Asymptotic Weight Distributions of Irregular Repeat-Accumulate Codes

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***Abstract*— In this paper, the average input-parity weight enumerator (AIPWE) of regular/irregular repeat-accumulate (RA) ensembles is derived, by viewing an RA code as a serial concatenation of an outer low-density generator matrix (LDGM) code and an inner accumulator. The exact average weight distribution (AWD) of the systematic and nonsystematic versions of the RA ensembles are then obtained from their AIPWE's. We further derive the asymptotic growth exponent of the AWD's, which are then used to bound the ensemble performance under maximum likelihood (ML) decoding. It is shown that simple nonsystematic regular RA ensembles outperform systematic regular RA and regular low-density parity-check (LDPC) ensembles and have distance spectrum which closely resembles that of the random ensemble.**

I. Introduction

Ten years after the introduction of turbo codes [1] and forty years after Gallager's introduction of low-density parity-check (LDPC) codes [2], it is now well established that turbo-like codes practically approach the capacity of many channels with linear decoding complexity. This result is corroborated by a vast amount of mostly experimental evidence. Several attempts have been made to prove rigorously that indeed these codes are capacity achieving. Successful examples in this direction include the works of [3–5] and [6,7], which showed that irregular LDPC codes and irregular repeat-accumulate (RA) codes, respectively, achieve the capacity of the binary erasure channel (BEC) with message-passing decoding. All these works rely on the powerful density evolution (DE) method [8] to prove the capacity-achieving property of these families of codes. Unfortunately, analysis using DE becomes an infinite dimensional problem on channels other than the BEC. This is the reason why very little progress has been made to prove that turbo-like codes achieve capacity even on memoryless binary-input output-symmetric (MBIOS) channels (a notable exception is the work of [9] that introduces capacity-achieving families of codes, albeit with decoding complexity per information bit increasing exponentially with the gap from capacity).

In this paper, we initiate an effort towards proving that turbo-like codes achieve capacity on MBIOS channels. As a first step, we investigate if certain families of codes achieve capacity with optimal maximum likelihood (ML) decoding. The motivation for looking at ML decoding is twofold. First, achieving capacity with ML decoding provides a necessary condition for achieving capacity with suboptimal message-passing decoding. Furthermore, it was recently shown [10, 11] that there are efficient methods to approach ML performance by improved iterative decoding algorithms. Thus, there is hope that the problem can be approached without resorting to DE which, as mentioned earlier, is the main difficulty in extending the results from the BEC to MBIOS channels. Among the various promising candidates of capacity achieving codes on MBIOS channels, we choose to work with RA codes for two reasons[1]. First, it was recently shown [7] that irregular RA codes achieve the capacity of the BEC with a bounded average number of edges per information bit, and thus with bounded decoding complexity under iterative decoding. Second, these codes have linear encoding complexity as compared to the quadratic encoding complexity of the LDPC codes.

The main difficulty of analyzing the ML performance of RA codes is that they **are not serial concatenations of simple constituent codes** if the check node degree is greater than 3 as shown in Fig. 1(a). This is true regardless of whether the RA code is regular or irregular. In this paper, we solve this problem by viewing an RA code as a serial concatenated code with an outer nonsystematic low-density generator matrix (LDGM) code [13] and an inner accumulator code. Based on this decomposition, we derive the average input-parity weight enumerator (AIPWE) of regular/irregular RA ensembles, which is then used to derive the average weight distribution (AWD) of systematic and nonsystematic versions of the ensembles. Asymptotic growth exponents of the AWD's of RA ensembles are also calculated, which can be immediately used to obtain various ML performance bounds as in [14–16]. As a side result, the asymptotic growth exponents of the AWD of LDGM ensembles are derived in the process and the role of the inner accumulator in spectral thinning is demonstrated. Our approach shows that simple nonsystematic RA codes outperform random codes (which are known to be capacity achieving) when Divsalar's bound [15] is used on the binary input AWGN channel.

II. LDGM and RA codes

Consider the irregular LDGM and RA codes as shown in Fig. 1. As can be seen in the figure, both of them have two different set of variable nodes, i.e., the information nodes and the parity nodes. The systematic version of them uses all the variable nodes as its codeword, while the nonsystematic one uses only the parity nodes. Therefore, letting k denote the number of information bits and n denote the number of parity bits, the rate R of the systematic and nonsystematic codes is $k/(n+k)$ and k/n, respectively.

Let λ_i be the fraction of edges between the information and check nodes that are connected to an information node with i check node neighbors, and ρ_i be the fraction of the same edges

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[1] Similar results can be found in [12] for LDPC ensembles with ML decoding.

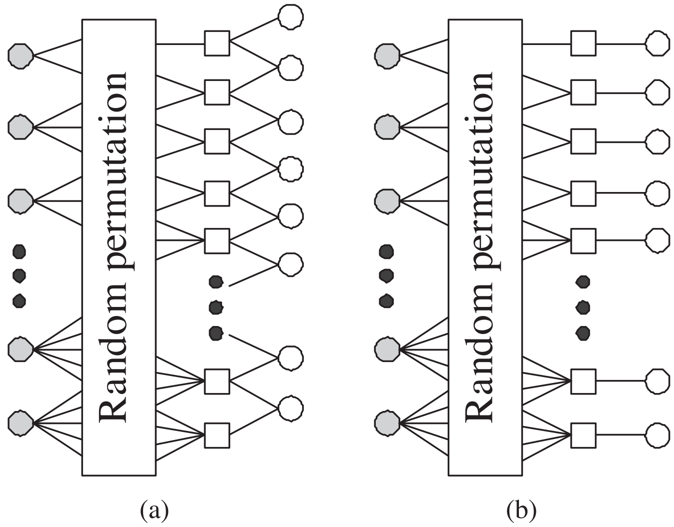

Fig. 1. Factor graph for (a) irregular RA and (b) irregular LDGM codes. Information bits are denoted by filled gray circles, parity bits by open circles, and check nodes by squares.

that are connected to a check node with i information node neighbors. Furthermore, define

$$\lambda(x) \triangleq \sum_{i=1}^{\infty} \lambda_i x^{i-1}, \text{and} \quad \rho(x) \triangleq \sum_{i=1}^{\infty} \rho_i x^{i-1} \qquad (1)$$

to be the generating functions of λ_i's and ρ_i's. These two functions are used to specify the ensembles of irregular LDGM and RA codes assuming random permutation of edges between information and check nodes within each ensemble. A special case is the "(c, d) regular" code ensemble defined by $\lambda(x) = \lambda_c x^{c-1}$, $\rho(x) = \rho_d x^{d-1}$. Note that, in the following, when we use the term "RA codes", we are referring to both irregular and regular RA codes, **but not the original RA codes introduced in [17]** which were restricted to have $\rho(x) = \rho_1$.

The above degree distribution pair (λ, ρ) is from the edge perspective. It can facilitate our following analysis if we also have an equivalent description from the node perspective. Let $\tilde{\lambda}_i$(respectively $\tilde{\rho}_i$) be the fraction of information (check) nodes that are connected to i check (information) nodes. Then we have

$$\tilde{\lambda}_i = \frac{\lambda_i / i}{\sum_{j=1}^{\infty} \lambda_j / j}, \text{and} \quad \tilde{\rho}_i = \frac{\rho_i / i}{\sum_{j=1}^{\infty} \rho_j / j}. \qquad (2)$$

III. Average input-parity weight enumerator of LDGM and RA code ensembles

The input-output weight enumerator (IOWE) $A_{w,h}$ of a binary linear block code $\mathcal{C}$ is defined to be the number of codewords in $\mathcal{C}$ with input hamming weight w and output hamming weight h. Similarly, we can define the input-parity weight enumerator (IPWE) $Z_{w,h}$ for LDGM and RA codes to denote the number of codewords with input weight w and parity weight h. Note that IPWE and IOWE are the same for nonsystematic LDGM and RA codes, but different for systematic ones.

In this section, we calculate the average IPWE (AIPWE) $\overline{Z_{w,h}}$ of LDGM and RA ensembles, which is then used in the next section to obtain the average weight distribution (AWD) of systematic and nonsystematic versions of the respective ensembles.

A. AIPWE of LDGM ensembles

Consider the (λ, ρ) irregular LDGM ensemble. Let W, and H be the random variables denoting the input and parity weight, respectively, of a randomly chosen codeword of a code drawn randomly from the ensemble. Furthermore, let E be the random variable denoting the total number of edges emanating from the information nodes that are equal to 1, of the aforementioned codeword. Moreover, define

$$t \triangleq k \sum_{i=1}^{\infty} i \tilde{\lambda}_i \qquad (3)$$

to be the total number of edges between information and parity nodes. We have

$$\begin{aligned}
\overline{Z_{w,h}^{(LDGM)}} &= 2^k P(H = h, W = w) \\
&= 2^k P(W = w) \sum_{e=0}^{t} P(H = h, E = e | W = w) \\
&= \binom{k}{w} \sum_{e=0}^{t} P(E = e | W = w) \\
&\quad P(H = h | E = e, W = w) \qquad (4)
\end{aligned}$$

The number of ways of having exactly e edges emanating from w information nodes, out of a total of $\binom{k}{w}$ possibilities, is equal to

$$\text{coef}(\prod_{i=1}^{\infty} (1 + x^i y)^{k \tilde{\lambda}_i}, x^e y^w), \qquad (5)$$

where coef$(f(x, y), x^a y^b)$ denotes the coefficient of $x^a y^b$ in the polynomial $f(x, y)$. Therefore, we have

$$P(E = e | W = w) = \frac{\text{coef}(\prod_{i=1}^{\infty} (1 + x^i y)^{k \tilde{\lambda}_i}, x^e y^w)}{\binom{k}{w}}. \qquad (6)$$

On the other hand, given that the number of edges from the information nodes equal to 1 is e, the output weight is h if and only if exactly h check nodes are connected to an odd number of such edges, and the remaining $n - h$ check nodes are connected to an even number of them. Counting the number of ways of connecting e edges to t check node sockets such that exactly h check nodes have an odd number of connections we see that the value is equal to

$$\text{coef}(\prod_{j=1}^{\infty} [f_-(x, j) y + f_+(x, j)]^{n \tilde{\rho}_j}, x^e y^h), \qquad (7)$$

where

$$f_-(x, j) \triangleq \frac{1}{2} [(1 + x)^j - (1 - x)^j] \qquad (8)$$

$$f_+(x, j) \triangleq \frac{1}{2} [(1 + x)^j + (1 - x)^j] \qquad (9)$$

to simplify notation. Since the total number of ways of connecting e edges to t sockets is equal to $\binom{t}{e}$, we have

$$
\begin{aligned}
&P(H=h|E=e,W=w)\\
=&\frac{\text{coef}(\prod_{j=1}^{\infty}[f_-(x,j)y+f_+(x,j)]^{n\tilde{\rho}_j},x^ey^h)}{\binom{t}{e}},
\end{aligned} \tag{10}
$$

which is not related to the exact input weight w. Combining (4), (6) and (10), we obtain the AIPWE of the (λ,ρ) irregular LDGM ensemble

$$
\begin{aligned}
\overline{Z_{w,h}^{(LDGM)}} =&\sum_{e=0}^{\infty}\frac{1}{\binom{t}{e}}\text{coef}(\prod_{i=1}^{\infty}(1+x^iy)^{k\tilde{\lambda}_i},x^ey^w)\\
&\text{coef}(\prod_{j=1}^{\infty}[f_-(x,j)y+f_+(x,j)]^{n\tilde{\rho}_j},x^ey^h).
\end{aligned} \tag{11}
$$

In particular, (11) simplifies to the following AIPWE for the (c,d) regular LDGM ensemble

$$
\overline{Z_{w,h}^{(LDGM)}}=\frac{\binom{k}{w}}{\binom{ck}{cw}}\binom{n}{h}\text{coef}(f_-(x,d)^hf_+(x,d)^{n-h},x^{cw}). \tag{12}
$$

B. AIPWE of RA ensembles

Since a (λ,ρ) irregular RA code can be viewed as a concatenated code with an outer nonsystematic (λ,ρ) irregular LDGM code and an inner accumulator code, we have

$$
\overline{Z_{w,h}^{(RA)}}=\sum_{s=0}^{n}\frac{\overline{Z_{w,s}^{(LDGM)}}A_{s,h}^{(acc)}}{\binom{n}{s}} \tag{13}
$$

by randomness of the ensemble construction, where $A_{w,h}^{(acc)}$ denotes the IOWE of the accumulator code, which is given in [17] to be

$$
A_{w,h}^{(acc)}=
\begin{cases}
\binom{n-h}{\lfloor w/2\rfloor}\binom{h-1}{\lceil w/2\rceil-1} & \text{if } \lfloor w/2\rfloor\le n-h \text{ and } \lceil w/2\rceil\le h,\\
0 & \text{else.}
\end{cases} \tag{14}
$$

Hence, from (13), (14) and (11), we obtain the AIPWE of the (λ,ρ) irregular RA ensemble as

$$
\begin{aligned}
\overline{Z_{w,h}^{(RA)}} =&\sum_{s\ge0,\lfloor s/2\rfloor\le n-h,\lceil s/2\rceil\le h}\frac{\binom{n-h}{\lfloor s/2\rfloor}\binom{h-1}{\lceil s/2\rceil-1}}{\binom{n}{s}}\\
&\sum_{e=0}^{\infty}\frac{1}{\binom{t}{e}}\text{coef}(\prod_{i=1}^{\infty}(1+x^iy)^{k\tilde{\lambda}_i},x^ey^w)\\
&\text{coef}(\prod_{j=1}^{\infty}[f_-(x,j)y+f_+(x,j)]^{n\tilde{\rho}_j},x^ey^s).
\end{aligned} \tag{15}
$$

Similarly, from (13), (14) and (12), we obtain the AIPWE of the (c,d) regular RA ensemble as

$$
\begin{aligned}
\overline{Z_{w,h}^{(RA)}} =&\frac{\binom{k}{w}}{\binom{ck}{cw}}\sum_{s\ge0,\lfloor s/2\rfloor\le n-h,\lceil s/2\rceil\le h}\binom{n-h}{\lfloor s/2\rfloor}\binom{h-1}{\lceil s/2\rceil-1}\\
&\text{coef}(f_-(x,d)^sf_+(x,d)^{n-s},x^{cw}).
\end{aligned} \tag{16}
$$

IV. Asymptotic average weight distribution of LDGM and RA ensembles

Consider an LDGM or RA ensemble with AIPWE $\overline{Z_{w,h}}$. Let X_l be the average number of codewords of weight l in a randomly drawn code from the ensemble. Then for the systematic version of the ensemble, the AWD is given by

$$
X_l^{(sys)}=\sum_{w=\max(0,l-n)}^{\min(k,l)}\overline{Z_{w,l-w}}. \tag{17}
$$

On the other hand, for the nonsystematic version of the ensemble, it is given by

$$
X_l^{(non)}=\sum_{w=0}^{k}\overline{Z_{w,l}}. \tag{18}
$$

To analyze the asymptotic behavior of AWD of an ensemble, we use the following two important equations proven in [18]

$$
\lim_{\substack{n\to\infty\\ \text{coef}(f(x),x^{\alpha n})\neq0}}\frac{1}{n}\log\text{coef}(f(x),x^{\alpha n})=\inf_{x>0}\log\frac{f(x)}{x^\alpha} \tag{19}
$$

$$
\begin{aligned}
&\lim_{\substack{n\to\infty\\ \text{coef}(f(x,y),x^{\alpha n}y^{\beta n})\neq0}}\frac{1}{n}\log\text{coef}(f(x,y),x^{\alpha n}y^{\beta n})\\
&=\inf_{x>0,y>0}\log\frac{f(x,y)}{x^\alpha y^\beta}
\end{aligned} \tag{20}
$$

where $0<\alpha,\beta<1$, $f(x)$ and $f(x,y)$ are polynomials with nonnegative coefficients, and log is base-2. Also, we use the well known property of binomial coefficients

$$
\lim_{n\to\infty}\log\binom{n}{\alpha n}=h(\alpha) \tag{21}
$$

where $0\le\alpha\le1$, and $h(x)\triangleq -x\log x-(1-x)\log(1-x)$ is the binary entropy function.

Now, let us consider the nonsystematic (c,d) regular LDGM ensemble with rate $R=\frac{d}{c}$ and AWD X_l. Define the asymptotic growth exponent of the AWD of the ensemble to be

$$
y(\alpha)\triangleq\lim_{n\to\infty,X_{\alpha n}\neq0}\frac{1}{n}\log X_{\alpha n}. \tag{22}
$$

Then, from (18) and (12), we have

$$
\begin{aligned}
y(\alpha)=&\lim_{n\to\infty,X_{\alpha n}\neq0}\frac{1}{n}\log X_{\alpha n}^{(non)}\\
=&\lim_{n\to\infty,X_{\alpha n}\neq0}\frac{1}{n}\log\sum_{w=0}^{k}\overline{Z_{w,\alpha n}^{(LDGM)}}\\
=&\lim_{n\to\infty,X_{\alpha n}\neq0}\frac{1}{n}\log\\
&\sum_{\beta k=0}^{k}\frac{\binom{k}{\beta k}\binom{n}{\alpha n}\text{coef}(f_-(x,d)^{\alpha n}f_+(x,d)^{(1-\alpha)n},x^{\beta ck})}{\binom{ck}{\beta ck}}\\
=&\lim_{n\to\infty,X_{\alpha n}\neq0}\frac{1}{n}\log\Bigg[\binom{n}{\alpha n}\max_{0\le\beta\le1}\\
&\frac{\binom{\frac{d}{c}n}{\frac{\beta d}{c}n}\text{coef}(f_-(x,d)^{\alpha n}f_+(x,d)^{(1-\alpha)n},x^{\beta dn})}{\binom{dn}{\beta dn}}\Bigg]+o(1)
\end{aligned} \tag{23}
$$

where $o(1)$ is a function of n that converges to 0 as n goes to infinity. By (19) and (21) we obtain the following result.

Theorem 1 *The asymptotic growth exponent of the AWD of the nonsystematic (c, d) regular LDGM ensemble satisfies*

$$y(\alpha) = h(\alpha) + \sup_{0\leq\beta\leq 1} R(1-c)h(\beta) + \inf_{x>0} \log \frac{f_-(x,d)^{\alpha} f_+(x,d)^{1-\alpha}}{x^{\beta d}} \quad (24)$$

Similarly, from (12), (16), (17), (18), (19) and (21), we can obtain the following results.

Theorem 2 *The asymptotic growth exponent of the AWD of the nonsystematic (c, d) regular RA ensemble, the systematic (c, d) regular LDGM ensemble and the systematic (c, d) regular RA ensemble are given, respectively, by*

$$y(\alpha) = \sup_{0\leq\gamma\leq\min(2(1-\alpha),2\alpha)} (1-\alpha)h(\frac{\gamma}{2(1-\alpha)}) + \alpha h(\frac{\gamma}{2\alpha}) + \sup_{0\leq\beta\leq 1} R(1-c)h(\beta) + \inf_{x>0} \log \frac{f_-(x,d)^{\gamma} f_+(x,d)^{1-\gamma}}{x^{\beta d}} \quad (25)$$

$$y(\alpha) = \sup_{\max(0,\frac{\alpha-1+R}{R})\leq\beta\leq\min(1,\frac{\alpha}{R})} R(1-c)h(\beta) + (1-R) \left(h(\frac{\alpha-\beta R}{1-R}) + \inf_{x>0} \log \frac{f_-(x,d)^{\frac{\alpha-\beta R}{1-R}} f_+(x,d)^{1-\frac{\alpha-\beta R}{1-R}}}{x^{\beta d}} \right) \quad (26)$$

$$y(\alpha) = \sup_{\max(0,\frac{\alpha-1+R}{R})\leq\beta\leq\min(1,\frac{\alpha}{R})} R(1-c)h(\beta) + \sup_{0\leq\gamma\leq\min(2(1-\frac{\alpha-\beta R}{1-R}),2(\frac{\alpha-\beta R}{1-R}))} (1-R) \left((1-\frac{\alpha-\beta R}{1-R})h(\frac{\gamma}{2(1-\frac{\alpha-\beta R}{1-R})}) + \frac{\alpha-\beta R}{1-R}h(\frac{\gamma}{2(\frac{\alpha-\beta R}{1-R})}) \inf_{x>0} \log \frac{f_-(x,d)^{\gamma} f_+(x,d)^{1-\gamma}}{x^{\beta d}} \right) \quad (27)$$

Analogous results for the irregular LDGM and RA ensembles can also be obtained in a straightforward manner from (11), (15), (17), (18), (20) and (21), and are omitted in this paper.

V. Numerical results

Fig. 2 depicts the asymptotic growth exponent of the AWD of the nonsystematic (10,5) regular LDGM and RA ensembles, and compares them with those of the (7,14) regular LDPC ensemble given in [18, 19] and the rate-$1/2$ random ensemble. As can be seen in the figure, the RA ensemble has a more concentrated AWD than the LDGM ensemble. This shows that the rate-1 accumulator code really helps eliminate low weight codewords in the nonsystematic LDGM codes. Moreover, this figure shows that the asymptotic AWD of the nonsystematic

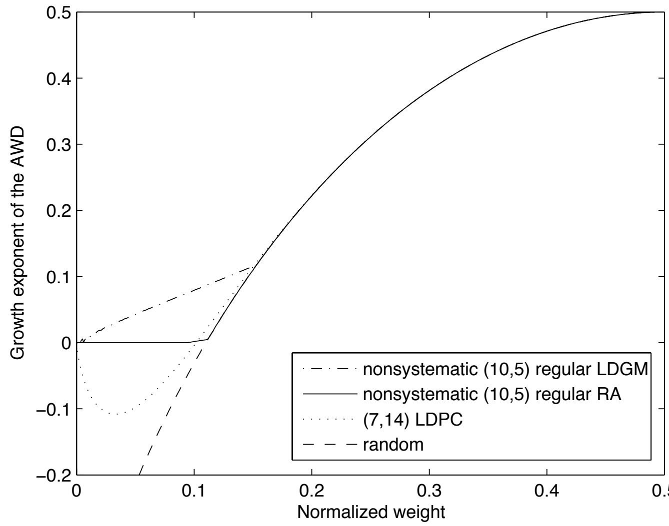

Fig. 2. The asymptotic AWD of the nonsystematic (10,5) regular LDGM ensemble, nonsystematic (10,5) regular RA ensemble, (7,14) regular LDPC ensemble, and the random ensemble. All of them have rate 1/2.

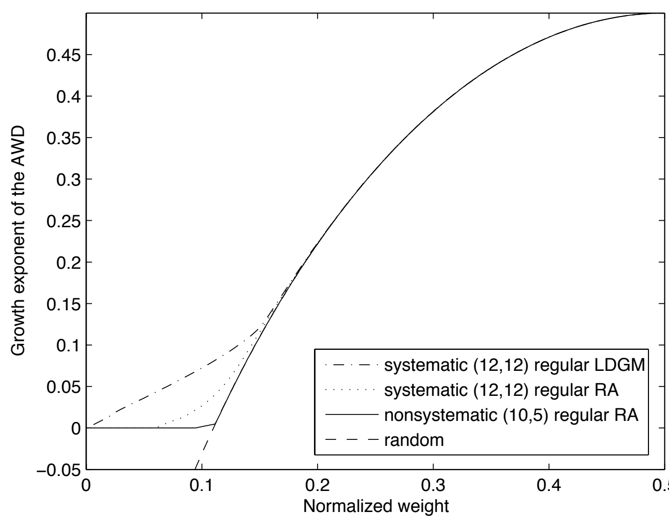

Fig. 3. The asymptotic AWD of the systematic (12,12) regular LDGM ensemble, systematic (12,12) regular RA ensemble, nonsystematic (10,5) regular RA ensemble, and the random ensemble. All of them have rate 1/2.

(10,5) regular RA ensemble well approximates that of the random ensemble with the same rate.

The effect of the accumulator code is also evident for the systematic (12, 12) regular RA ensemble as shown in Fig. 3. However, we see in the figure that the asymptotic AWD of the systematic RA ensemble is not as good as that of the nonsystematic one with the same average number of edges per information bits. This seems to signify the superiority of nonsystematic RA ensembles over systematic ones and is consistent with the results of [7] for the BEC.

To gain an immediate idea of how the asymptotic growth exponents of the AWD's are related to ML performance of codes and how regular RA ensembles perform on AWGN channel with ML decoding, we invoke Divsalar's bound [15] on the minimum bit signal to noise ratio (SNR) $(\frac{E_b}{N_0})^*$ required for reliable communication as follows

$$(\frac{E_b}{N_0})^* \leq \frac{1}{R} \max_{0\leq\alpha\leq1} \left\{ \frac{(1-2^{-2y(\alpha)})(1-\alpha)}{2\alpha} \right\}. \quad (28)$$

The results for rate-$1/2$ nonsystematic regular RA, systematic regular RA and regular LDPC ensembles are summarized in table I. Note that the comparison is based on the same average number of edges per information bit, denoted by $\overline{e}$ in the table, and the threshold corresponding to capacity is 0.184dB in this case. As can be seen in the table, all nonsystematic regular RA ensembles with a check node degree greater than 3 yield the same performance bound, which is better than that of their corresponding systematic regular RA and regular LDPC ensembles. Furthermore, the evaluated bound on $(E_b/N_0)^*$ of nonsystematic regular RA ensembles is even better than that of the random ensemble. This is not a contradiction, but a direct consequence of the fact that Divsalar's bound in (28) is simple but not tight for rate-$1/2$ codes. On the contrary, this result seems to suggest that nonsystematic regular RA codes with small check node degree can come very close to the capacity or even achieve the capacity on the AWGN channel. However, for a rigorous proof of such a statement we will need to resort to a tighter ML performance bound.

TABLE I

COMPARISON OF $(\frac{E_b}{N_0})^*$ AS GIVEN IN (28) FOR SEVERAL ENSEMBLES WITH RATE 1/2.

Ensemble	$\overline{e}$	$(\frac{E_b}{N_0})^*(dB)$
nonsystematic (4,2) regular RA	8	0.3076
Systematic (6,6) regular RA	8	0.4443
(4,8) regular LDPC	8	0.4264
nonsystematic (6,3) regular RA	10	0.3076
Systematic (8,8) regular RA	10	0.3434
(5,10) regular LDPC	10	0.3412
nonsystematic (8,4) regular RA	12	0.3076
Systematic (10,10) regular RA	12	0.3175
(6,12) regular LDPC	12	0.3180
nonsystematic (10,5) regular RA	14	0.3076
Systematic (12,12) regular RA	14	0.3110
(7,14) regular LDPC	14	0.3112
random		0.3081

VI. CONCLUSION

The problem of deriving the AIPWE for RA codes is solved in this paper by viewing a regular/irregular RA code as a serial concatenated code with an outer LDGM code and an inner accumulator code. The resulting AIPWE is then used to derive AWD for systematic and nonsystematic versions of RA codes, whose asymptotic growth exponent is also calculated. By numerically plotting the asymptotic growth exponent of the ensembles, we acquire the following three observations. First, the accumulator code plays an important role in eliminating low weight codewords for RA ensembles. Second, the nonsystematic regular RA ensembles have more concentrated AWD's than their corresponding systematic ones with the same average number of edges per information bit. Third, the nonsystematic regular RA ensembles with moderate check node degrees have asymptotic growth exponent of AWD's extremely close to that of the random ensemble for all growth exponent values greater than zero. In the last part of the paper, bit SNR thresholds on the AWGN channel based on Divsalar's bound are obtained to show that nonsystematic regular RA ensembles with small check node degrees have a better guaranteed ML performance than the corresponding systematic regular RA and regular LDPC ensembles with the same average number of edges per information bit and even better than the random ensemble. These promising results make nonsystematic RA ensembles strong candidates of capacity achieving codes on noisy channels.

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