuits n), (2) the bandwidth of the filter b, (3) the length of the input pulse in time d, and (4) the time of the center of the input pulse c.

Table E.1 shows a typical series of runs. These were made to study the effect of varying the time c and pulse length d on the response of a two circuit filter. The bandwidth of the filter can conveniently be normalized in terms of the reciprocal of the pulse width. This series is for

$$bd = 2\pi. \tag{E.9}$$

Table E.1 Typical Series of Runs

2 circuits $bd = 2\pi$ $s = .0710 \text{ rad/sec}^2$

d in seconds	5	10	20
c in seconds	- 25	- 20	- 15
	- 10	- 10	- 10
	- 5	- 5	- 5
	0	0	0
	+ 5	+ 5	+ 5
	+ 10	+ 10	+ 10
	+ 25	+ 20	+ 15

In accordance with Eq 4.10, the bandwidth of the individual circuits is

$$b_o = \frac{b}{.636} = \frac{3.11\pi}{d}, \tag{E.10}$$

and from Eq E. 4, the damping potentiometer settings are given by

$$P_r = \frac{778\pi}{d} \text{ divisions.} \tag{E.11}$$

For each run the time of the start of the pulse must be determined relative to the start of the run. For every run the time t = 0 of Fig. 4.4 occurred 55.3 seconds after the run started. If c = -15 seconds and d = 20 seconds, the signal is connected to the filter input $55.3 - \left[15 + \frac{20}{2}\right] = 30.3$ seconds after the run begins. The signal is disconnected from the filter input by the electronics interval timer 20 seconds later. The proper time to connect the signal is determined by observing the Brush recorder tape, which is driven by a synchronous motor at a rate of 5 millimeters per second. The input pulse is always recorded on one of the four output channels so that a check on this time is available.

E.2.2 Extraction and Processing of Differential Analyzer Output Data

A data tape as recorded at the output of the differential analyzer is shown in Fig. E.3. Two data are extracted from each tape as indicated in the figure: the maximum amplitude of the pulse envelope A' and the length of the output pulse in time t_o. The time t_o is measured between the points at which the envelope amplitude is $A'/\sqrt{2}$. Throughout this study a comparison of the behavior of various filters was made in terms of three quantities: A, W, and B.[1]

[1] See Section 2.3 for a more detailed discussion.

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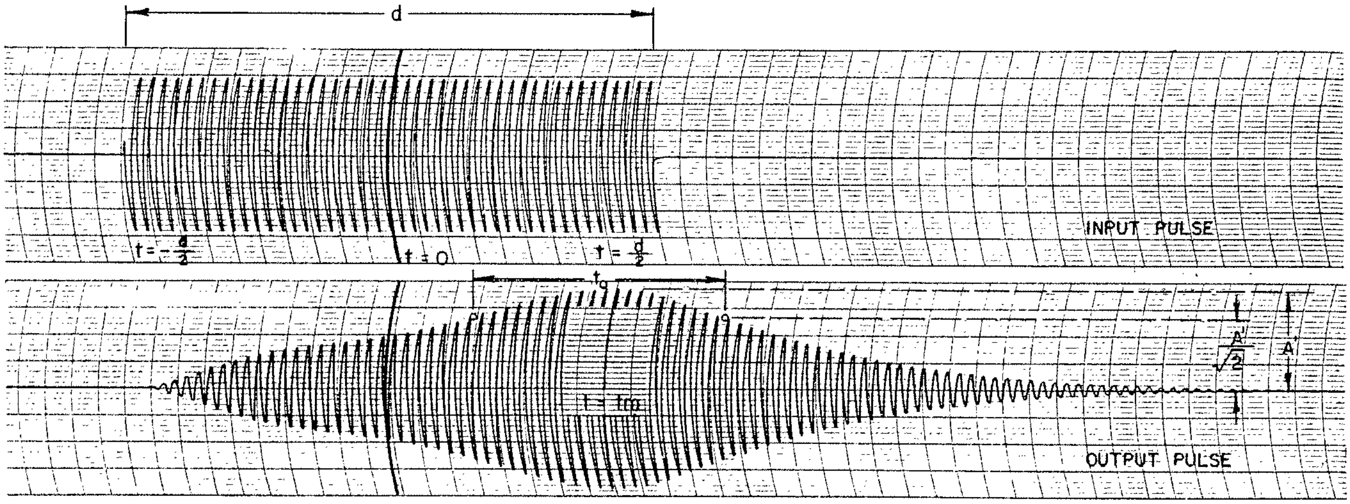
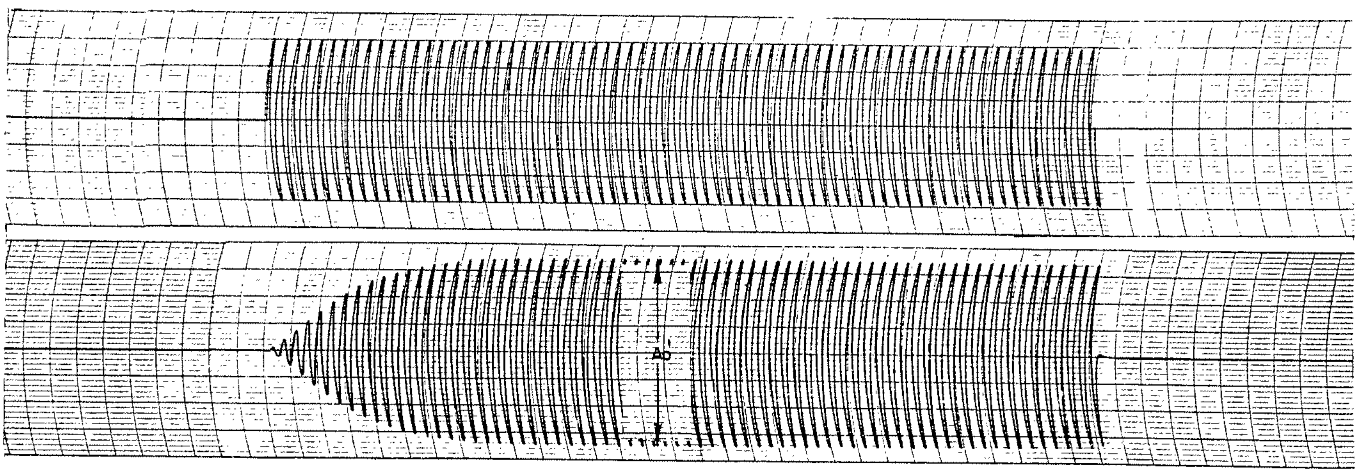


(a) INPUT AND OUTPUT PULSES FOR A TWO CIRCUIT FILTER
$d = 20$, $bd = 2\pi$ and $c = 0$



(b) STEADY STATE RESPONSE TAPE

FIG. E.3
TYPICAL RESULTS OF A RUN AS OBSERVED AT THE DIFFERENTIAL ANALYZER OUTPUT

A is the peak amplitude of the output pulse relative to the steady-state output amplitude of the filter when a sinusoidal signal is applied having the frequency a. The steady-state response is a function of the filter bandwidth, and therefore three steady-state runs have to be made for the series indicated in Table E.1. The differential analyzer output for one such steady-state run is shown in Fig. E.3b. The steady-state amplitude A_o' is determined from this tape, and then the normalized amplitude is given by

$$A = \frac{A'}{A_o'} \quad . \tag{E.12}$$

The output pulse width W is the width of the output pulse in frequency relative to the bandwidth of the filter b. Since the frequency of the input pulse depends linearly on time as given by Eq 4.3, the width of the output pulse in frequency is st_o, and

$$W = \frac{st_o}{b} \; . \tag{E.13}$$

The determination of the apparent bandwidth B of the filter circuit is a somewhat more involved process. Two steps are required: (1) Plots of the normalized amplitude A versus $\frac{sc}{b}$ are made, (sc is the difference between the center frequency of the input pulse and the center frequency of the filter). The data taken from the runs indicated in Table E.1 yield the three plots shown in Fig. E.4. The normalized response is a maximum when the pulse outer frequency equals the filter frequency and drops off at higher and lower frequencies. The apparent bandwidth B can be measured directly from the curves of Fig. E.4; it is the width of these curves between the points at which response is 70.7 per

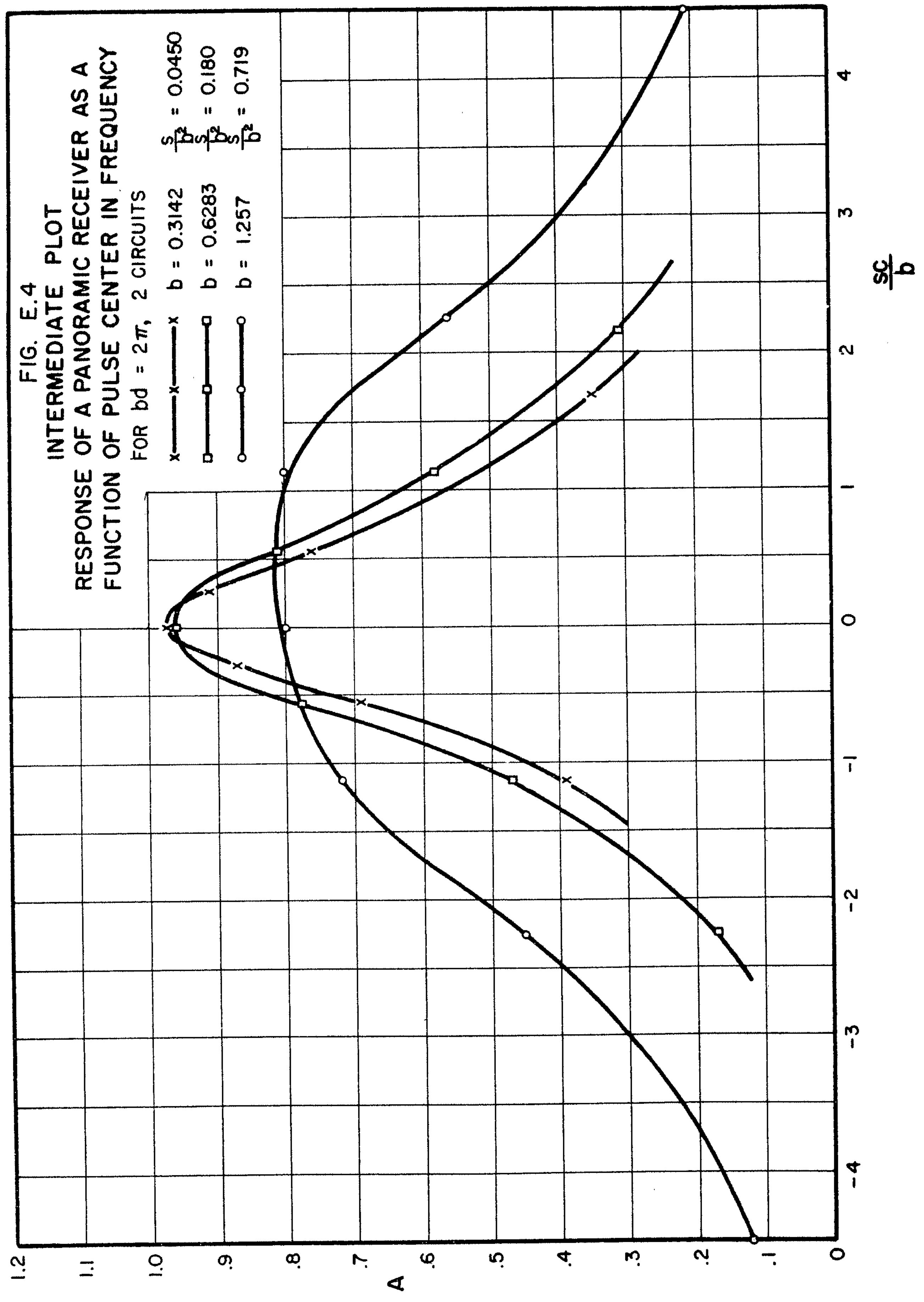


FIG. E.4
INTERMEDIATE PLOT
RESPONSE OF A PANORAMIC RECEIVER AS A FUNCTION OF PULSE CENTER IN FREQUENCY

cent of the response at sc = 0. The three values of B measured from the intermediate plots of Fig. E.4 are plotted against the normalized sweep-rate $\frac{s}{b^2}$ in Fig. E.5.

E.3 Discussion of Errors

E.3.1 Machine Errors The circuit resistor and condenser values are correct to within ± 0.1% of their nominal values. Trimmer resistances were used to provide fine adjustment of the resonant frequencies of the individual circuit loops. These frequencies are adjusted to be within ± 0.1% of one another and of the input signal frequency, when taking steady-state tapes.

The potentiometers which were used to control bandwidth and amplitude are 10 revolution Helipots marked with 1000 divisions. These potentiometer settings are correct within one division. The errors which have the greatest effect on the data taken were those which occurred in making bandwidth settings, particularly at the smaller values of b. The values of b_o are accurate to within ± .01 radians per second; thus the percentage error ranges from ± .01% for b_o = 1.257 radians per second to ± 15% for b = 0.07854 radians per second.

The pulse length and the center of pulses in time are correct within ± 0.3 second, except for the centered pulse group of runs, in which the tolerance was ± 1 second.

E.3.2 Data Processing Errors Readings from the tapes could be made within one-fourth division out of twenty. This is ± 1.25% at full scale, and within ± 5% at one-fourth of the full deflection. Readings at less than one-fourth full deflection were usually avoided.

Some idea of the overall reproducibility of the data may be gained by comparing plots from two sets of data for the same parameters, taken at different times, as shown in Fig. E.6.

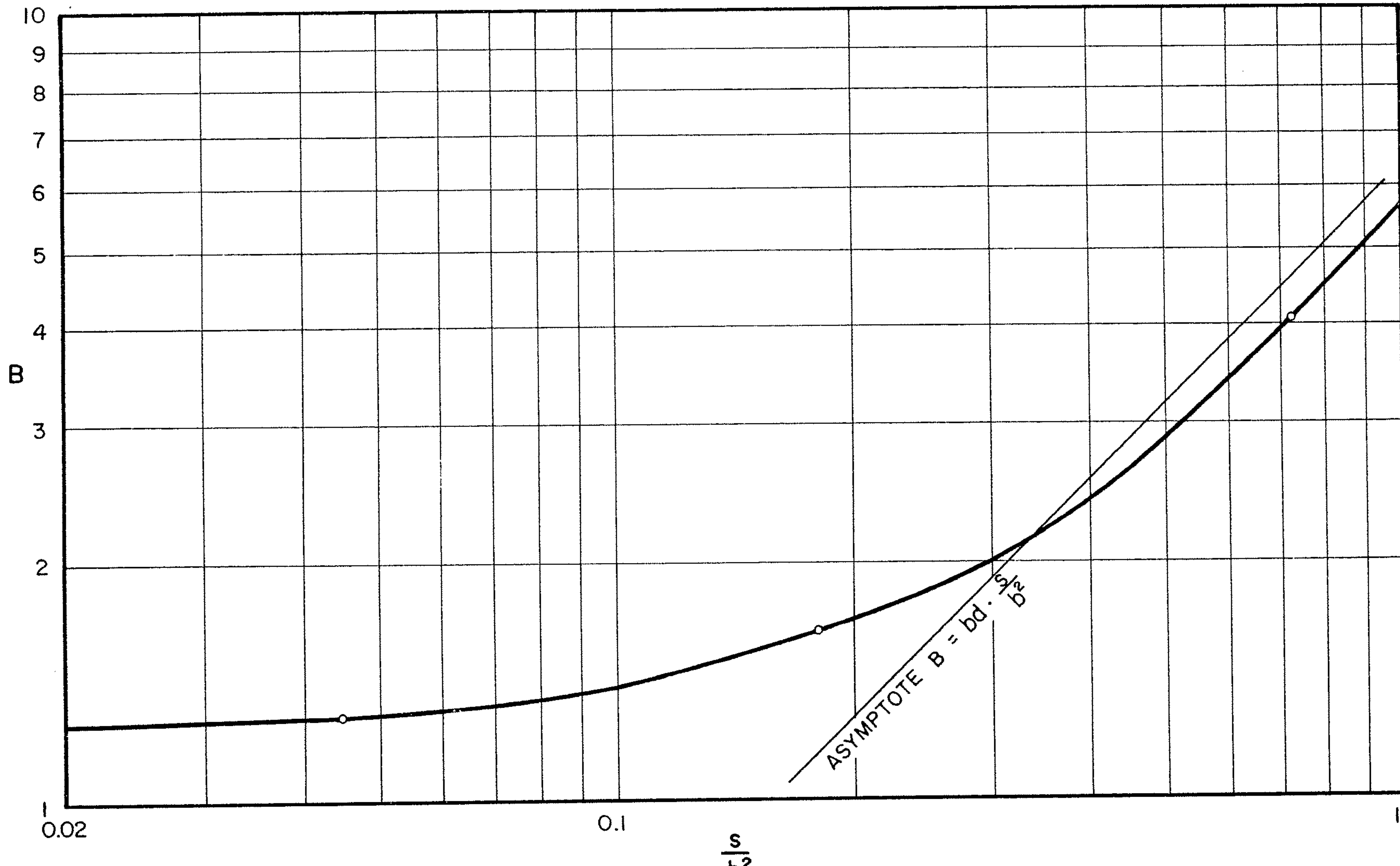


FIG. E.5

THE APPARENT BANDWIDTH OF A TWO CIRCUIT FILTER AS A FUNCTION OF SWEEP-RATE

(DATA FROM DIFFERENTIAL ANALYZER) $bd = 2\pi$

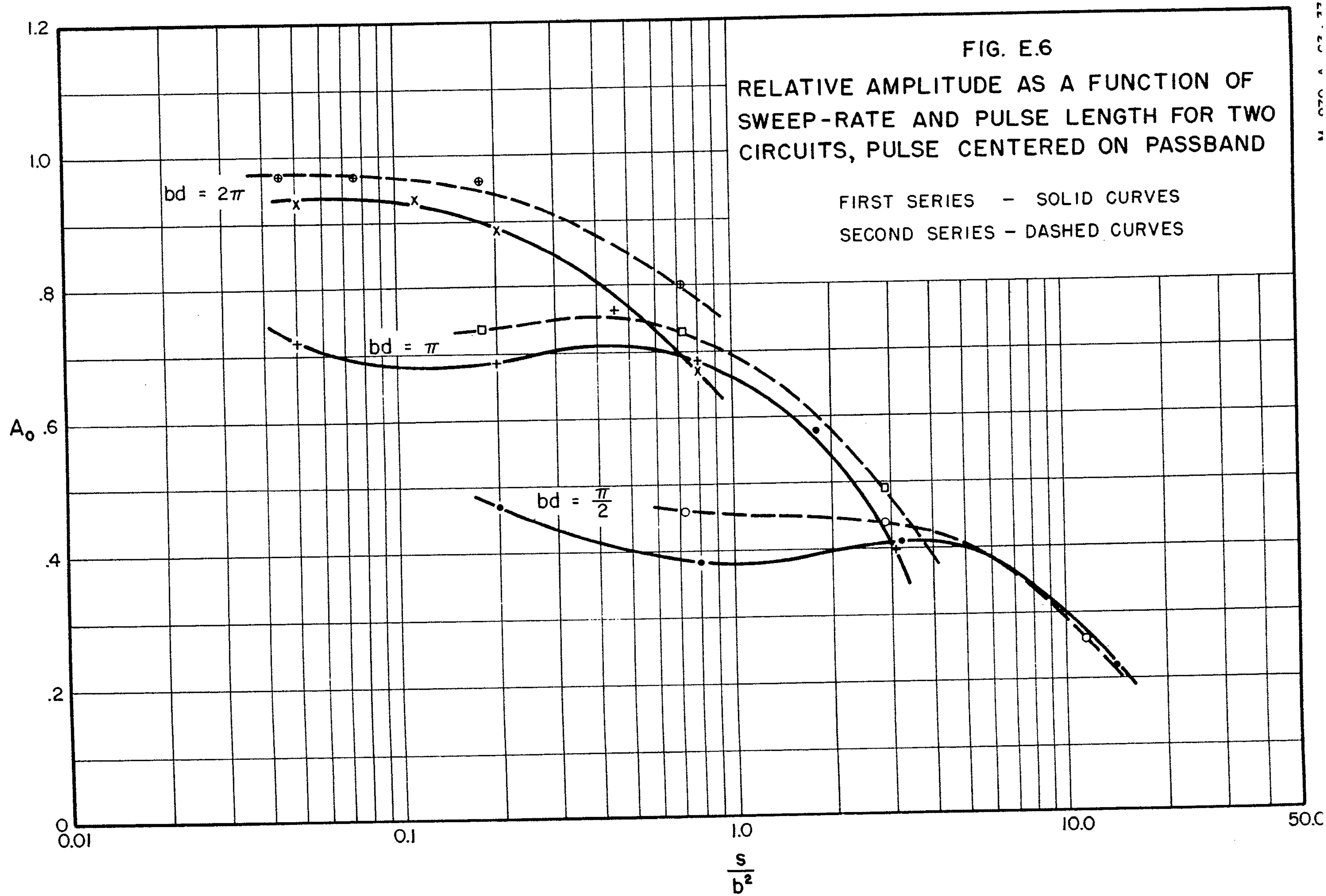


FIG. E.6

RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE AND PULSE LENGTH FOR TWO CIRCUITS, PULSE CENTERED ON PASSBAND

FIRST SERIES – SOLID CURVES

SECOND SERIES – DASHED CURVES

APPENDIX F

Differential Analyzer Data

F.1 Examples of Differential Analyzer Solutions

Section 4.3 includes a discussion of several typical solutions taken from the differential analyzer. Additional runs showing the form of the responses and certain anomalous effects are included in this appendix.

Figure F.1 shows how the response to pulses changes with the pulse position for one, two, and four circuits. The middle pulse is centered on the filter passband and the other two pulses are each about midway between the center of the passband and the 3 db point. The tendency of the envelope of the response to approach a Gaussian shape with an increase in the number of circuits is evident in this figure.

Figure F.2 gives the response of one circuit with various bandwidths to a cw signal. The output pulse width is clearly a minimum for some intermediate bandwidth. The undulatory nature of the response that predominates in single circuit filters and narrow passbands is evident here. Figure F.3 shows a similar set of curves for the four circuit case. These responses are delayed in time and depressed in amplitude by comparison with the single circuit solutions of the same bandwidth. The output pulse width is also seen to be longer for the greater number of circuits, and the undulatory character has been suppressed.

Figure F.4 shows an interesting anomalous effect that can be attributed to the beat phenomena. Figure F.4a indicates that a change of 25% in the input pulse length can cause the output pulse width to triple. Figure F.4b shows a similar situation for the two circuit case. The solutions in Fig. F.5

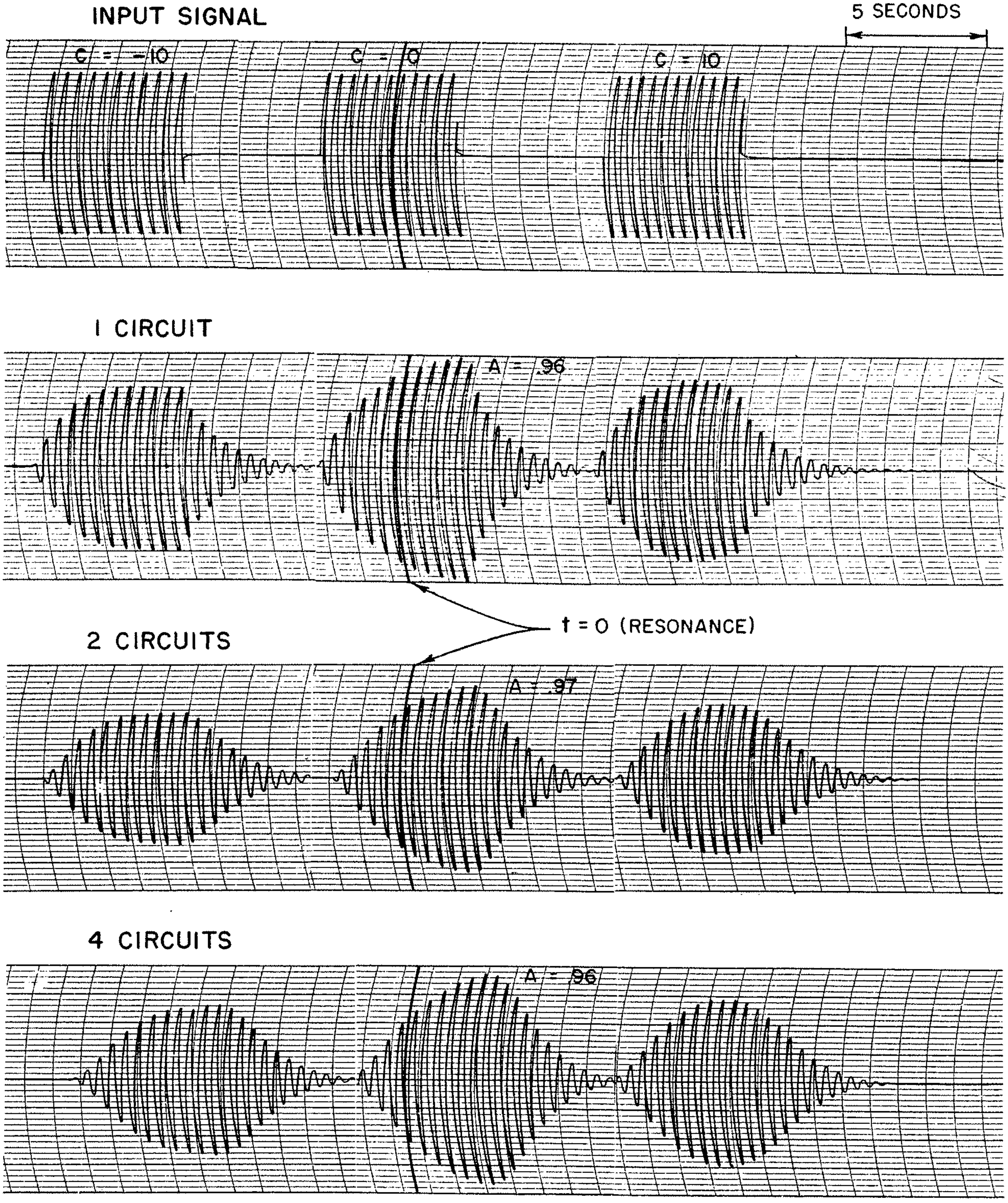
FIG. F.1

RESPONSES FOR VARIOUS NUMBERS OF CIRCUITS & INPUT PULSE POSITIONS

b = 1.26

d = 5

s = .0710



M-970 C-P3-10 5-7-52

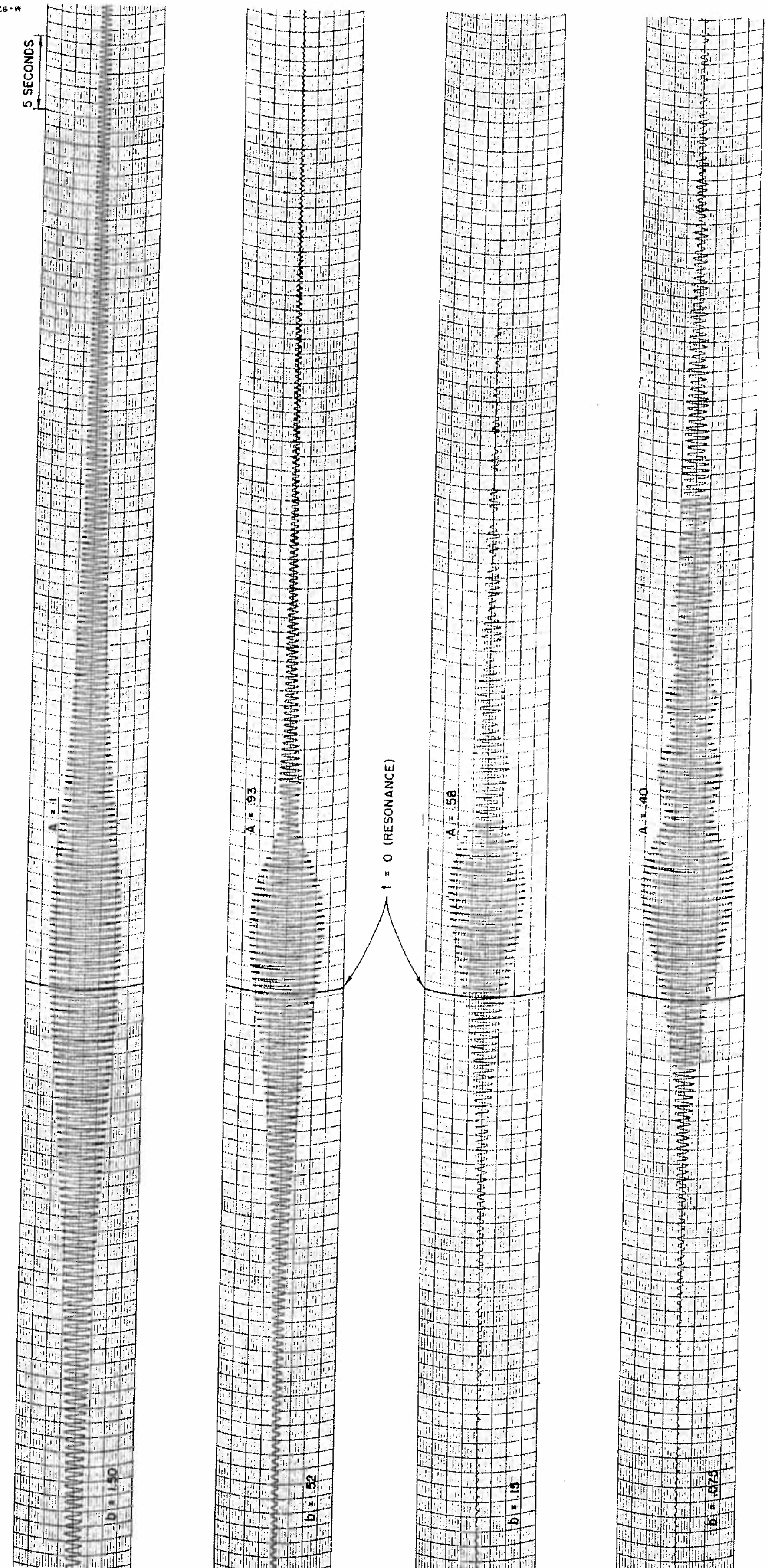
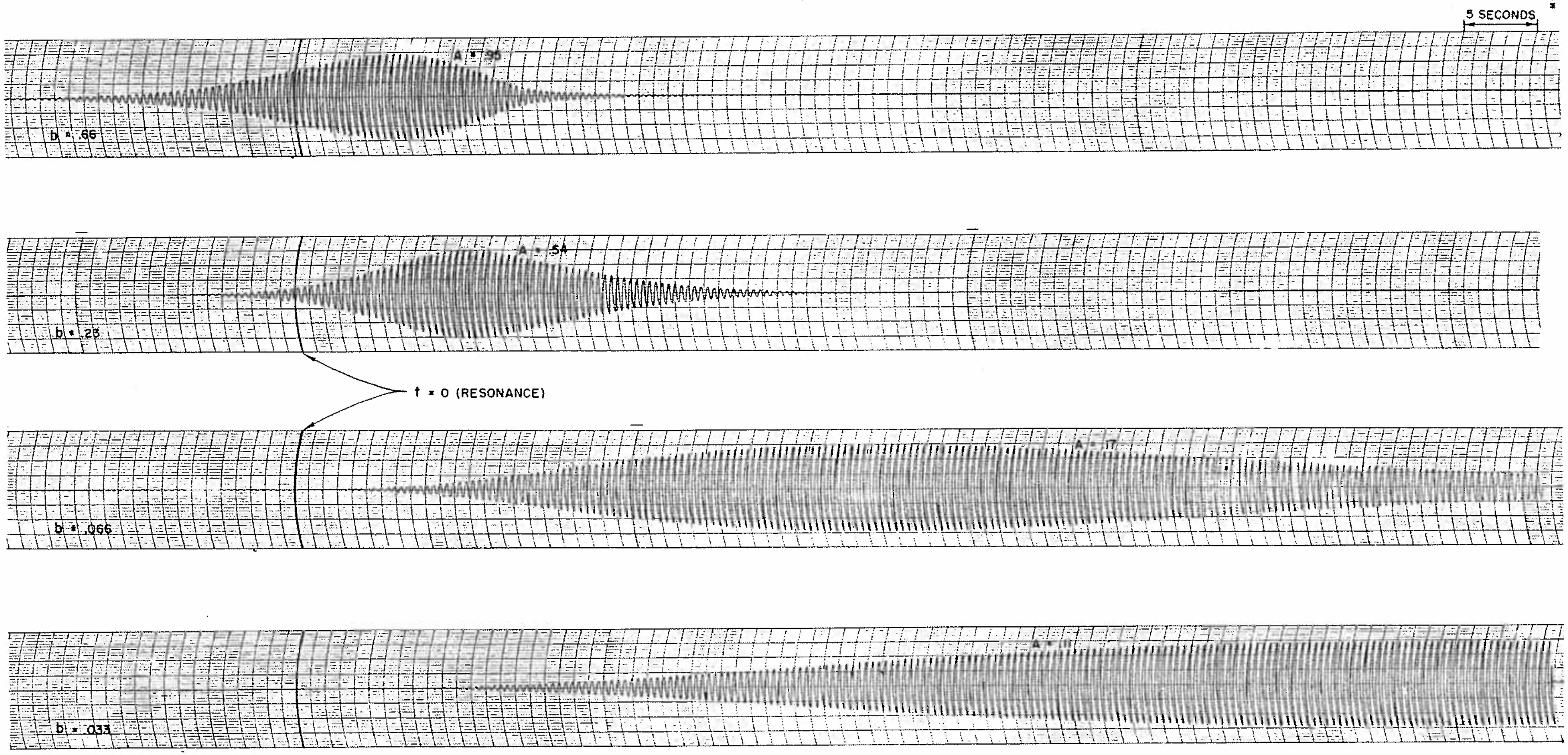


FIG. F.2
RESPONSES FOR VARIOUS BANDWIDTHS
I CIRCUIT
$d = \infty$
$s = .0710$

FIG. F.3
RESPONSES FOR VARIOUS BANDWIDTHS
4 CIRCUITS
$d = \infty$
$s = .0710$



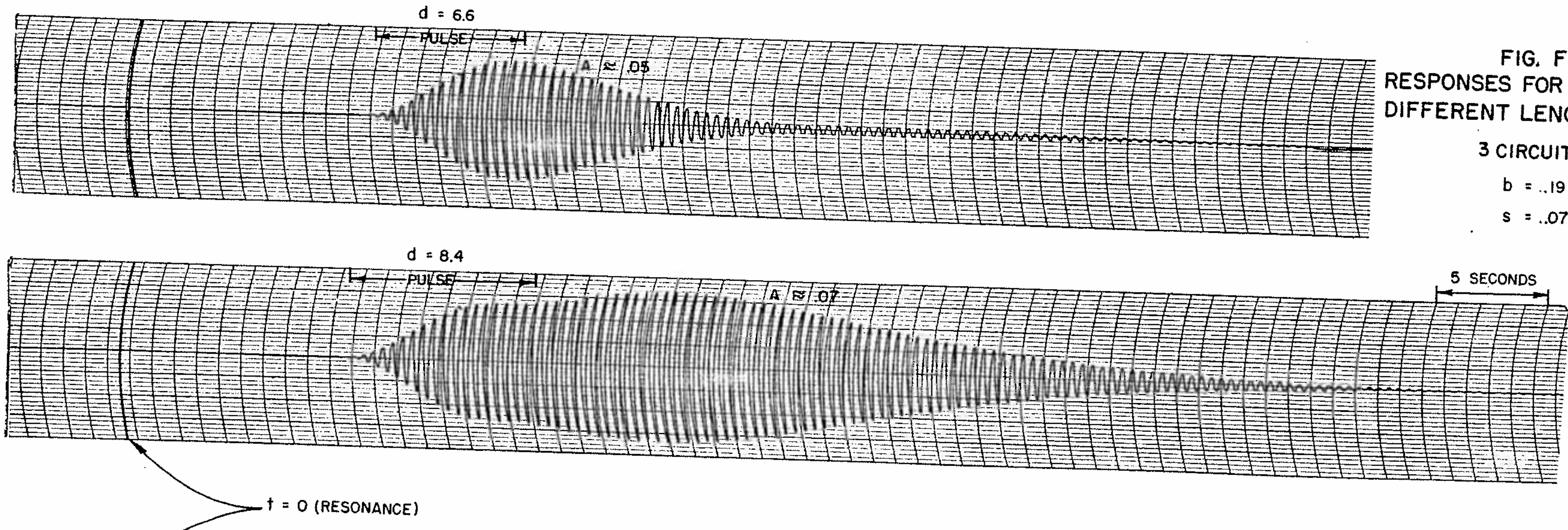


FIG. F.4a
RESPONSES FOR SLIGHTLY DIFFERENT LENGTH PULSES

3 CIRCUITS

b = ..19

s = ..0710

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M-970 B-P3-6

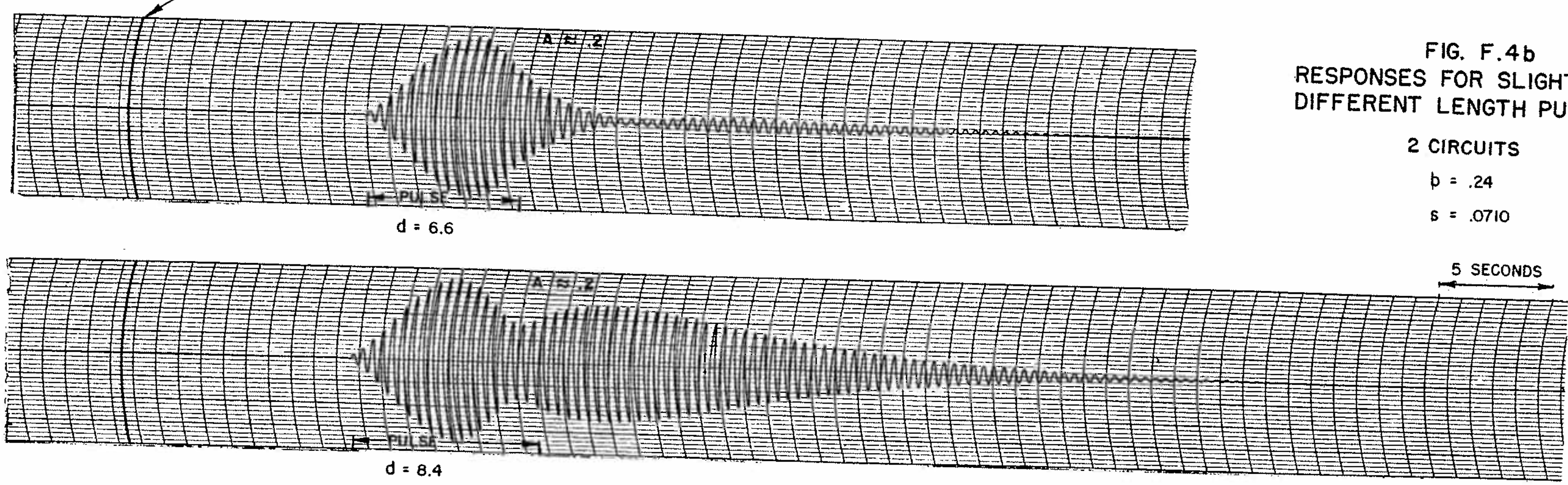


FIG. F.4b
RESPONSES FOR SLIGHTLY DIFFERENT LENGTH PULSES

2 CIRCUITS

b = .24

s = .0710

are also included because they show an anomalous effect. Note that the beat phenomenon is much more pronounced in the two circuit solution that in the single circuit case. By the fourth circuit the ripple has disappeared and the envelope approximates the Gaussian shape.

F.2 Curves for the Factors A_o, B, and W

Most of the solutions compiled from the differential analyzer are summarized in this section. The objective is to present fairly complete data on the relative amplitude, effective bandwidth, and output pulse width as a function of the number of circuits, pulse width, bandwidth and the sweep-rate. Table F.1 summarizes the curves of A_o, B, and W obtained from the differential analyzer solutions. Since examples of each type of curve have been discussed previously, the compiled results are included without comment.

[1]Table F.1 Location of Curves of A_o, B, and W for Differential Analyzer Solutions

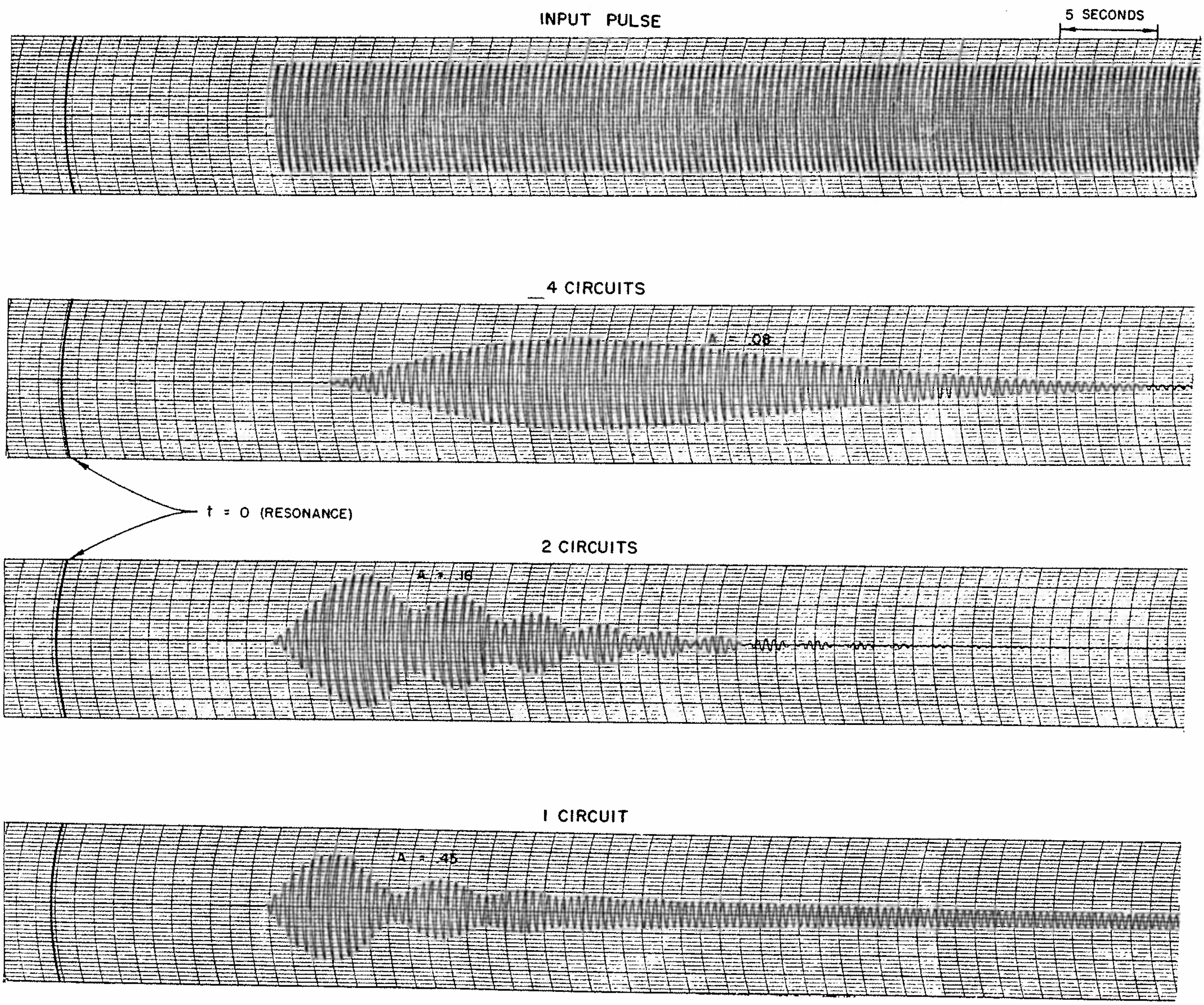
Type of Filter	1, 2, and 4 Circuits						2 Circuits
bd	1	2	$\frac{\pi}{2}$	π	2π	∞	$\frac{\pi}{2}$, π, 2π, and ∞
A_o	F.6	F.7	F.8	F.9	F.10	4.10	4.11
W	F.11	F.12	4.13	F.13	F.14	F.15	4.12
B			4.15				

[1]Numbers in table are figure numbers.

M-970 B-P3-11 RKL 5-7-52

FIG. F.5
RESPONSES FOR DIFFERENT NUMBERS OF CIRCUITS

b = .38
d = ∞
s = .0710



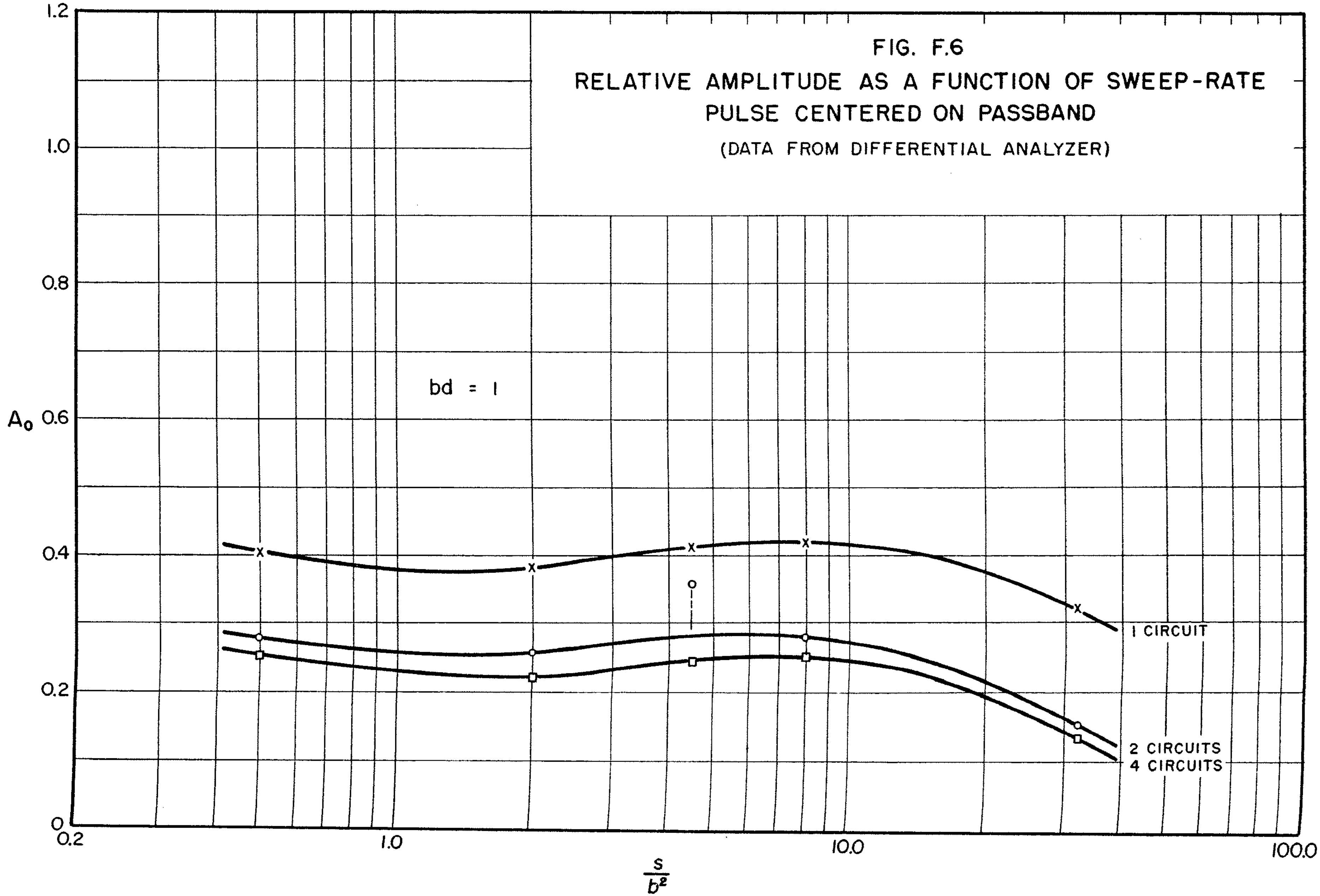


FIG. F.6
RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE
PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)

M-970 A-63-40 RKL 6-11-

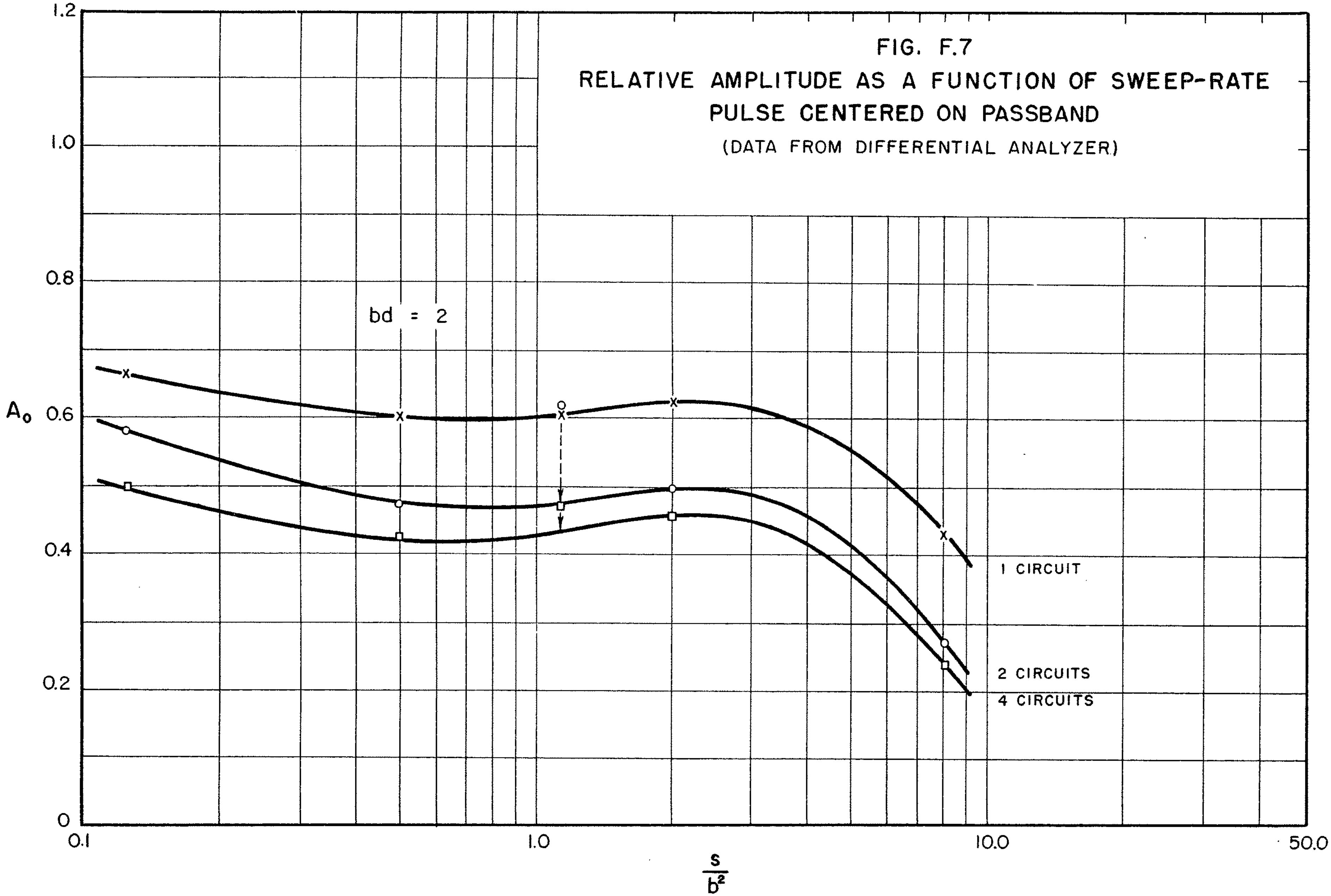


FIG. F.7
RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)

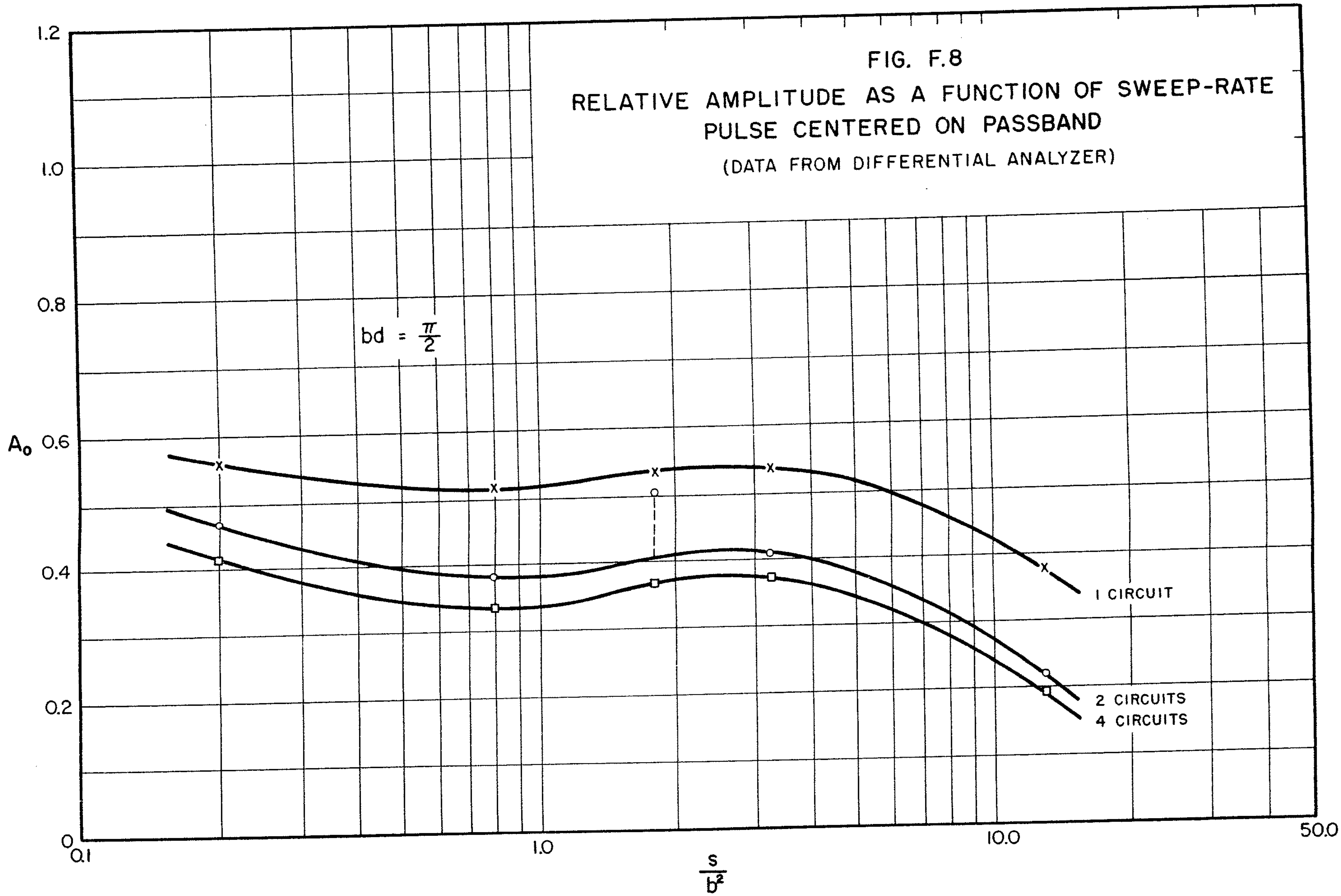


FIG. F.8
RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE
PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)
1.2
1.0
0.8
A0 0.6
0.4
0.2
0
0.1
1.0
10.0
50.0
s/b²
bd = π/2
1 CIRCUIT
2 CIRCUITS
4 CIRCUITS

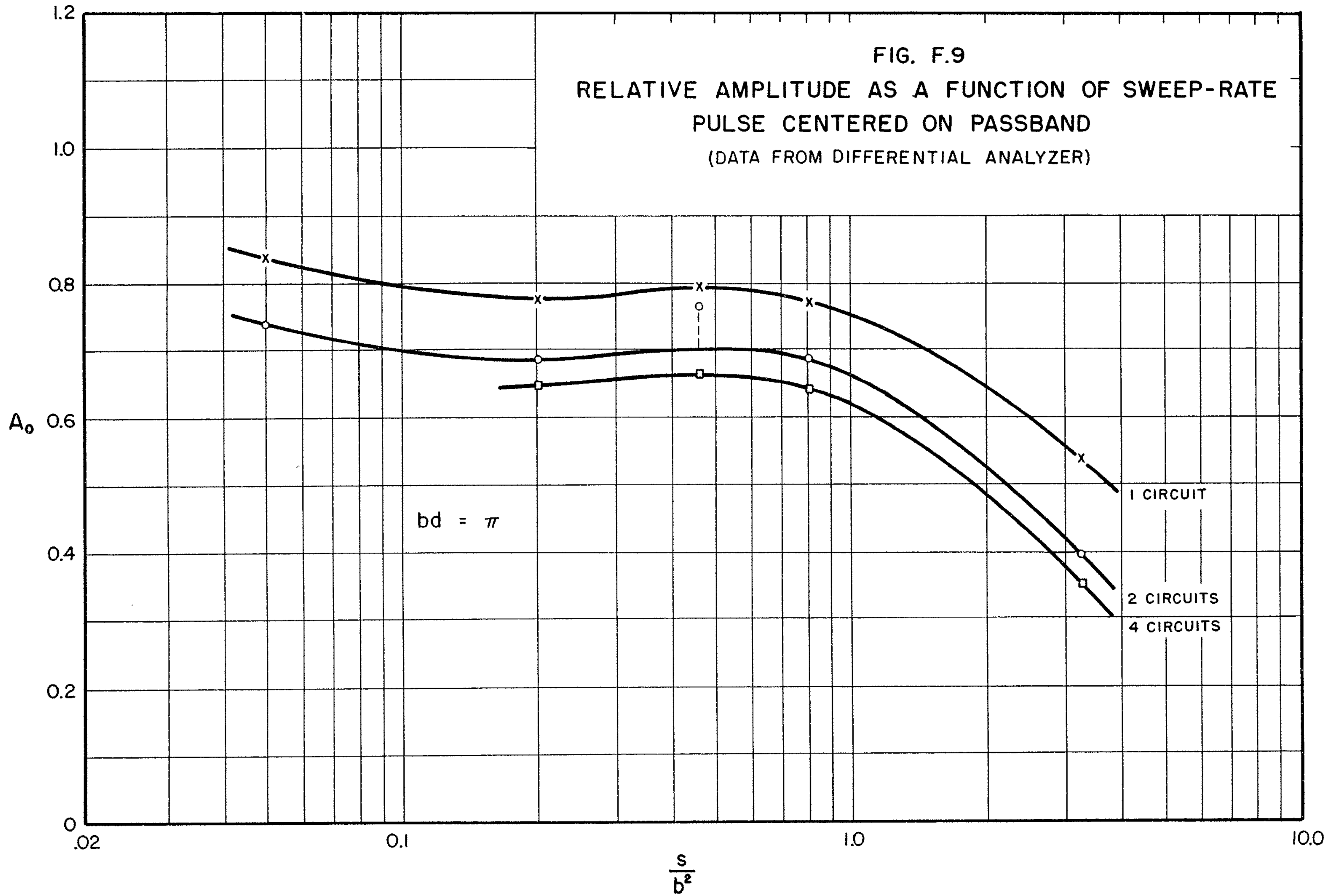


FIG. F.9
RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE
PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)
1.2
1.0
0.8
0.6
0.4
0.2
0
A_o
bd = π
1 CIRCUIT
2 CIRCUITS
4 CIRCUITS
.02
0.1
1.0
10.0
$\frac{s}{b^2}$

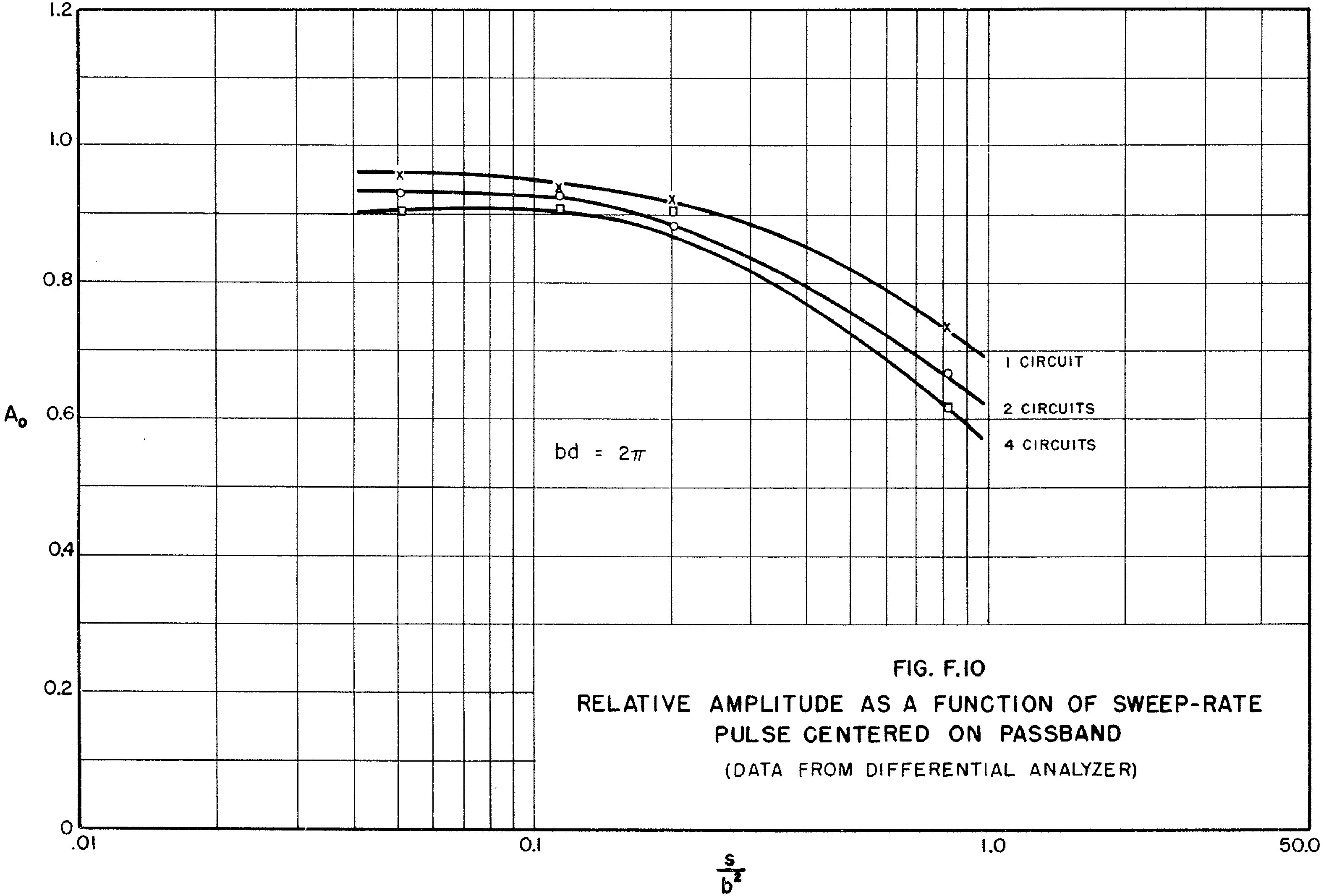


FIG. F.10

RELATIVE AMPLITUDE AS A FUNCTION OF SWEEP-RATE PULSE CENTERED ON PASSBAND

(DATA FROM DIFFERENTIAL ANALYZER)

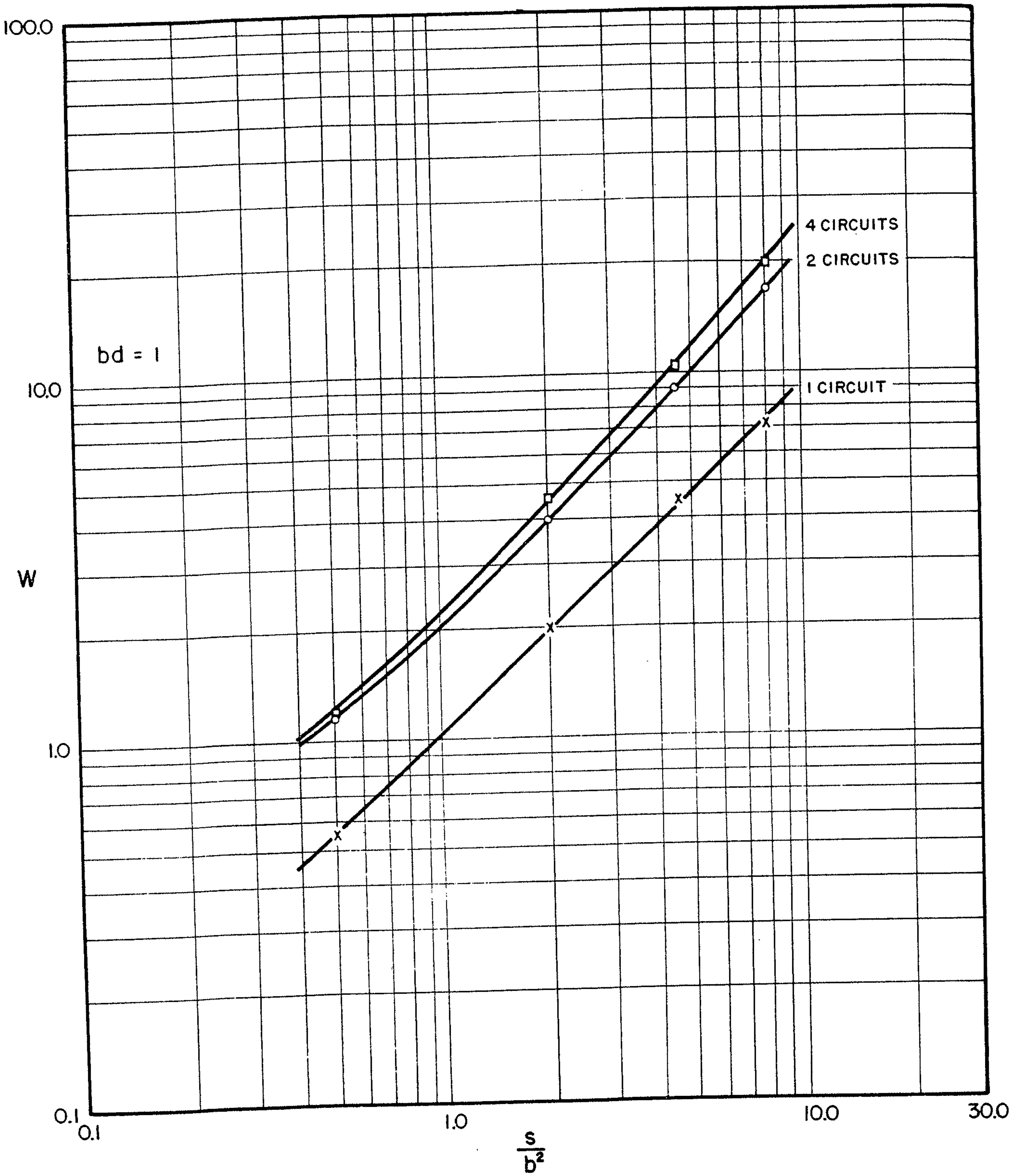


FIG. F.11

OUTPUT PULSE WIDTH AS A FUNCTION OF SWEEP-RATE
PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)

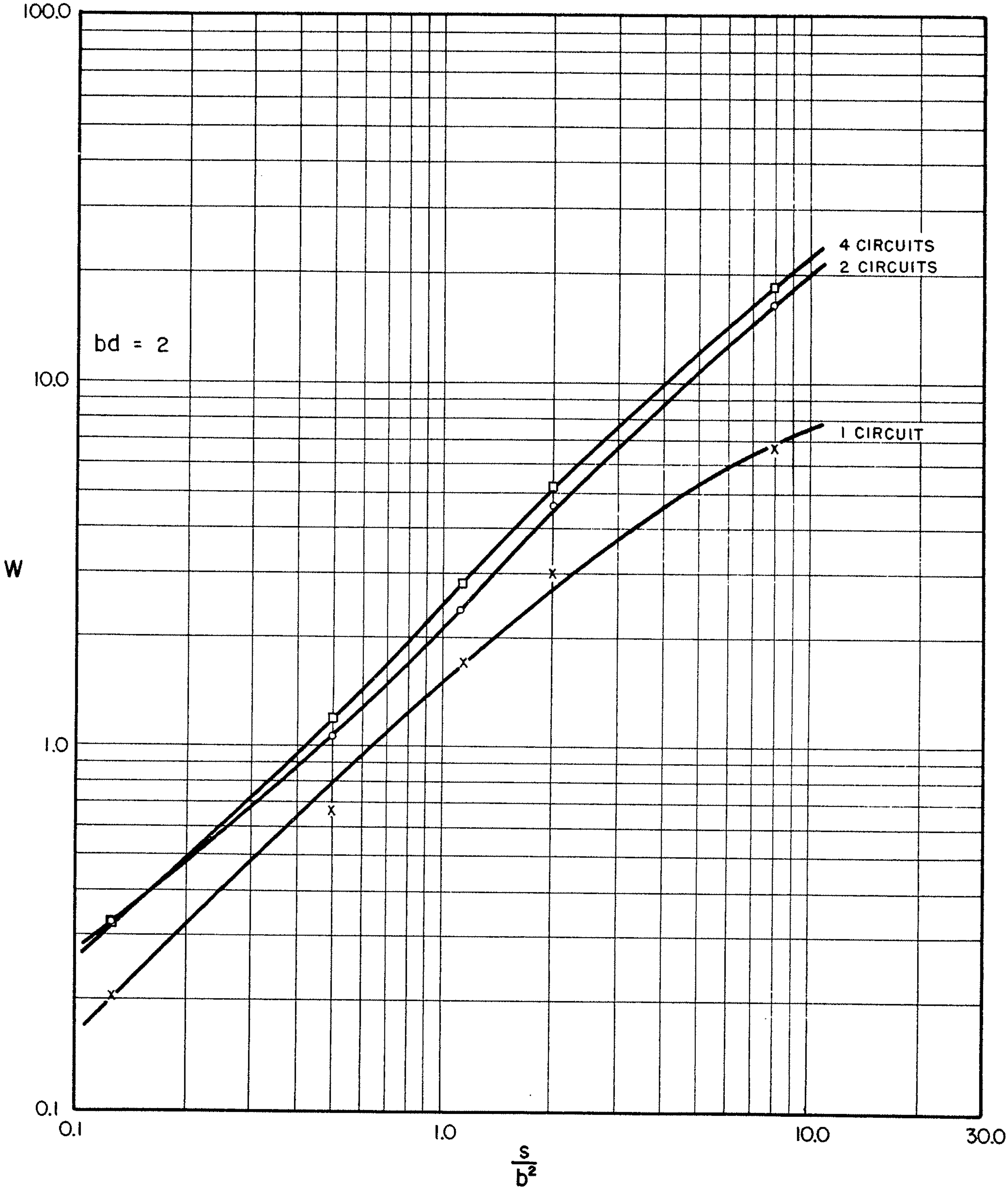


FIG. F.12
OUTPUT PULSE WIDTH AS A FUNCTION OF SWEEP-RATE
PULSE CENTERED ON PASSBAND
(DATA FROM DIFFERENTIAL ANALYZER)

M-970 A-63-37 RKL 6-10-52

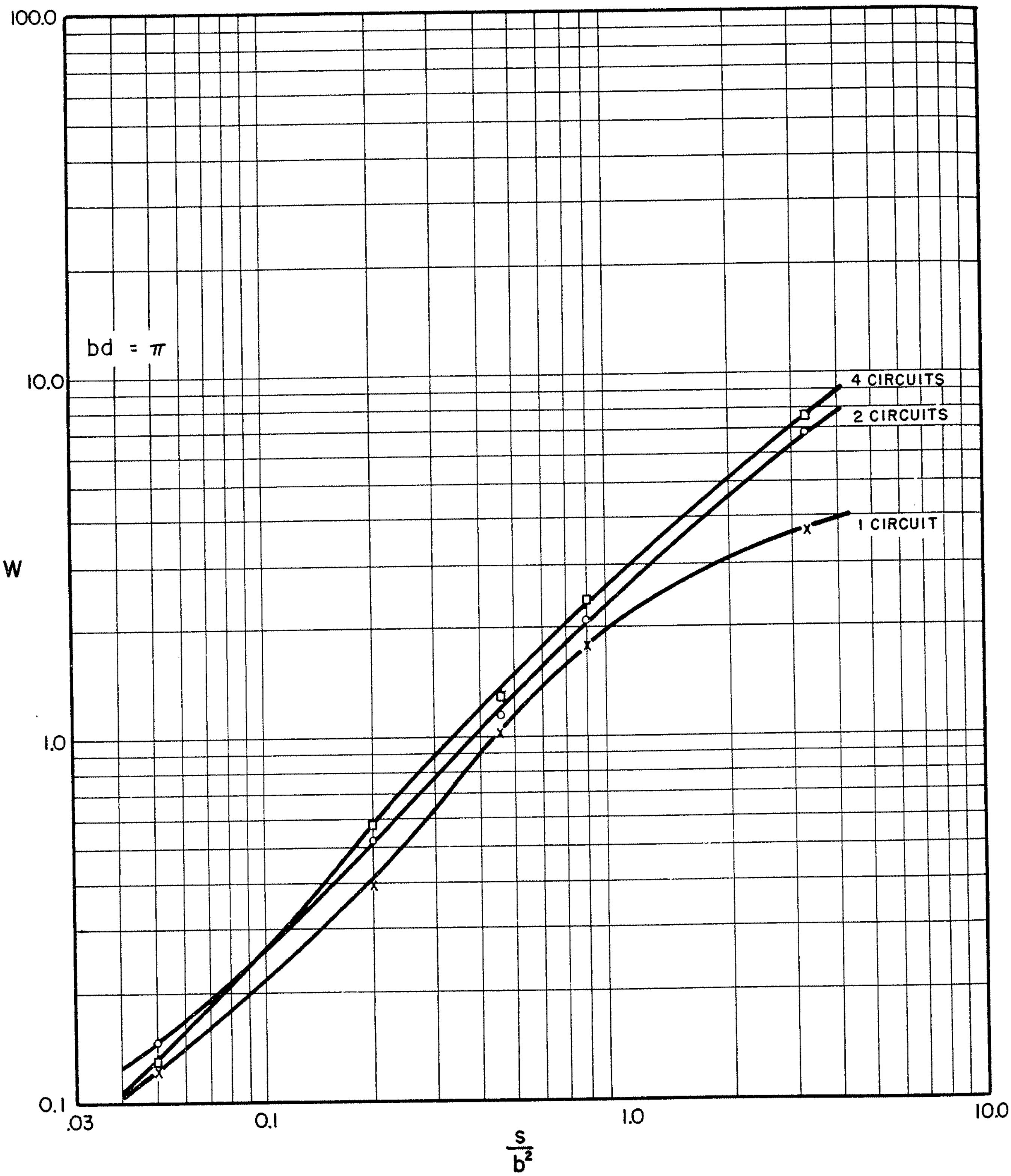


FIG. F.13

OUTPUT PULSE WIDTH AS A FUNCTION OF SWEEP-RATE

PULSE CENTERED ON PASSBAND

(DATA FROM DIFFERENTIAL ANALYZER)

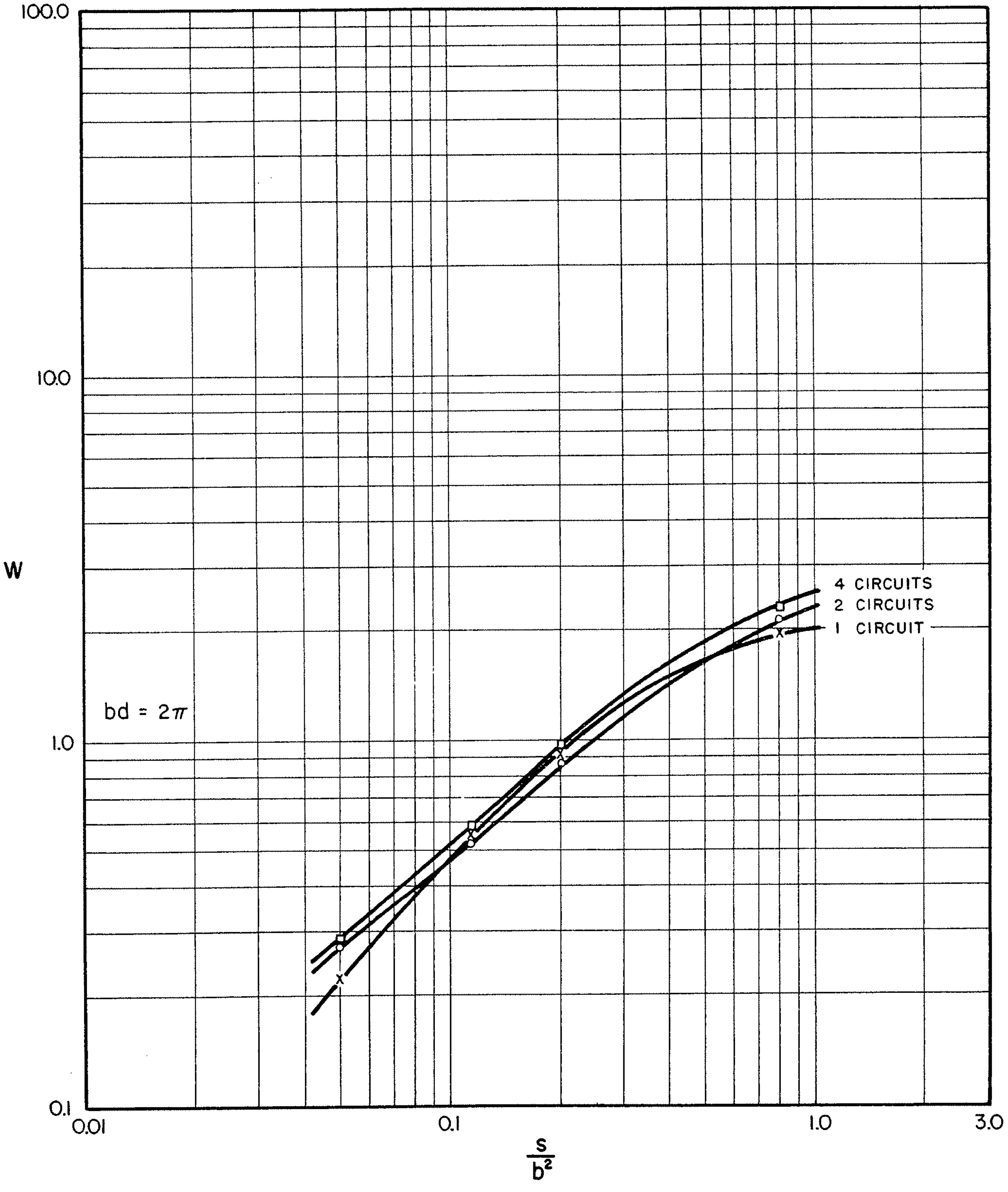


FIG. F.14

OUTPUT PULSE WIDTH AS A FUNCTION OF SWEEP-RATE PULSE CENTERED ON PASSBAND

(DATA FROM DIFFERENTIAL ANALYZER)

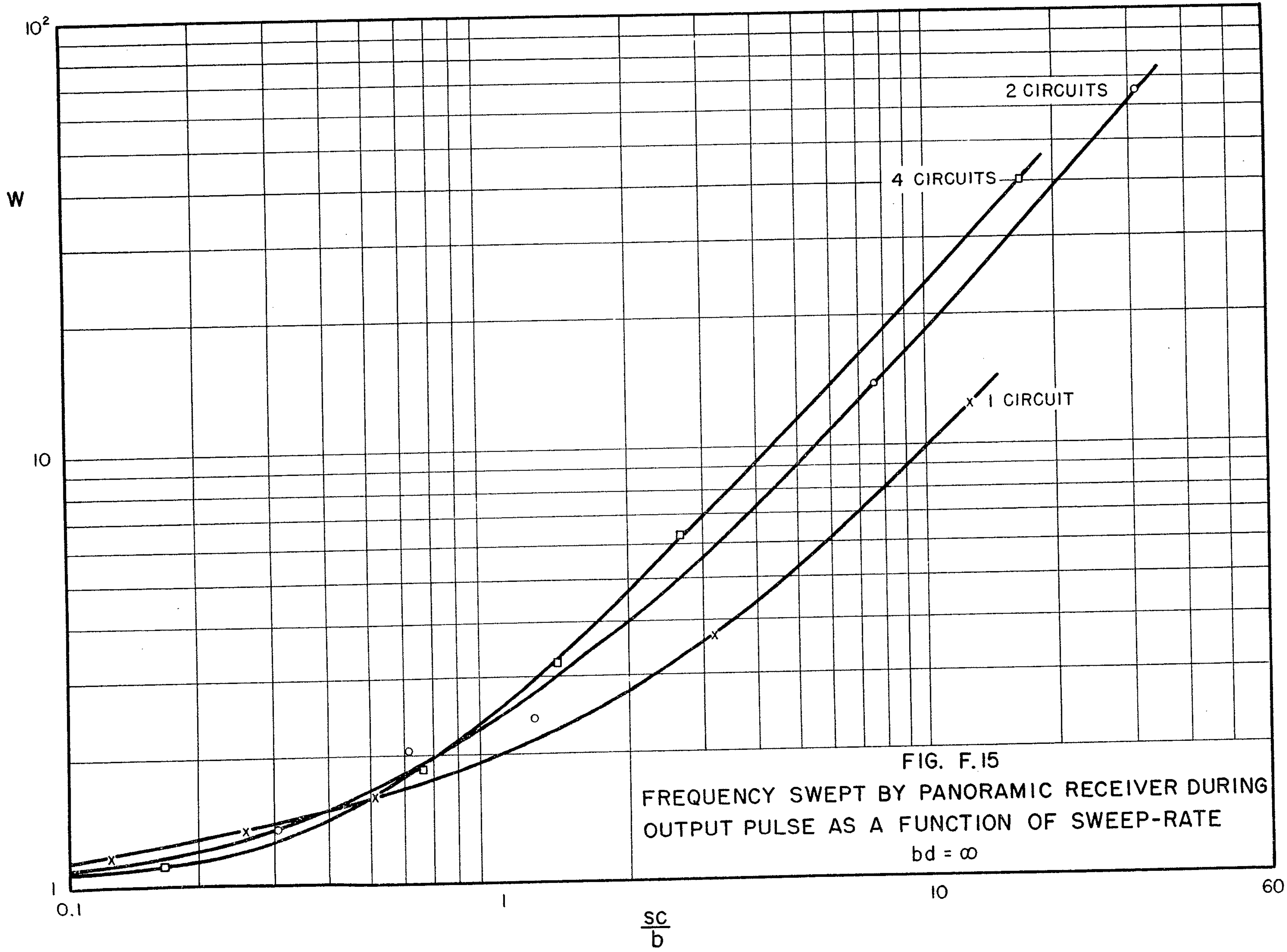


FIG. F.15

FREQUENCY SWEPT BY PANORAMIC RECEIVER DURING OUTPUT PULSE AS A FUNCTION OF SWEEP-RATE

$bd = \infty$

M-970 A-G3-6 RKL 5-9

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It is pointed out in Appendix A that the derivation in that section is essentially contained in Hok's paper. In addition, Hok gives a general form for the response relation when the sweep is not necessarily linear, derives an apparent bandwidth, and presents suitable curves so that the output phase as well as amplitude for the single resonant circuit can be calculated in the case of cw input and a linear sweep. Many interesting curves are included.

2. Lewis, F.M., "Vibration During Acceleration Through a Critical Speed," Trans. Am. Soc. Mech. Eng., APM-54-24, pp. 253-261; 1932.

This paper presents a solution of the fundamentally similar mechanical problem of "running a system having a single degree of freedom and linear damping through its critical speed from rest at a uniform acceleration."

3. Barber, N.F. and Ursell, F., "The Response of a Resonant System to a Gliding Tone," London Philosophical Magazine, Vol. 39, pp. 345-361; May 1948.

This study is similar to the work of Hok and that of Lewis. The same integral is derived for the response and discussed in some detail. The discussion of the case of varying the natural period of the system rather than the frequency of the signal is carried farther in this paper. Interesting diagrams are presented and good experimental agreement is shown.

4. Williams, E.M., "Radio-Frequency Spectrum Analyzers," Proc. I.R.E., Vol. 34, pp. 18-22; January 1946.

This paper is an experimental study of resolution and the bandwidth corresponding to optimum resolution. The author concludes that the best resolution defined as the frequency between 3 db points of the output is $1.3\sqrt{s/2\pi}$ and that this resolution is achieved for a bandwidth of $\sqrt{s/4\pi}$ cps.

5. Barlow, H.M. and Cullen, A.L., Microwave Measurements, pp. 320-332; 1950.

In the section on microwave spectrum analysis, the authors discuss the response of a fixed resonant system to a signal of varying frequency as it applies to microwave spectrum analyzers. For resolution defined as the frequency between 3 db points of the output, the optimum Q is given as

$$Q_{opt} = \frac{a}{\sqrt{2\pi s}}, \text{ i.e., } b = \sqrt{2\pi s}$$

6. Montgomery, C.G., Editor, Technique of Microwave Measurements, Vol. 11, MIT Radiation Laboratory Series, pp. 408-455; 1947.

Chapter 7, "The Measurement of Frequency Spectrum and Pulse Shape," discusses the principles and design of spectrum analyzers, giving descriptions and circuit diagrams of representative spectrum analyzers. The author also discusses the response of a Gaussian filter to signals of constant amplitude and linearly varying frequency in connection with the generation of pulses (Section 4.6).

7. Marique, J., "Response of a Circuit to a Linear-Frequency-Sweep Voltage," Onde Elect., Vol. 31, pp. 313-315; July 1951. (in French)

The universal response curves of Hok are transformed into a set of curves which can be applied more readily in practice. These are based on circuit bandwidth corresponding to a 3 db fall in response, and show response as a function of sweep-rate. As this increases the current maximum decreases and occurs later, while the passband (3 db below the maximum) increases.

8. Hatton, W.L., "Simplified FM Transient Response," Technical Report No. 196, MIT; April 1951.

This paper derives the response of a single tuned circuit when the input current makes a sudden jump in frequency. The paper examines the instantaneous output frequency, the rise time, and the maximum overshoot.

There are also earlier papers by D.A. Bell, H. Salinger, and C.C. Eaglesfield which treat the special problem of transient response with a sudden change in frequency.

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11. Moulic, W.E., "Panoramic Principles," Electronic Industries, Vol. III, No. 7, pp. 86-88, 206; July 1944.

A description is given of the basic components in the panoramic system and applications to industrial and other problems are suggested.

12. Thomasson, D.W., "Panoramic Display - Design Considerations," Electronic Engineering, Vol. 21, No. 257, pp. 259-261; July 1949.

Basic design principles of panoramic display are presented. General limitations are pointed out and several applications are suggested.

13. Bush, V. and Caldwell, S.H., "A New Type of Differential Analyzer," Journal of the Franklin Inst., Vol. 240, pp. 255-325; October 1945.

14. Ragazzini, J.R., Randall, R.H., and Russell, F.A., "Analysis of Problems in Dynamics by Electronic Circuits," Proc. I.R.E., Vol. 35, pp. 444-452; May 1947.

15. Macnee, A.B., "An Electronic Differential Analyzer," Proc. I.R.E., Vol. 37, pp. 1315-1324; November 1949.

16. Salinger, H., "On the Theory of Frequency Analysis by Means of a Searching Tone," Elektrische Nachrichten Technik, Vol. 6, pp. 293-302; August 1929 (in German).

This paper describes an automatic spectrum analyzer; the response of low-pass filter to signal sweeping linearly in time is presented. The author considers an "ideal" filter with a rectangular amplitude response curve and a linear phase response curve. He concludes that sweeping will not seriously affect the spectrum analyzer output provided

$$\frac{b}{\sqrt{s}} \geqq \sqrt{128\pi} .$$

[1]LIST OF SYMBOLS

a, the instantaneous frequency in radians per second of the filter input signal at time, $t = 0$.

a, the center frequency of the filter in radians per second.

A(relative amplitude), the peak response to a pulse per unit of centered cw signal with the same input amplitude (see Section 2.3).

A_o(relative amplitude), the value of A for centered pulses (see Section 2.3).

b, the bandwidth of the filter in radians per second.

$B = \frac{s(c_2 - c_1)}{b}$ (effective bandwidth), the range of frequencies of the input pulse per unit bandwidth such that the response is at least 0.707 of the maximum (see Section 2.3).

c, the center of the input pulse in time (seconds).

c_1, c_2, the centers in time of the input pulse such that the peak response is 0.707 of its maximum value.

d, the input pulse width in time (seconds).

s, the sweep-rate in radians per second per second.

sc, the distance in radians per second from the filter to the center of the input pulse.

t, the time in seconds.

t_m, the time of maximum response in seconds.

t_o, the width of the output pulse in seconds.

$W = \frac{st_o}{b}$ (output pulse width), the width of the output pulse in frequency per unit of bandwidth (see Section 3.2).

ω, the frequency in radians per second.

[1]Additional symbols of less significance are defined where they are used in the text.

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