Error Exponent Region for Gaussian Multiple Access Channels and Gaussian Broadcast Channels

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1 Introduction

Error exponents for single-user additive white Gaussian noise (AWGN) channels provide bounds on the rate of exponential decay of the average probability of error as a function of the block length of the random codebooks. In this paper, we only consider Gallager-type lower bounds of error exponents [1, 2]. For simplicity, we use the term "error exponent" to mean the maximum of random coding exponent and expurgated exponent throughout this paper, though actually should be called a lower bound of error exponents. The concept of error exponents was extended to Gaussian MAC channels [3, 4], where random coding exponents were derived. Recently, Zheng *et al.* considered error exponents, in high signal-to-noise ratio (SNR), for single-user wireless multi-input-multi-output (MIMO) channels [5], and for wireless MIMO multiple access channels [6][1]. Conceptually, the error exponent is a function of the channel capacity C and the transmission rate R in a single user channel (see Fig. 1(a)). However, error exponents for a multi-user network is quite different from the error exponent for a single-user channel. Each user in a multi-user network is associated with his own error exponent. Therefore, there are multiple error exponents for a given multi-user channel. In contrast to all previous works, however, we make the following observations. Consider the capacity region of a two-user multiple access channel as shown in Fig. 1(b). As expected, the error exponents for the two users are functions of both the transmission rate point A and the channel capacity. However, unlike the case in a single user channel where channel capacity boundary is a single point, in a multi-user channel we have multiple points on the capacity boundary (e.g. A_1,A_2 in Fig. 1(b)). Thus it is expected that one can get different error exponents depending on which particular point on the capacity boundary is considered. Furthermore, it might be possible to trade off error exponents between users by considering different points on the capacity boundary. For instance, consider a rate point A with respect to a boundary point A_1 in Fig. 1(b). It is intuitive to expect that the error exponent for user 1 is smaller than that of user 2. On the other hand, if we consider point A with respect to point A_2, we then expect error exponent for user 1 is larger than that of user 2. Therefore, tradeoff of error exponent between users might be possible by considering different points on the capacity boundary. Similar behavior can be observed in other multi-user channels, e.g. the broadcast channels in Fig. 1(c). It is our intention in this paper to formalize these ideas and show that such tradeoff indeed exists and suggest a constructive strategy to achieve it.

[1]Zheng *et al.* use the term "diversity gain", but it's just another form of the error exponent in high SNR.

rate point, the error exponent region consists all achievable error exponents when the channel is transmitted at that rate point. For example, the error exponent region for a single user channel with transmission rate R is a line segment from the origin to the error exponent $E(R)$ (see Fig. 2(a)). For a multiple access channel operated at rate point A (see Fig. 1(b)), the error exponent region is a two-dimensional region which depends on rates R_1 and R_2 (see Fig. 2(b)). The concept of the error exponent region is very similar to the channel capacity region (CCR). In the EER, it is possible to increase user 1 error exponent by decreasing user 2 error exponent. This is similar to increasing user 1 transmission rate by reducing user 2 transmission rate in the CCR. However, there is a fundamental difference between CCR and EER. For a given channel, there is only one CCR. One the other hand, an EER depends on the channel operating rate point, and for a given channel, there are numerous EERs depending on which operating point we consider. Therefore, when we refer to an EER, we need to specify both the channel and the channel operating rate point.

The rest of the paper is structured as follows. In section 2, we review error exponents for a two-user Gaussian MAC channel and we compare the error exponent region derived by superposition with the the error exponent regions derived by other schemes. We extend this result to Gaussian broadcast channels in section 3, and conclude our work in section 4.

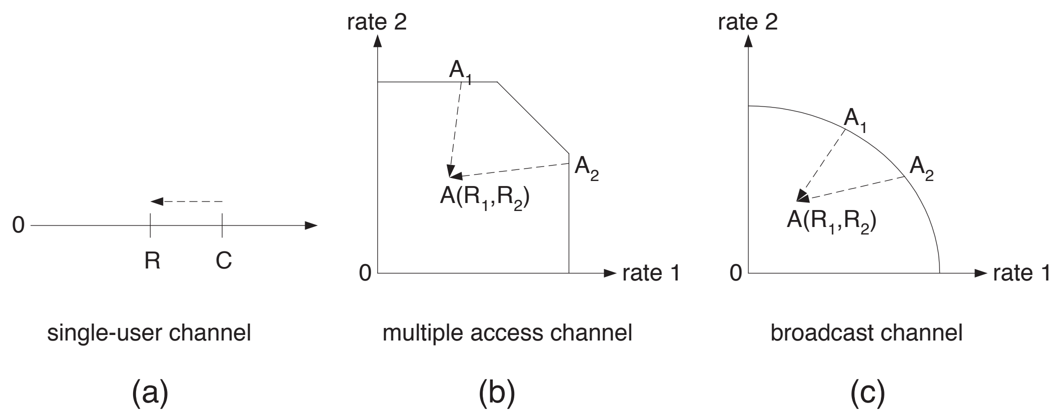


Figure 1: Capacity region for single user , multiple access , and broadcast channels.

2 Error exponent region for Gaussian MAC channels

Error exponent for a single-user AWGN channel is the maximum of random coding exponent and expurgated exponent [1,2]. Consider a two-user Gaussian MAC channel

$$Y = X_1 + X_2 + Z, \tag{1}$$

where X_1 and X_2 are channel inputs for user 1 and user 2 and Z is white Gaussian noise. Gallager derived random coding exponent for this channel using maximum-likelihood joint decoding [3]. There are three types of error using jointly decoding. Type 1 error denotes when user 1 codeword is decoded wrong, but user 2 codeword is decoded correctly. Type 2 error denotes when user 2 codeword is decoded wrong, but user 1 codeword is decoded correctly. Type 3 error denotes when both users' codewords are decoded as wrong codewords. Denote SNR_1 and SNR_2 as the signal-to-noise ratios for user 1 and user2, and $SNR =$

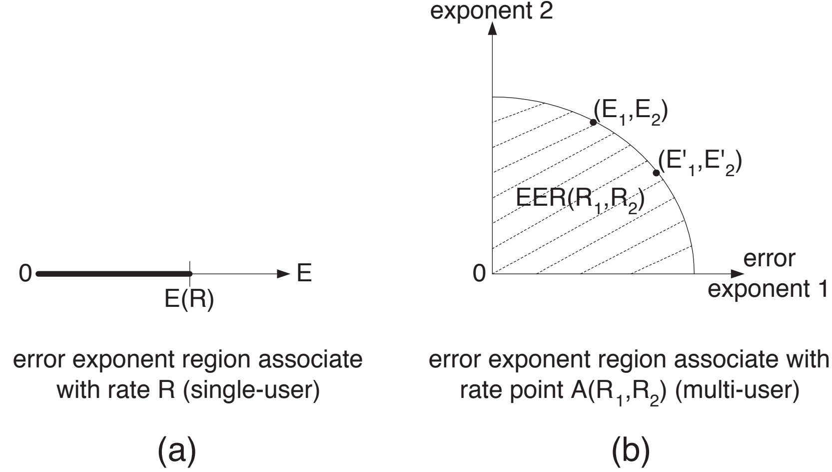


Figure 2: Error exponent region.

$SNR_1 + SNR_2$ (noise power is normalized to 1). We use the notation $E(R, SNR)$ to denote the error exponent for a single user channel at rate R with power constraint SNR. Similarly, we use the notation $E_{t3}(R_1 + R_2, SNR_1, SNR_2)$ to denote the error exponent for type 3 error in a two-user MAC channels[2]. Therefore, the error exponents for user 1 and user 2 using superposition encoding are

$$E_1^s = \min\{E(R_1, SNR_1), E_{t3}(R_1 + R_2, SNR_1, SNR_2)\} \quad (2)$$
$$E_2^s = \min\{E(R_2, SNR_2), E_{t3}(R_1 + R_2, SNR_1, SNR_2)\}. \quad (3)$$

An explicit expression for E_{t3} is given in the appendix. For an important case when $SNR_1 = SNR_2$, there is closed form formula for E_{t3} (hence E_1^s and E_2^s). In general ($SNR_1 \neq SNR_2$), E_{t3} need to be solved numerically.

The capacity region of a two-user Gaussian MAC channel can be divided into four regions R_{12}, R_{13}, R_{23}, and R_3 depending whether type 1 error exponent E_{t1}, type 2 error exponent E_{t2}, or type 3 error exponent E_{t3} dominates (see Fig. 3). In the region R_{12}, $E_{t1} \leq E_{t3}$ and $E_{t2} \leq E_{t3}$. The error exponent for user 1 is $E_1^s = E(R_1, SNR_1)$ and for user 2 is $E_2^s = E(R_2, SNR_2)$. This is the region where the appearance of the second (first) user doesn't affect the error exponent of the first (second) user. In this region, we can not increase error exponents for either of the users. In the region R_{13}, $E_{t1} \leq E_{t3} \leq E_{t3}$. The error exponent for user 1 is $E_1^s = E(R_1, SNR_1)$ and for user 2 is $E_2^s = E_{t3}(R_1 + R_2, SNR_1, SNR_2)$. In this region, although we can not increase first user's error exponent by reducing the second user's error exponent, it is possible to increase the second user's error exponent by reducing the first user error exponent because the dominant error for user 2 is a type 3 error. A similar result also holds for region R_{23} by changing the role of user 1 and user 2 in the R_{13} region. In the region R_3, $E_{t3} \leq E_{t1}$ and $E_{t3} \leq E_{t2}$. The error exponents are $E_1^s = E_2^s = E_{t3}(R_1 + R_2, SNR_1, SNR_2)$. In this region, type 3 error is dominant over both type 1 and type 2 errors. It is possible to increase the first (second) user's error exponent by reducing the second (first) user's error exponent.

As explained in section 1, error exponent is a function of both the transmission rate and the channel capacity. For a multiple access channel, one might get different error exponents

[2]Error exponent functions for type 1 error and type 2 error are the same as the single-user error exponent function, i.e. $E_{t1}(R_1, SNR_1) = E(R_1, SNR_1)$ and $E_{t2}(R_2, SNR_2) = E(R_2, SNR_2)$.

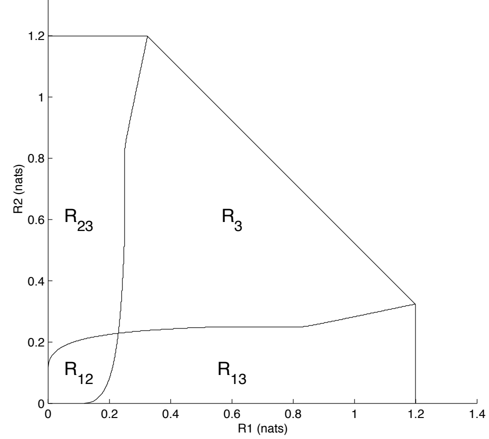


Figure 3: Regions where E_{ti} dominates; SNR = 20.

depending on which particular point on the capacity boundary is considered. For a Gaussian MAC channel (or a wireless MIMO MAC channel), however, the capacity boundary happen to be achieved by the same input distributions. That's why all previous works in Gaussian MAC channels and wireless MIMO MAC channels derived only one single point on the error exponent region (i.e. error exponents for user 1 and user 2 are fixed by the data rates for user 1 and for user 2). That single point derived by previous works in fact implies a rectangle error exponent region achieved by superposition encoding. Consider a two-user Gaussian MAC channel with equal power constraint for user 1 and user 2 (i.e. $SNR_1 = SNR_2$). In Fig. 4(a)), the dashed square is the achievable error exponent region (EER) by superposition. Although the (EER) is still unknown for Gaussian MAC channels, we do propose a simple scheme (time-division) to enlarge the achievable EER beyond what has been achieved by superposition encoding. The error exponents for user 1 and user 2 by time-division are

$$E_1^{td} = \alpha E\left(\frac{R_1}{\alpha}, \frac{SNR_1}{\alpha}\right) \tag{4}$$

$$E_2^{td} = (1-\alpha)E\left(\frac{R_2}{1-\alpha}, \frac{SNR_2}{1-\alpha}\right), \tag{5}$$

where $0 < \alpha < 1$. In Fig. 4(a), the solid curve is the achievable EER by time-division. The union of the superposition achievable EER and the time-division achievable EER is an inner bound for the EER of a Gaussian MAC channel. We summarize in the following theorem.

Theorem 1: For a two-user Gaussian MAC channel with power constraints SNR_1 and SNR_2 for user 1 and user 2 (noise power is normalized to one), the achievable $EER(R_1, R_2)$ (inner bound) is $EER_s(R_1, R_2) \cup EER_{td}(R_1, R_2)$, where superposition achievable region $EER_s(R_1, R_2)$ and time-division achievable region $EER_{td}(R_1, R_2)$ are

$$EER_s(R_1, R_2) = \{(e_1, e_2) : e_1 \leq E_1^s, e_2 \leq E_2^s\} \tag{6}$$

$$EER_{td}(R_1, R_2) = \{(e_1, e_2) : e_1 \leq \alpha E(\frac{R_1}{\alpha}, \frac{SNR_1}{\alpha}), e_2 \leq (1-\alpha)E(\frac{R_2}{1-\alpha}, \frac{SNR_2}{1-\alpha})\}, \tag{7}$$

where E_1^s and E_2^s are defined in (2), (3).

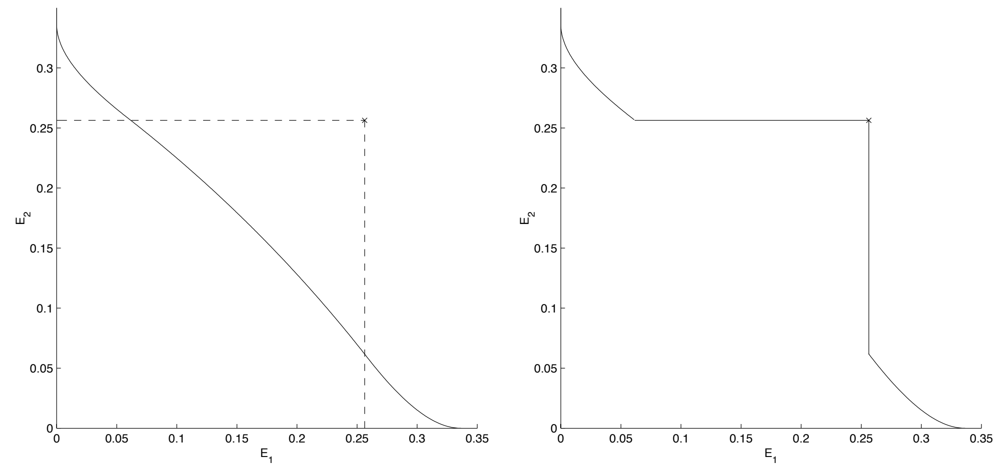
for rate points inside region R_{12} (see Fig. 3). What's the maximum rate region we can increase EER_s by time-division? If we adjust α to make E_2^{td} arbitrary small (but still positive), then we get the largest error exponent $E_{1,max}^{td}$ for user 1 by time-sharing method (see (4),(5)). In Fig. 5, the shaded region is the region where $E_{1,max}^{td} > E_1^s$. Similarly, we can obtain another region where $E_{2,max}^{td} > E_2^s$ by reducing E_1^{td} close to zero. The union of these two regions is the rate region where EER_s can be enlarge by time-division.

Error exponents for user 1 and user 2 by time-sharing method are

$$E_1^{ts} = \alpha E\left(\frac{R_1}{\alpha}, SNR_1\right) \tag{8}$$

$$E_2^{ts} = (1-\alpha) E\left(\frac{R_2}{1-\alpha}, SNR_2\right), \tag{9}$$

where $0 < \alpha < 1$. We can also enlarge EER_s by time-sharing instead of time-division, but the shaded region by time-sharing in Fig. 5 is smaller than that by time-division.



(a) Error exponent achievable regions by superposition and time-division (b) Inner bound for the error exponent region

Figure 4: Error exponent achievable region (R_1 = 0.5, R_2 = 0.5) by time-division and superposition; SNR = 20.

3 Error exponent region for Gaussian Broadcast channels

Consider a two-user Gaussian broadcast channel

$$Y_1 = X + Z_1 \tag{10}$$

$$Y_2 = X + Z_2, \tag{11}$$

where X is the channel input, and Y_1 and Y_2 are the channel outputs for user 1 and user 2. Assume noise power for Z_1 is $\sigma_1^2 = 1$ (normalized) and noise power for Z_2 is $\sigma_2^2 = \beta\sigma_1^2 = \beta$ (normalized). In contrast to Gaussian MAC channels, capacity boundary for Gaussian broadcast channels is achieved by different input distributions. In Fig. 1(c), the rate point A_1 is achieved

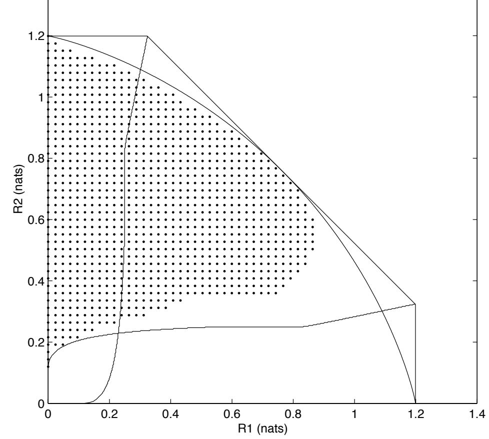


Figure 5: Rate region where $E_{1,max}^{td} > E_1^s$; SNR = 20.

by input distributions $\mathcal{N}(0, \alpha_1 SNR)$ and $\mathcal{N}(0, (1-\alpha_1)SNR)$, but the point A_2 is achieved by another input distributions $\mathcal{N}(0, \alpha_2 SNR)$ and $\mathcal{N}(0, (1-\alpha_2)SNR)$ $(0 < \alpha_1 < \alpha_2 < 1)$. Therefore, we expect the error exponents for rate point A evaluated with respect to A_1 are different from those evaluated with respect to A_2. We can also derive achievable EER by superposition encoding for a two-user Gaussian broadcast channels. We summarize the result in the following theorem.

Theorem 2: For a two-user Gaussian broadcast channel with power constraint SNR (noise powers are normalized to 1 and β), the achievable $EER(R_1, R_2)$ (inner bound) by superposition is

$$\begin{aligned}
\{(e_1, e_2) \quad : \quad & e_1 \le \min\{E(R_1, \alpha SNR), E_{t3}(R_1 + R_2, \alpha SNR, (1-\alpha)SNR)\}, && (12) \\
& e_2 \le \min\{E(R_2, \frac{(1-\alpha)SNR}{\beta}), E_{t3}(R_1 + R_2, \frac{\alpha SNR}{\beta}, \frac{(1-\alpha)SNR}{\beta})\}, && (13) \\
& 0 \le \alpha \le 1\}. && (14)
\end{aligned}$$

4 Summary and Conclusion

In this paper, we consider error region for Gaussian MAC channels and Gaussian broadcast channels. For a Gaussian MAC channel, Gallager-type superposition encoding derives only one single point inside the error exponent region. For a Gaussian broadcast channel, however, the channel capacity boundary corresponds to an error-exponent curve (instead of one single point). A simple scheme (time-division) is used to increase achievable error exponent region by superposition for Gaussian MAC channels. The concept of error exponent region is general and it's possible to extend the results in this paper to other channel models, like wireless MIMO MAC channels and wireless MIMO broadcast channels.

5 Appendix

Type 3 error exponent $E_{t3}(R_3, SNR_1, SNR_2)$ is the maximum of type 3 random coding exponent $E_{t3,r}(R_3, SNR_1, SNR_2)$ and type 3 expurgated exponent $E_{t3,ex}(R_3, SNR_1, SNR_2)$. The

$$E_{t3,r}(R_3, SNR_1, SNR_2) = \max_{\rho,\theta_1,\theta_2} \{E_{t3,0}(\rho,\theta_1,\theta_2) - \rho R_3\} \quad (15)$$

$$E_{t3,0}(\rho,\theta_1,\theta_2) = (1+\rho)\ln\left[\frac{e\sqrt{\theta_1\theta_2}}{1+\rho}\right] - \frac{\theta_1+\theta_2}{2} + \frac{\rho}{2}\ln\left[1 + \frac{SNR_1}{\theta_1} + \frac{SNR_2}{\theta_2}\right], \quad (16)$$

where maximization is over $0 \le \rho \le 1$, and $0 < \theta_1, \theta_2 \le 1+\rho$ [3]. The expression for $E_{t3,ex}(R_3, SNR_1, SNR_2)$ is

$$E_{t3,ex}(R_3, SNR_1, SNR_2) = \max_{\rho,r_1,r_2} \{E_{t3,x}(\rho, r_1, r_2) - \rho R_3\} \quad (17)$$

$$E_{t3,x}(\rho, r_1, r_2) = 2\rho(r_1 SNR_1 + r_2 SNR_2) + \rho\ln[(1-2r_1 SNR_1)(1-2r_2 SNR_2)] + \frac{\rho}{2}\ln\left[1 + \frac{SNR_1}{2\rho(1-2r_1 SNR_1)} + \frac{SNR_2}{2\rho(1-2r_2 SNR_2)}\right], \quad (18)$$

where maximization is over $\rho \ge 1, 0 \le r_1 < \frac{1}{2SNR_1}$, and $0 \le r_2 < \frac{1}{2SNR_2}$. $E_{t3}(R_3, SNR_1, SNR_2)$ has a closed form solution (i.e. no need to maximize over dummy parameters ρ, θ_1, etc.) when $SNR_1 = SNR_2$

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