Lattice Constellations for Both Rayleigh Fading and Gaussian Channels

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Signal constellations having lattice structure are commonly accepted as good means of transmitting with high spectral efficiency. The linear and highly symmetrical structure of lattices usually simplifies the labelling of signal points. The problem of finding good signal constellations over a Gaussian channel is tantamount to the lattice packing of spheres. For the Rayleigh channel the basic rules remain the same. But in general a good signal constellation for Gaussian channel is bad when used over a Rayleigh channel. On the other hand the signal constellations matched to the Rayleigh fading channel are usually bad over Gaussian channels. We review the work done in [1] [2] and [3] which discuss using algebraic number theory to find lattices which have good performance over Gaussian and Rayleigh channels.

Papers Reviewed

[1] J. Bautros, E. Viterbo, C. Rastello, and J.C. Belfiore, "Good Lattice Constellations for Both Rayleigh Fading and Gaussian Channels" , *IEEE Trans. on Information Theory* Vol. 42, No. 2, March 1996.

[2] X. Giraud, E. Boutillon, and J.C. Belfiore, "Algebraic Tools to Build Modulation Schemes for Fading Channels", *IEEE Trans. on Information Theory*, Vol. 43, No. 3, May 1997.

[3] J. Bautros, E. Viterbo, "Signal Space diversity: A power and bandwidth efficient diversity technique for the Rayleigh fading channel", *IEEE Trans. on Information Theory* Vol. 44, No. 4, July-1998.

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1 Preliminaries

➜ Let $\mathcal{C}$ be the constellation containing points from an n-dimensional space $\mathcal{F}^n$. If the pairwise error probability $p(\boldsymbol{x}_i \to \boldsymbol{x}_j)$ between any two different signals is smaller than ε then, by the union bound, the symbol error probability is upper bounded by $|\mathcal{C}| \times \varepsilon$

➜ For any $\boldsymbol{x} \in \mathcal{F}^n$, we define the set of non-admissible points with respect to $\boldsymbol{x}$ as

$$S^{\varepsilon}_{\boldsymbol{x}} = \{ \, \boldsymbol{t} \in \mathcal{F}^n \mid p(\boldsymbol{x} \to \boldsymbol{t}) > \varepsilon \, \}$$

We assume that $S^{\varepsilon}_{\boldsymbol{x}}$ is an open set, if it is not the case, we consider the topological interior.

➜ **Admissible Set:** The signal set $\mathcal{C}$ is ε-admissible if each region $S^{\varepsilon}_{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathcal{C}$, contains no other point from $\mathcal{C}$.

$$S^{\varepsilon}_{\boldsymbol{x}} \cap \mathcal{C} = \{ \, x \, \}$$

If the constellation is ε-admissible, then we can bound the error probability by $|\mathcal{C}| \times \varepsilon$.

➜ **Packing Formulation:** Let $\mathcal{S}$ be an open subset of $\mathcal{F}^n$. The system consisting of translates $(\mathcal{S} + \boldsymbol{x})_{\boldsymbol{x} \in \mathcal{C}}$ is called a $(\mathcal{S}, \mathcal{C})$-packing of $\mathcal{S}$ if for any two

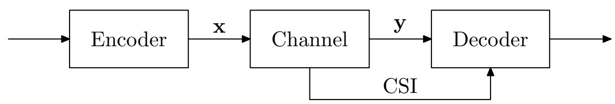

Figure 1: A general baseband transmission model.

distinct points $\boldsymbol{x}, \boldsymbol{t} \in \mathcal{C}$, the sets $\boldsymbol{x} + \mathcal{S}$ and $\boldsymbol{t} + \mathcal{S}$ are disjoint.

➜ A signal set $\mathcal{C}$ is ε-admissible if and only if $\left(\frac{1}{2}\mathcal{S}^{\varepsilon}_{\boldsymbol{o}}, \mathcal{C}\right)$ is a packing of $\frac{1}{2}\mathcal{S}^{\varepsilon}_{\boldsymbol{o}}$

➜ **Admissible Lattice:** A lattice Λ is ε-admissible if and only if $\mathcal{S}^{\varepsilon}_{\boldsymbol{o}} \cap \Lambda = \{\, \boldsymbol{o} \,\}$

➜ **Critical Lattice:** The $\mathcal{S}^{\varepsilon}_{\boldsymbol{o}}$-admissible lattice with the minimal determinant is called the critical lattice. Thus, *the lattice design problem is tantamount to determining a critical lattice of* $\mathcal{S}^{\varepsilon}_{\boldsymbol{o}}$

2 System Model

A general transmission model is shown in Figure 1. Corresponding to the input vector $\boldsymbol{x} = \left(x_1, \cdots, x_n\right)$, the channel outputs the sequence $\boldsymbol{y}$ whose kth coordinate is related to x_k by

$$y_k = \alpha_k x_k + w_k \tag{1}$$

where α_k is Rician distributed and w_k is zero mean white Gaussian random variable.

Note The pdf of a Rician variable is given by $\rho(x) = \dfrac{x}{b}\exp\Big(-\dfrac{x^2 + A^2}{2b}\Big) I_0\Big(\dfrac{xA}{b}\Big)$ and $A^2 + 2b = 1$. The model is characterized by the ratio $K = {}^{A^2}\!/_{2b}$. When $K = 0$, we have Rayleigh statistical model and when $K \to \infty$, we obtain AWGN model.

Assumptions

- We assume that the channel is memoryless by means of perfect interleaving/deinterleaving. The components are real i.e., $\mathcal{F} = \mathbb{R}$ when the interleaving is performed at coordinate level and complex, i.e., $\mathcal{F} = \mathbb{C}$, when the interleaving is done over complex symbols.
- The channel state information is ideally known.

3 Problem Formulation

We need to find a signal constellation carved out of a lattice that gives good performance over different values of K of the channel. In particular it should perform well in AWGN and Rayleigh fading channels.

Optimal maximum likelihood detection implies maximization of the following metric

$$m(\boldsymbol{t}, \boldsymbol{y}) = \sum_{i=1}^{n} |y_i - \alpha_i t_i|^2$$

Using Chernoff bound, the pairwise probability of error can be upper bounded as

$$p(\boldsymbol{x} \to \boldsymbol{s}) \le f(K, \boldsymbol{u}) \tag{2}$$

where

$$f(K, \boldsymbol{u}) = \prod_{i=1}^{n} \frac{1 + K}{1 + K + |u_i|^2} \exp\left(\frac{-K|u_i|^2}{1 + K + |u_i|^2}\right)$$
$$\boldsymbol{u} = \sqrt{\frac{N_0}{2}}(\boldsymbol{x} - \boldsymbol{s})$$

Thus one is interested in finding critical lattice

$$\mathcal{S}^{\varepsilon}_{K} = \{\, \boldsymbol{x} \in \mathcal{F}^n, f(K, \boldsymbol{x}) > \varepsilon \,\}, \quad \text{where } \mathcal{F} = \mathbb{R} \text{ or } \mathcal{F} = \mathbb{C}$$

The size of $\mathcal{S}^{\varepsilon}_{K}$ is actually larger than needed. The function $f(K, \boldsymbol{x})$ may be viewed as the pairwise error probability of some fictitious channel worse than the fading channel.

The Lattice Design Problem

We consider the three cases $K = 0$, $0 < K < \infty$ and $K = \infty$. The critical lattices S^{ε}_{K} for different values of K and ε are shown in Figure 2

AWGN Channel ($K = \infty$)

For an AWGN channel, $K = \infty$ and (2) simplifies to

$$p(\boldsymbol{x} \to \boldsymbol{s}) \le \prod_{i=1}^{n} \exp\left(-|u_i|^2\right) \tag{3}$$

which is the Bhattacharyya bound. Here the lattice design problem reduces to a sphere packing problem.

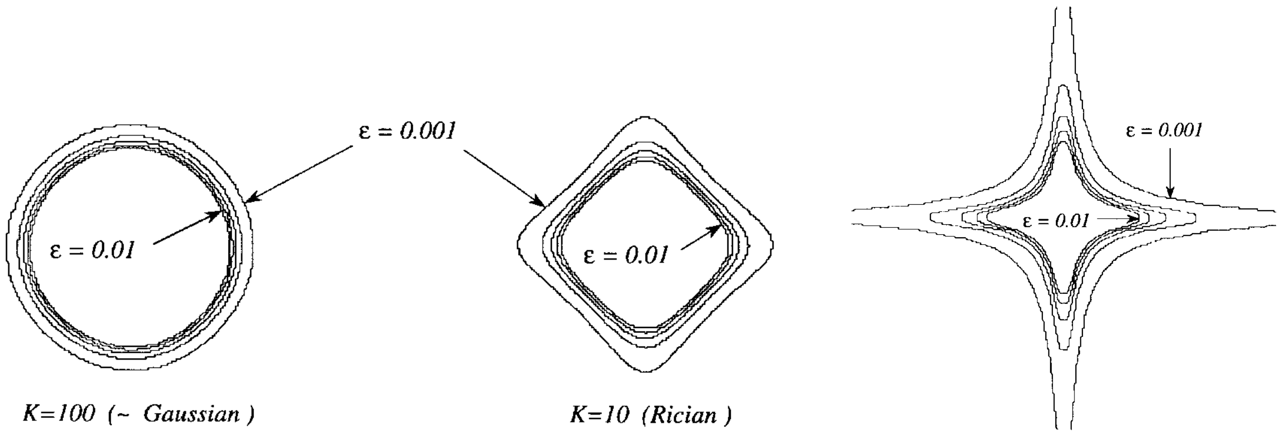

Figure 2: Behavior of $\mathcal{S}_K^\varepsilon$

Rician Channel ($0 < K < \infty$)

For a Rician channel with $K = 10$, set of non-admissible points with respect to the origin is bounded by a diamond

$$P_2 = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; |x| + |y| < 1 \right\}$$

A critical lattice of the diamond is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbb{Z}^2$. For the n-dimensional case, the critical lattice is bounded by the n-dimensional crosspolytope

$$P_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} |x_i| < 1 \right\}$$

that is, the unit ball for the l_1-norm.

Rayleigh Channel ($K = 0$)

For a Rayleigh Channel, $K = 0$ and the critical lattice simplifies to

$$\mathcal{S}^\varepsilon = \left\{ \boldsymbol{x} \in \mathcal{F}^n \;\middle|\; \prod_{i=1}^{n} \left(|x_i|^2 + \varepsilon^{1/n} \right) < 1 \right\}$$

Observe that $\mathcal{S}^\varepsilon \subset \mathcal{S}$ where

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \mathcal{F}^n \;\middle|\; \prod_{i=1}^{n} |x_i|^2 < 1 \right\} \tag{4}$$

If Λ is $\mathcal{S}$-admissible, then it is $\mathcal{S}^\varepsilon$-admissible and $\mathcal{S}^\varepsilon$ goes to $\mathcal{S}$ as ε decreases (SNR increases).

If the components are real (the interleaving is done over the coordinates) we have

$$\mathcal{S}_r = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \prod_{i=1}^{n} |x_i|^2 < 1 \right\} \tag{5}$$

When the interleaving is done over 2-D symbols, we have

$$\mathcal{S}_c = \left\{ \boldsymbol{x} = (\boldsymbol{a} + j\boldsymbol{b}) \in \mathbb{C}^n \;\middle|\; \prod_{i=1}^{n} |a_i^2 + b_i^2|^2 < 1 \right\} \tag{6}$$

The effective construction of $\mathcal{S}$-admissible lattice is not easy. [1] and [2] propose techniques from number theory to find solutions of this problem. We present the main results in the next section.

4 A Family of $\mathcal{S}$-Admissible Lattices

The lattice design problem leads to consider a multiplicative norm on each point $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$ as given by (4)

$$N(\boldsymbol{x}) = x_1 \times \cdots \times x_n$$

We make this statement more concrete using the following construction from algebraic number theory.

Some Definitions from Number Theory

Let $\mathcal{R}$ denote the Euclidean Domain $\mathbb{Z}$ of integers when $\mathcal{F} = \mathbb{R}$ and $\mathbb{Z}[j]$ or $\mathbb{Z}[\omega]$ where $\omega \triangleq e^{j2\pi/3}$ when $\mathcal{F} = \mathbb{C}$. Let $\mathcal{K}$ denote the quotient field of $\mathcal{R}$ i.e.,

$\mathcal{R}$	$\mathbb{Z}$	$\mathbb{Z}[j]$	$\mathbb{Z}[\omega]$
$\mathcal{K}$	$\mathbb{Q}$	$\mathbb{Q}[j]$	$\mathbb{Q}[\omega]$

Let $\theta \in \mathbb{C}$ be **algebraic** over $\mathcal{K}$, i.e., it is the root of some polynomial $P \in \mathcal{K}[X]$. If P can be chosen to be monic with coefficients in $\mathcal{R}$, the number θ is **integral** over $\mathcal{R}$.

$\mathbb{Q}$-Homomorphism: Let $\mathcal{L}$ and $\mathcal{L}'$ be two fields containing $\mathbb{Q}$. A mapping $\phi : \mathcal{L} \to \mathcal{L}'$ is called a $\mathbb{Q}$-homomorphism if for each $q \in \mathbb{Q}$, $\phi(q) = q$. If $\mathcal{L}' = \mathbb{C}$, the field of complex numbers, a $\mathbb{Q}$-homomorphism $\phi : \mathcal{L} \to \mathbb{C}$ is called an **embedding** of $\mathcal{L}$ into $\mathbb{C}$.

Some Results from Number Theory

Let θ be integral over $\mathcal{R}$ with $M_\theta \in \mathcal{K}[X]$ as the minimal polynomial of θ. Then, we have the following results

- If $n = \deg M_\theta$, the polynomial M_θ has n distinct roots in $\mathbb{C}$, say $\theta_1, \ldots, \theta_n$
- The extension field of $\mathcal{K}$ given by
$$\mathcal{K}[\theta] = \left\{ a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} \middle| (a_0, \ldots, a_{n-1}) \in \mathcal{K}^n \right\}$$
is an n-dimensional vector space over $\mathcal{K}$ with $(1, \theta, \ldots, \theta^{n-1})$ as a basis.
- There are exactly n embeddings of $\mathcal{K}$ in $\mathbb{C}$ which fix $\mathcal{K}$ pointwise. Each of them is uniquely determined by sending θ to one of its n conjugates, $\theta_1, \ldots, \theta_n$
$$\sigma_k : \mathcal{K}[\theta] \to \mathbb{C}$$
$$x = \sum_{i=0}^{n-1} x_i\theta^i \to \sigma_k(x) = \sum_{i=0}^{n-1} x_i\theta_k^i$$
- A mapping $\sigma : \mathcal{K}[\theta] \to \mathbb{C}^n$ can be obtained by sending each $\phi \in \mathcal{K}[\theta]$ in the n-tuple $\big(\sigma_1(\phi), \cdots, \sigma_n(\phi)\big)$. This mapping is an additive homomorphism with trivial kernel.
- The norm of $\phi \in \mathcal{K}[\theta]$ is defined as
$$N(\phi) \triangleq \sigma_1(\phi) \times \cdots \times \sigma_n(\phi)$$

These results can be used to prove the following theorem.

Theorem Let $\mathcal{O}$ be the set of elements of $\mathcal{K}[\theta]$ that are integral over $\mathcal{R}$. Then

(a) the norm of $\phi \in \mathcal{O} \setminus \{0\}$ is an element of $\mathcal{R} \setminus \{0\}$

(b) $\mathcal{O}$ is a ring, called the **number ring** of $\mathcal{K}[\theta]$ over $\mathcal{R}$.

(c) $\mathcal{O}$ is a free module of rank n over $\mathcal{R}$.

Cases of Interest

(1) When $\mathcal{R} = \mathbb{Z}$ and $\mathbb{Q}[\theta]$ is totally real, then σ ranges in $\mathbb{R}^n$ and the theorem implies

- the norm of $\phi \in \mathcal{O}\setminus\{0\}$ is a non-zero integer, hence
$$|N(\phi)| = |\sigma_1(\phi) \times \cdots \times \sigma_n(\phi)|$$
- $\sigma(\mathcal{O})$ is a free module of rank n over $\mathbb{Z}$, i.e., a lattice of $\mathbb{R}^n$, denoted by $\Lambda_\mathcal{O}$.

Combining these two results, we obtain that the embedding of the number ring of a totally real number field of degree n is $\mathcal{S}_r$-admissible and such lattices offer an nth-order diversity over Rayleigh fading channel.

(2) When $\mathcal{R} = \mathbb{Z}[j]$ or $\mathcal{R} = \mathbb{Z}[\omega]$, we have

- the norm of $\phi \in \mathcal{O} \setminus \{0\}$ belongs to $\mathcal{R}$ and it is nonzero, hence
$$|N(\phi)| = |\sigma_1(\phi) \times \cdots \times \sigma_n(\phi)|$$
- $\sigma(\mathcal{O})$, denoted by $\Lambda_\mathcal{O}$, is a free module of rank n over $\mathcal{R}$, i.e., it can be written as $\sigma(\mathcal{O}) = A\mathcal{R}^n$ where A is a $n \times n$ invertible matrix with complex coefficients, called a **generating matrix** of $\sigma(\mathcal{O})$.

Using these two results, we obtain a family of S_c-admissible lattices. A mapping $\mathbb{C}^n \to \mathbb{R}^{2n}$ is obtained by sending each n-tuple of complex numbers $(z_1, \cdots, z_n)$, where $z_i = x_i + jy_i$ to the $2n$-tuple of real numbers $(x_1, y_1, \cdots, x_n, y_n)$. Applying this mapping to $\Lambda_\mathcal{O}$ yields a lattice in $\mathbb{R}^{2n}$, denoted by $\Lambda_{\mathcal{O},r}$, having diversity of n over Rayleigh fading channel.

5 Lattices for the Fading Channel adapted to the Gaussian Channel

The embedding $\Lambda_{\mathcal{O}}$ of the number ring $\mathcal{O}$ of a totally real number field yields an $\mathcal{S}_r$-admissible lattice with low sphere packing density and thus is not useful over a AWGN channel. Giraud *et al.* in [2] have shown that one can characterize **equivalent lattices** on the Rayleigh fading channel and inside the set of lattices equivalent to $\Lambda_{\mathcal{O}}$ find the lattice with maximal sphere packing density. We briefly describe the method below.

Equivalent Lattices on Rayleigh Fading Channel

Let Λ be an $\mathcal{S}_r$-admissible lattice with generating matrix $A = [a_{i,j}]$. Let $\pi_n : (x_1, \ldots, x_n) \to x_1 \times \cdots x_n$ be the multiplicative norm. The generating matrix A determines a homogeneous form of degree n defined by

$$\pi_n \circ A : (x_1, \ldots, x_n) \to \prod_{i=1}^{n} (a_{i,1}x_1 + \cdots + a_{i,n}x_n)$$

We say that two lattices Λ_1 and Λ_2 are equivalent and denote it by $\Lambda_1 \sim \Lambda_2$ if there exists a generating matrix A_1 of Λ_1 and A_2 of Λ_2 such that $\pi_n \circ A_1 = \pi_n \circ A_2$.

One can show

Lemma Let $A = [a_{i,j}]$ be a $n \times n$ invertible matrix such that $\pi_n \circ A = \pi_n \circ \mathbf{I}_n$. Then $A = DP$ where D is a diagonal matrix with determinant 1 and P s a permutation matrix.

Theorem Two equivalent lattices Λ_1 and Λ_2 have the same performance over Rayleigh fading channel at high SNR.

Performance over Gaussian Channel

Since we wish to improve the properties of $\Lambda_{\mathcal{O}}$ over the Gaussian channel, we need to determine the densest lattice with respect to the Euclidean distance inside the set $\mathcal{E}$ of lattices equivalent to $\Lambda_{\mathcal{O}}$. The ratio

$$\gamma_2(\Lambda) \triangleq \frac{d^2_{\min}(\Lambda)}{\det{(\Lambda)}^{2/n}}$$

which gives the *length* of the spheres, can be used to evaluate the sphere packing density of a lattice Λ. One can show that the lattice $\Lambda_{\mathcal{O}}$ has the lowest sphere packing density inside $\mathcal{E}$, i.e. the sphere packing density of $\Lambda \in \mathcal{E}$ is lower-bounded by $\gamma_2(\Lambda_{\mathcal{O}})$.

Finding the densest sphere packing in $\mathcal{E}$

We now describe the method to find the densest sphere packing lattice from the class of equivalent lattices.

Let G denote the generating matrix of $\Lambda_{\mathcal{O}}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $D_{\boldsymbol{\lambda}} = \text{diag}(\lambda_1, \ldots, \lambda_n)$. We need to find $\boldsymbol{\lambda}$ such that

(i) $\prod_{i=1}^{n} \lambda_i = 1$ and

(ii) $d^2_{\min}\big(D_{\boldsymbol{\lambda}} G \mathbb{Z}^n\big)$ is as large as possible

i.e.,

$$\boxed{\boldsymbol{\lambda} = \arg\max_{N(\boldsymbol{\lambda})=1} d^2_{\min}\big(D_{\boldsymbol{\lambda}} G \mathbb{Z}^n\big)} \tag{7}$$

How the algorithm works? We first find a solution $\Lambda_{\mathcal{O}}$ for the critical lattice for Rayleigh channel. Then we move each point of the lattice along the hyperbole $abs\prod x_i = 1$, which is the boundary of the critical lattice so this rotation does not effect the performance over Rayleigh channel (this also explains why the equivalent lattices have same performance). Then from this equivalence class, we find the lattice with the maximum minimum-distance. As all the lattices have the same volume, this lattice has the densest packing, giving the best performance over Gaussian channel.

Why rotation improves performance?

In this section we give an intuitive reasoning, as explained in [3], of why rotation of a constellation improves performance over a fading channel. They define a new type of diversity called **signal space diversity** with the diversity order as the minimum number of distinct components between any two constellation points. In other words, the diversity order is the minimum Hamming distance between any two coordinate vectors of constellation points.

A key point to increase signal space diversity is to apply a certain rotation to a classical signal constellation in such a way that any two points achieve the

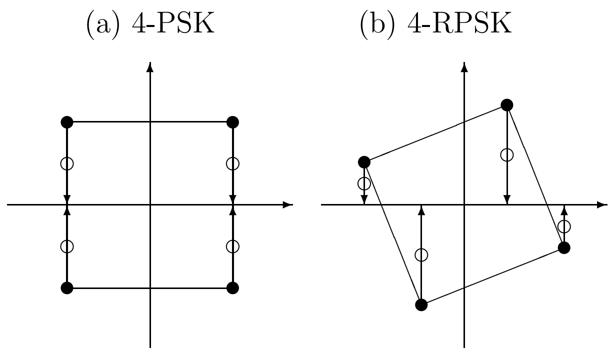

Figure 3: How rotation increases diversity (a) L = 1 (b) L = 2

maximum number of distinct components. Consider a 4-PSK constellation and its rotation as shown in Figure 3. Now suppose a deep fade hits one of the components of the transmitted signal vector, then we can see that the 'compressed' constellation (empty circles) in (b) offers more protection against the effects of noise since no two points collapse to the same point, as would happen in (a). This is the basis of the improved diversity. Note that the shape of the critical lattice for Rayleigh channel, shown in Figure 2 also implies that we should avoid the axis while choosing the lattice, which means that we have to look at some rotation of the lattice.

An Example

Consider the case of interleaving over components so that $\mathcal{R} = \mathbb{Z}$. Next we choose a $\theta \in \mathcal{R}$ which is integral over $\mathcal{R}$. $\theta = \sqrt{2}$ is one such choice with the minimal polynomial $M_\theta(x) = x^2 - 2$ whose roots are $(\sqrt{2}, -\sqrt{2})$. The extension field of $\mathcal{K} = \mathbb{Q}$ is

$$\mathbb{Q}[\sqrt{2}] = \left\{ a + b\sqrt{2} \;\middle|\; (a, b) \in \mathbb{Q}^2 \right\}$$

The ring of algebraic integers of $\mathbb{Q}[\sqrt{2}]$ is $\mathcal{O}_{\sqrt{2}} = \mathbb{Z} \oplus \sqrt{2}\mathbb{Z}$ and the embedding of $\mathbb{Q}[\sqrt{2}]$ in $\mathbb{R}^2$ is

$$\sigma : (a + b\sqrt{2}) \to (a + b\sqrt{2},\ a - b\sqrt{2})$$

Hence, the lattice $\Lambda_{\mathcal{O}_{\sqrt{2}}}$ is generated by $\sigma(1)$ and $\sigma(\sqrt{2})$, i.e.,

$$\Lambda_{\mathcal{O}_{\sqrt{2}}} = G\mathbb{Z}^2 \quad \text{with } G = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix}$$

The sphere packing density for this lattice, $\gamma_2(\Lambda_{\mathcal{O}_{\sqrt{2}}}) = {}^1/\sqrt{2}$. Now we multiply the lattice with $D_{\boldsymbol{\lambda}} = \text{diag}(\lambda, 1/\lambda)$, so the points are moved along the curves $x \times y = c$. The maximum sphere packing is obtained by

$$\lambda \in \left\{ (1 + \sqrt(2)^{n+1/2} \;\middle|\; n \in \mathbb{Z} \right\}$$

and is $\gamma_{2,\max} = 1$. These values yield two lattices congruent to $\mathbb{Z}^2$ (see Figure 4). Thus we have

$$\left(\mathbb{Z}^2\right)_2 = U_2\mathbb{Z}^2$$

where

$$U_2 = 2^{3/4} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and } \alpha = \pm\frac{\pi}{8}$$

6 Conclusion

The authors in [1] and [2] have proposed schemes to construct families of lattices that achieve good performance over both Gaussian and Rayleigh channels, with high spectral efficiency. Using these schemes we can find good lattices for an arbitrary signal space diversity. The advantage of this type of diversity is that it is traded only for a higher demodulator complexity; no additional power or bandwidth is required. However they do not have any fast and efficient decoding algorithm, so increasing the number of dimensions slows down the demodulation operation. Thus the utility of these schemes is limited by the ratio of system complexity over the practical gain. Also it needs to be analyzed how the performance is affected by imperfect CSI estimates.

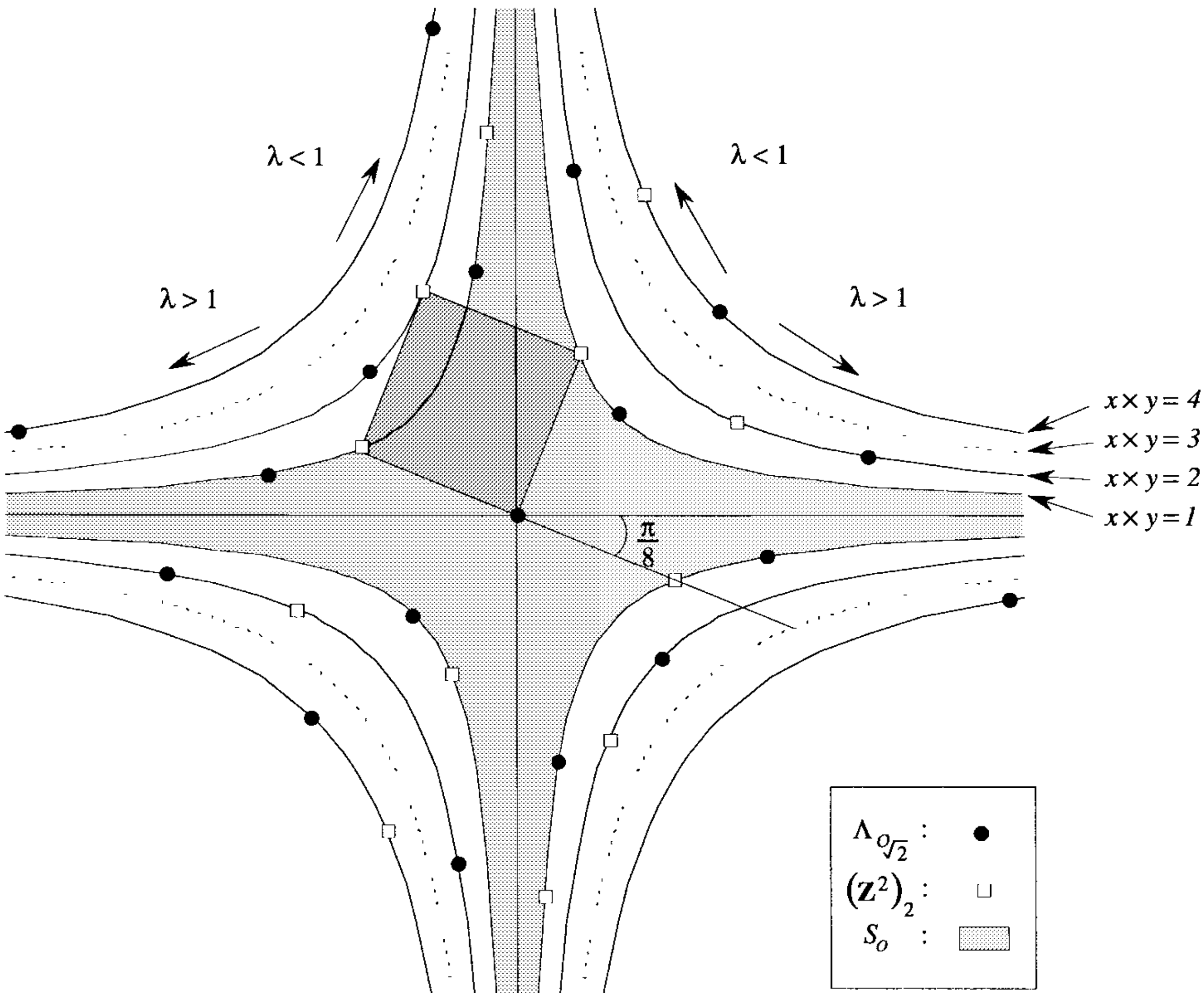

Figure 4: $\Lambda_{\mathcal{O}_{\sqrt{2}}}$ and the effect of multiplication by D_{λ}