Multiple Access Channels with Feedback

An Achievable Region

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Outline

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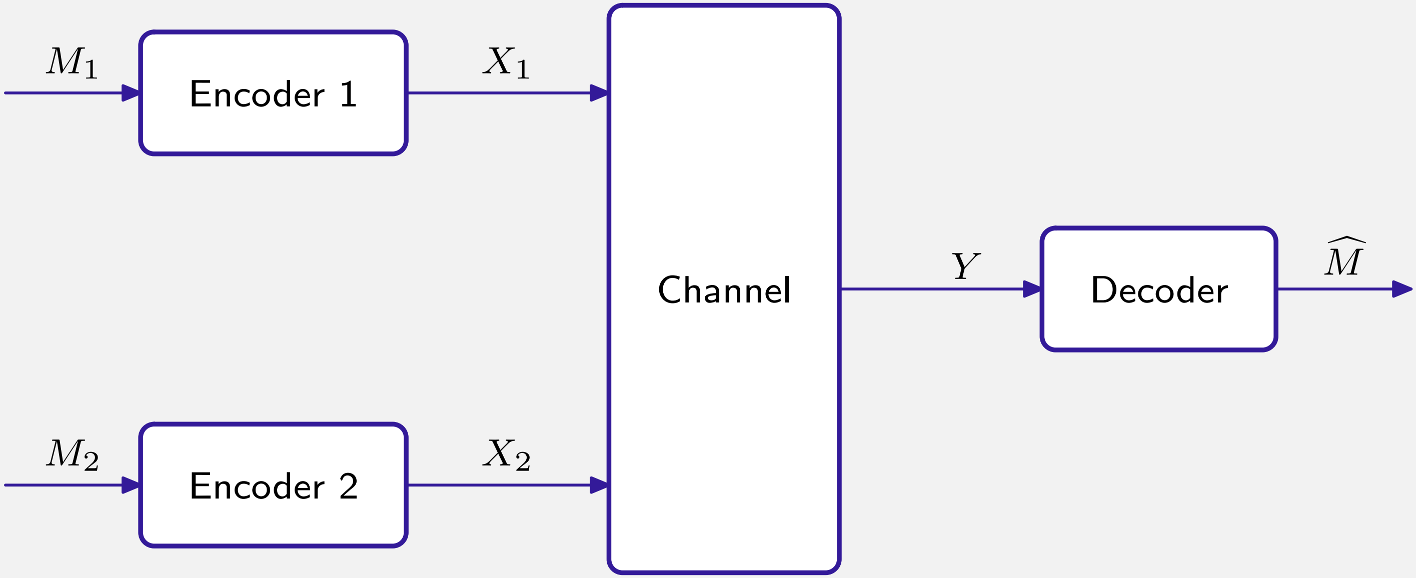
MAC Channel without feedback



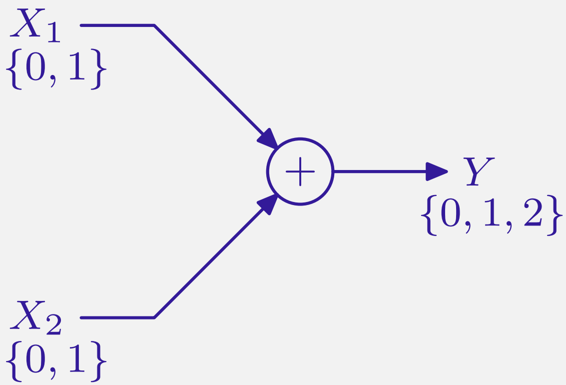
Let $\mathcal{R} = \bigcup_{P_{X_1}, P_{X_2}} \left\{ (R_1, R_2) : \begin{array}{r} R_1 < I(X_1; Y \mid X_2) \\ R_2 < I(X_2; Y \mid X_1) \\ R_1 + R_2 < I(X_1, X_2; Y) \end{array} \right\}$

➜ The capacity region is $\mathcal{R}^*$, the convex hull of $\mathcal{R}$.

(Ahlswede 1971, Liao 1972)

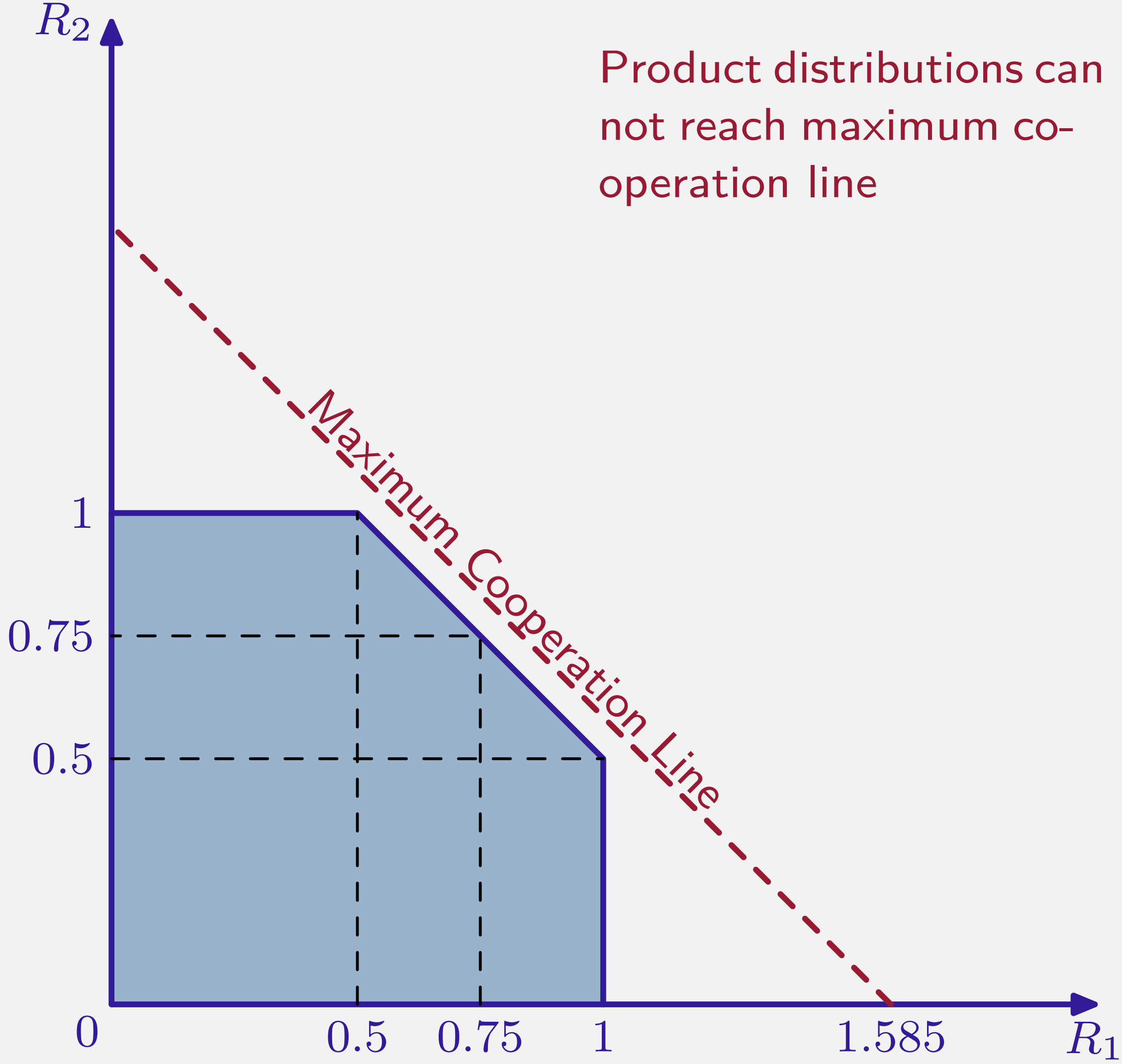
Example

Noiseless Binary Erasure Channel

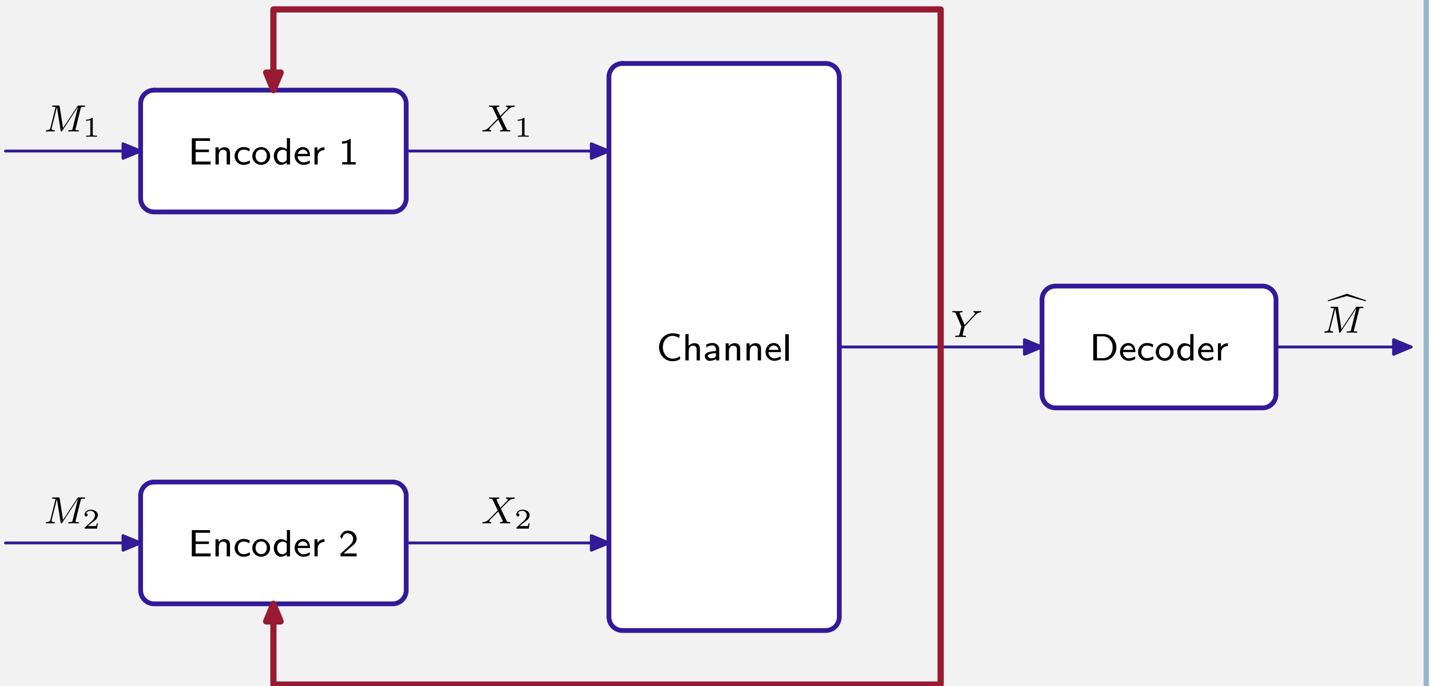


$$Y = X_1 + X_2$$

$$\mathcal{R}^* = \left\{ (R_1, R_2) : \begin{array}{r} R_1 < 1 \\ R_2 < 1 \\ R_1 + R_2 < 1.5 \end{array} \right\}$$


Product distributions can
not reach maximum co-
operation line
Maximum Cooperation Line
R_2
R_1
1
0.75
0.5
0
0.5
0.75
1
1.585

MAC Channel with feedback



Message

M_1 and M_2 ...independent and equally likely

Encoder

$$X_{1,n} = f_{1,n}(M_1, X_1^{n-1}, Y^{n-1})$$
$$X_{2,n} = f_{2,n}(M_2, X_2^{n-1}, Y^{n-1})$$

Discrete Memoryless Channel

$$P(y_n \mid x_1^n, x_2^n, y^{n-1}) = P(y_n \mid x_{1,n}, x_{2,n})$$

Error Event

$$P_e = P\big(\widehat{M} \neq (M_1, M_2)\big)$$

Main Idea

- Encoders can communicate using the feedback channel

Can this be used to increase capacity?

- Product distributions do not reach maximum cooperation limits
- Joint distribution need to be chosen
- Encoders can use the feedback channel to communicate with one another
- Step 1: Encoders communicate thier message to each other.
- Step 2: Communicate to decoder and maximum cooperation limit.

Example – Noiseless BEC

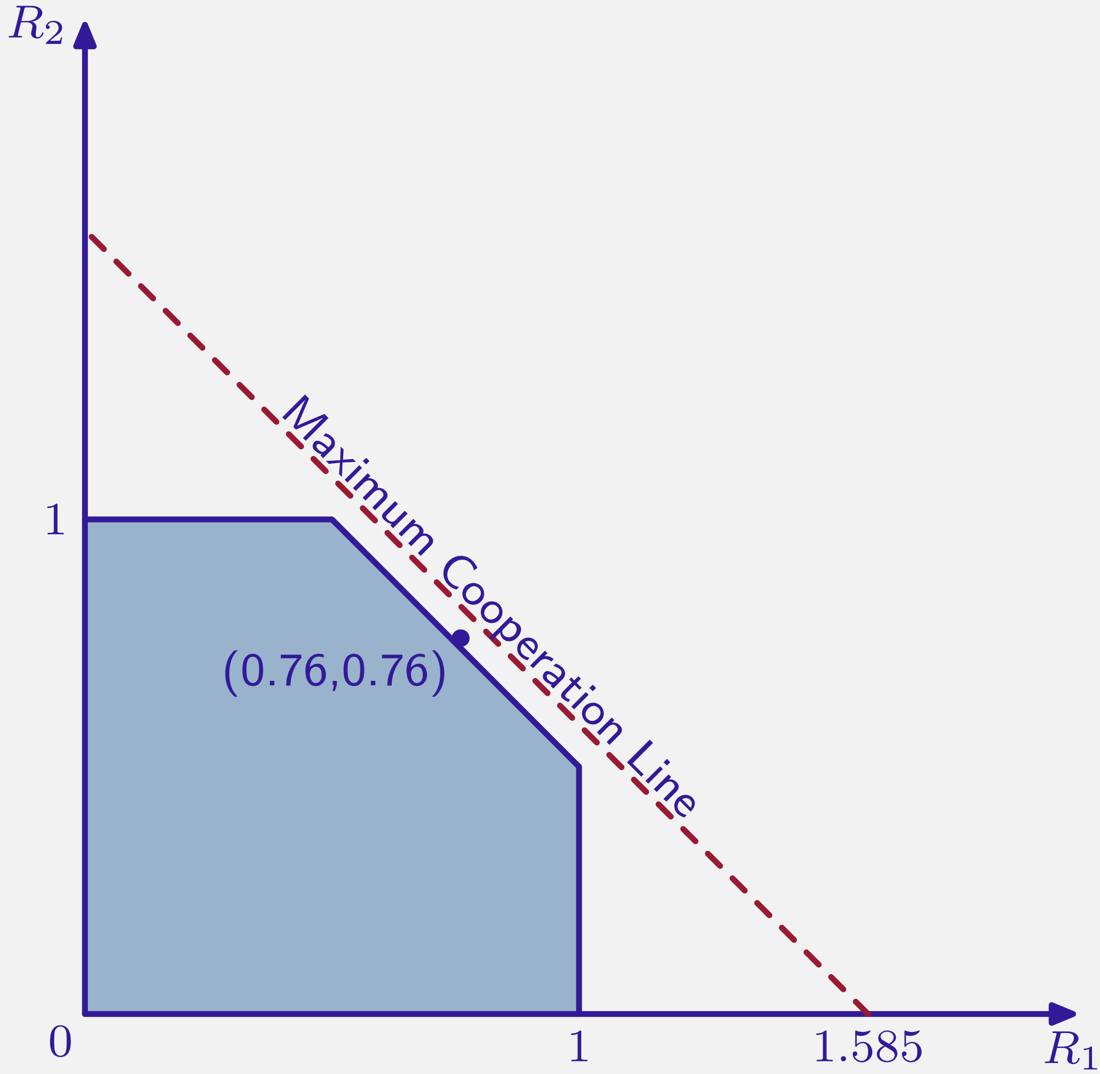
- Both encoders use first m bits to communicate message M_i.
- Decoder: $m/2$ decoded correctly, $m/2$ ambiguous.
- Feedback Channel: M_1, M_2 known at both encoders.
- Use cooperative encoding to for next $n = (\frac{m}{2}) \log 3$ channel uses.
- Rate: $R_1 = R_2$ and $R_1 + R_2 = \frac{(m+m)}{\text{m} + \frac{m}{2}\log 3} = 1.52037$.

First Stage (m)

Second Stage ($\frac{m}{2}\log 3$)

(0.76,0.76) is achievable.

(Gaarder and Wolf 1975)



R_2
1
Maximum Cooperation Line
(0.76,0.76)
0
1
1.585
R_1

Feedback can increase capacity!

➜ Can we do better?

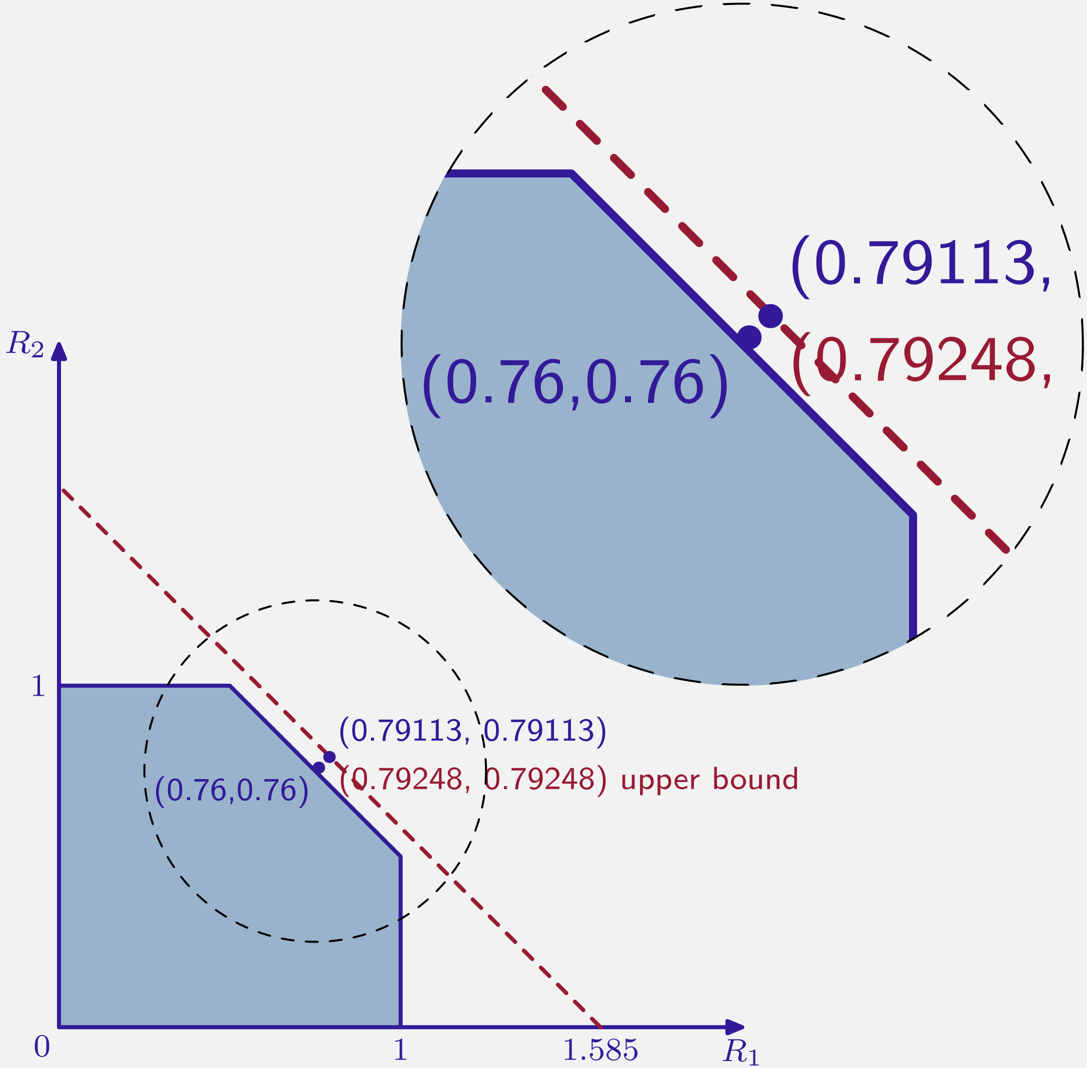
Cover and Leung Achievable Region

Let $\mathcal{R} = \bigcup_{P_{X_1}, P_{X_2}, \mathcal{U}} \left\{ (R_1, R_2) : \begin{array}{r} R_1 < I(X_1; Y \mid X_2, U) \\ R_2 < I(X_2; Y \mid X_1, U) \\ R_1 + R_2 < I(X_1, X_2; Y) \end{array} \right\}$

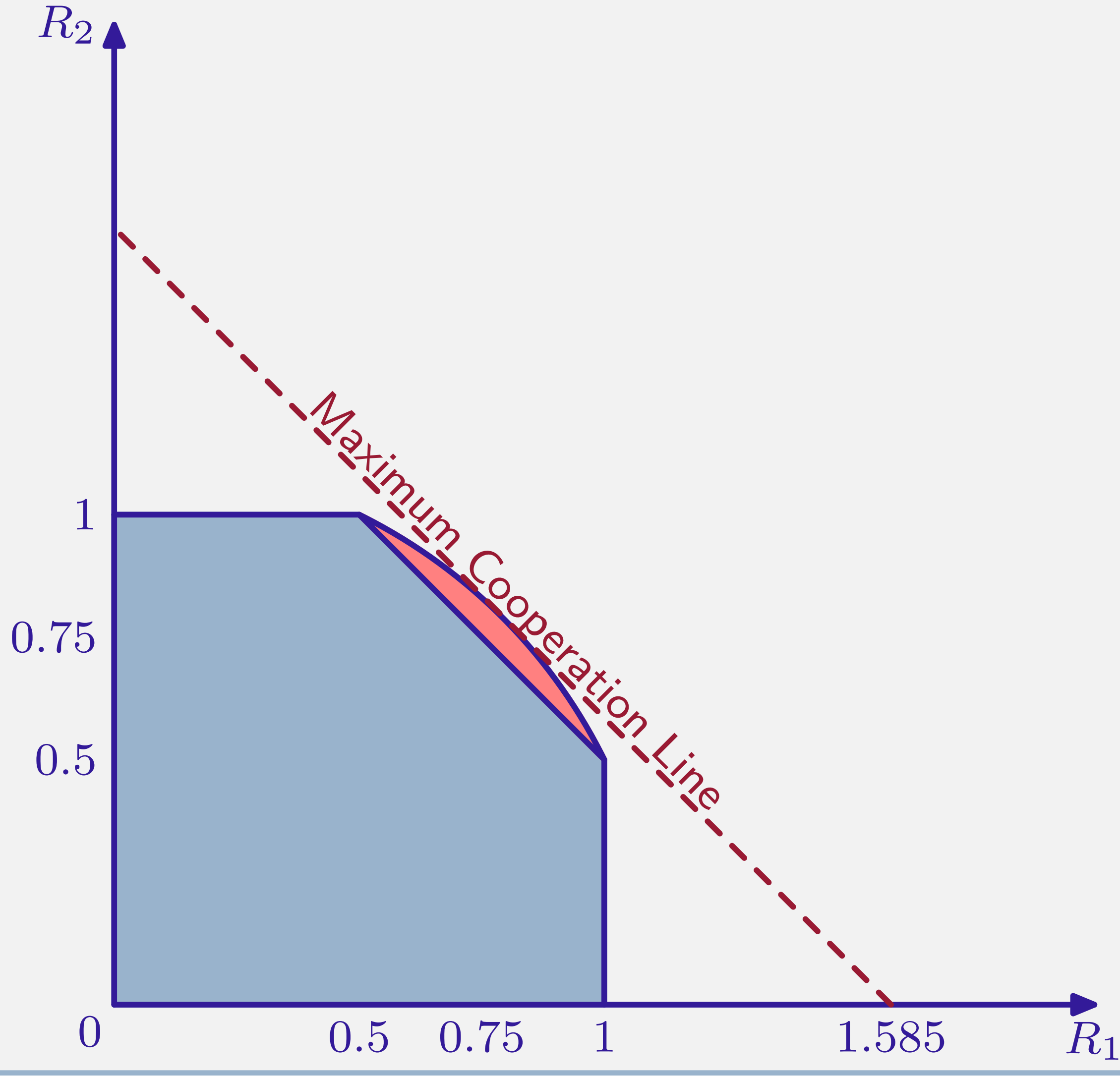
$\|\mathcal{U}\| \leq \min\{\|X_1\| \cdot \|X_2\|, \|Y\|\}.$

$\mathcal{R}^*$, the convex hull of $\mathcal{R}$ is achievable

(Cover and Leung 1981)



(0.79113,
(0.79248,
(0.76,0.76)
R_2
1
(0.79113, 0.79113)
(0.79248, 0.79248) upper bound
(0.76,0.76)
0
1
1.585
R_1



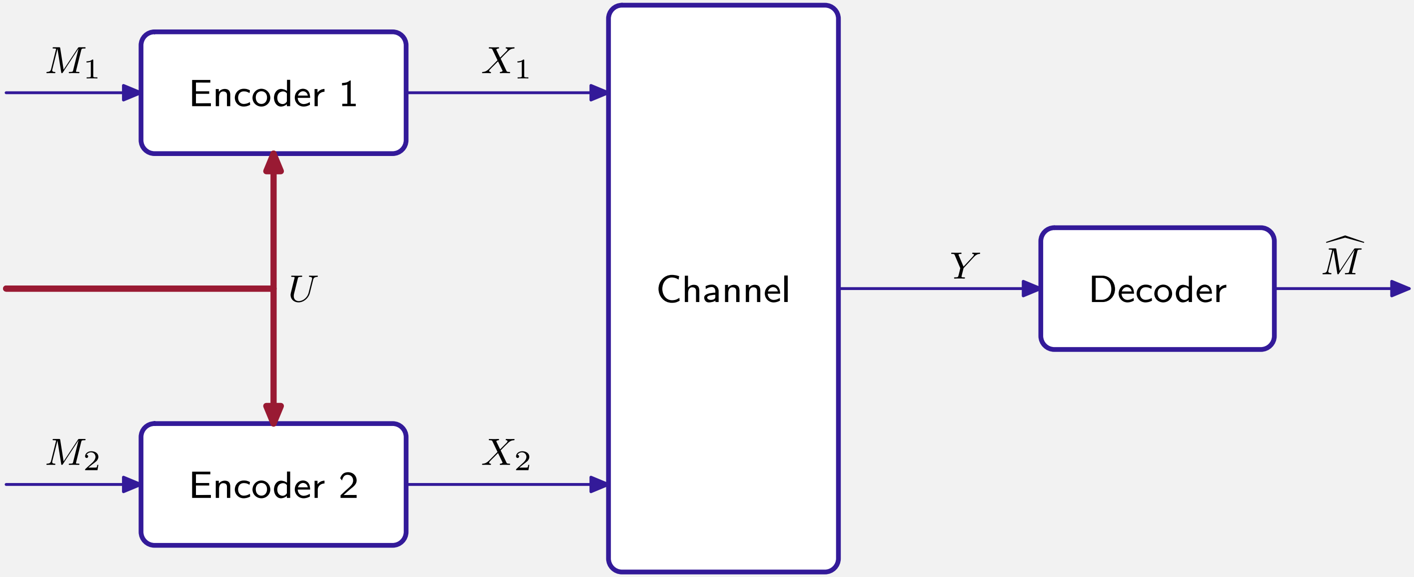
R_2
1
0.75
0.5
0
0.5
0.75
1
1.585
R_1
Maximum Cooperation Line

Main Idea of Cover and Leung

Combine

- MAC with correlated sources (Slepian Wolf)
- Block Stationary Coding and Sequential Decoding
- Superposition

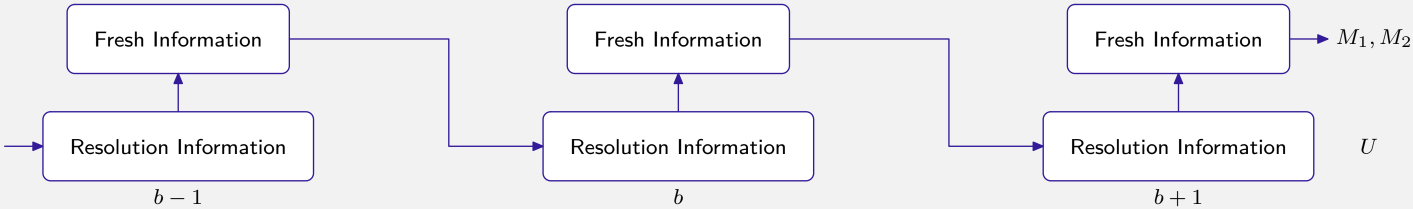
Main Idea of Cover and Leung



MAC with correlated sources

(Slepian Wolf 1973)

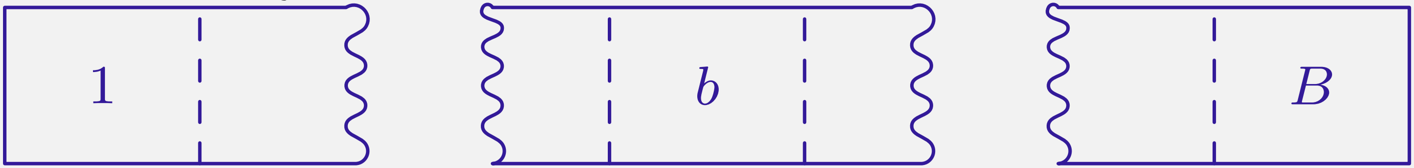
Main Idea of Cover and Leung



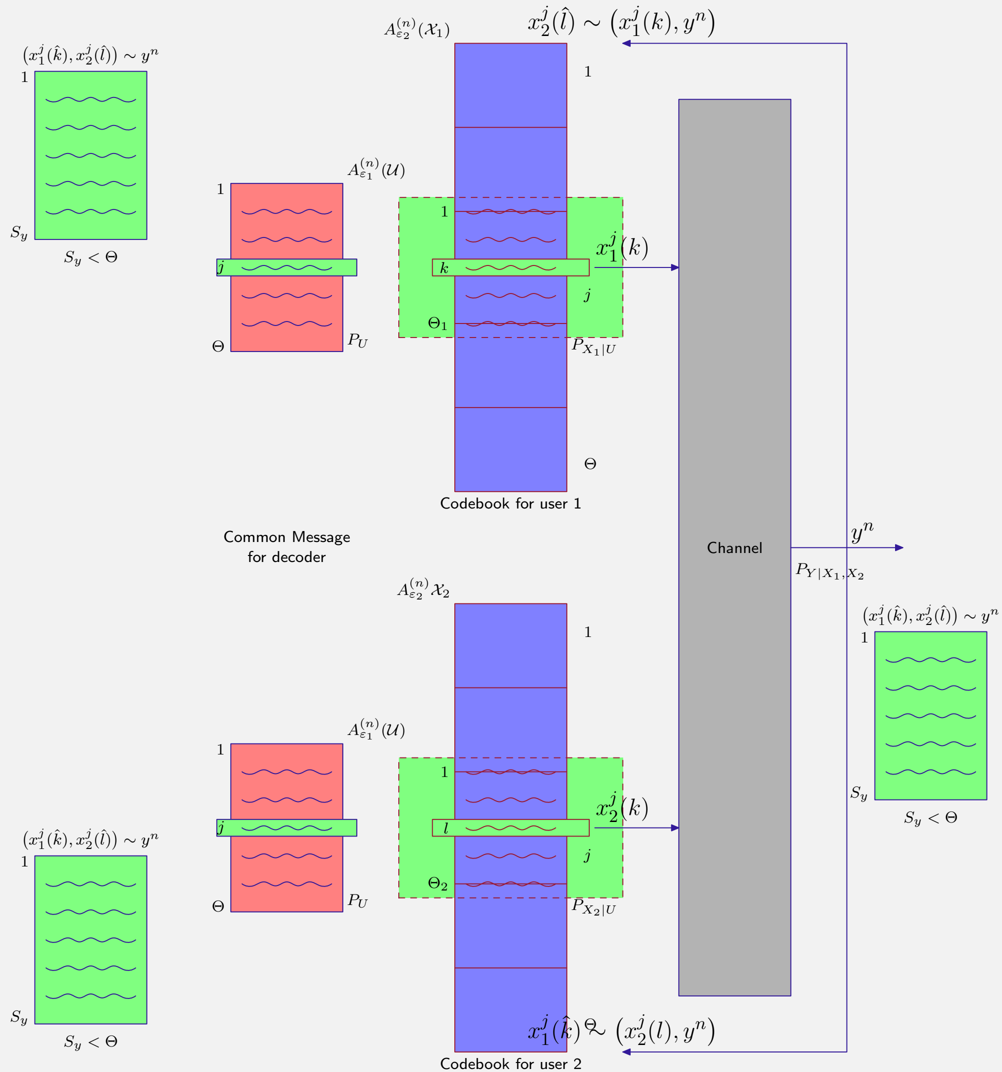
- transmit message to other encoder, overshoot for decoder
- introduce U for resolution information and send this in the next block
- Superposition block Markov coding

Coding Scheme

Block Stationary Scheme



- Generate a resolution codebook P_U
- Use binning to generate conditionally typical codebooks
- Keep track of the receiver's interpretation.



$(x_1^j(\hat{k}), x_2^j(\hat{l})) \sim y^n$
1
S_y
$S_y < \Theta$
$A_{\varepsilon_1}^{(n)}(\mathcal{U})$
1
j
Θ
P_U
$A_{\varepsilon_2}^{(n)}(\mathcal{X}_1)$
$x_2^j(\hat{l}) \sim (x_1^j(k), y^n)$
1
1
k
j
Θ_1
$P_{X_1|U}$
$x_1^j(k)$
Θ
Codebook for user 1
Common Message
for decoder
Channel
y^n
$P_{Y|X_1,X_2}$
$A_{\varepsilon_2}^{(n)}\mathcal{X}_2$
1
$(x_1^j(\hat{k}), x_2^j(\hat{l})) \sim y^n$
1
S_y
$S_y < \Theta$
$A_{\varepsilon_1}^{(n)}(\mathcal{U})$
1
j
Θ
P_U
1
l
j
Θ_2
$P_{X_2|U}$
$x_2^j(k)$
$(x_1^j(\hat{k}), x_2^j(\hat{l})) \sim y^n$
1
S_y
$S_y < \Theta$
$x_1^j(\hat{k}) \sim (x_2^j(l), y^n)$
Θ
Codebook for user 2

At Encoders

$$x_2^j(\hat{l}) \sim \left(x_1^j(k), y^n\right)$$

$$P_{X_1,X_2|Y} = P_{X_1|U}P_{X_2|U} \qquad P_U P_{X_1|U} P_{X_2|U} P_{Y|X_1,X_2}$$

$$\frac{1}{n}\log\Theta_1 < I(X_2;Y \mid X_1,U)$$
$$\frac{1}{n}\log\Theta_2 < I(X_1;Y \mid X_2,U)$$

At Decoder

There are two superimposed messages

$$u^n \sim y^n \qquad \frac{1}{n}\log\Theta < I(U;Y)$$

$$\|S_Y\| \qquad \frac{1}{n}\log\|S_Y\| > \frac{1}{n}\log\Theta_1 + \frac{1}{n}\log\Theta_2 - I(X_1,X_2;Y \mid U)$$

$$\|S_Y\| < \Theta, \ \therefore \frac{1}{n}\log\Theta_1 + \frac{1}{n}\log\Theta_2 < I(X_1,X_2;Y \mid U) + I(U;Y) = I(X_1,X_2;Y)$$

Bounding the Probability of Error

Error Events

$$
\begin{aligned}
E_1 &= \text{Tx 1 decodes } \hat{l} \text{ incorrectly} && < \tfrac{\varepsilon}{5\mathrm{B}} \\
E_2 &= \text{Tx 2 decodes } \hat{k} \text{ incorrectly} && < \tfrac{\varepsilon}{5\mathrm{B}} \\
E_3 &= \hat{j} \text{ decoded incorrectly at decoder} && < \tfrac{\varepsilon}{5\mathrm{B}} \\
E_4 &= \big(x_1^j(k), x_2^j(l)\big) \text{ and } y^n \text{ not jointly typical} && < \tfrac{\varepsilon}{5\mathrm{B}} \\
E_5 &= \|S_Y\| > \exp_2 n(R_1 + R_2 - I(X_1, X_2; Y \mid U)) && < \tfrac{\varepsilon}{5\mathrm{B}}
\end{aligned}
$$

$$
\begin{aligned}
P_{e,b} &\leq P(E_1) + \cdots + P(E_5) \\
&\leq \frac{\varepsilon}{5B} + \cdots + \frac{\varepsilon}{5B} = \frac{\varepsilon}{B} \\
P_e &\leq P_{e,1} + \cdots + P_{e,b} + \cdots P_{e,B} < \varepsilon
\end{aligned}
$$

Converse

In General CL Region is not tight

- Gaussian Channels .... (Ozarow 1984)
- Poisson Channels .... (Bross and Lapidoth 2005)

CL Region tight for

- Binary Erasure MAC .... (Willems 1984)
- $H(X_1 \mid Y, X_2) = 0$ .... (Willems 1982)
- Partial Feedback .... (Willems 1983)

Conclusion

- Feedback increases capacity of MAC channel
- Encoders use feedback channel to communicate
- New Idea of Block Markov Sequential Coding
- Not tight!

References

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- N. T. Gaarder and J. K. Wolf, "capacity region of a Multiple Access Discrete Memoryless Channel can increase with feedback", IT, Jan 1975.
- F. M. J. Willems "Theoretic Results for the Discrete Memoryless Multiple Access Channel", Ph.D. thesis, Oct 1982.
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- S. I. Bross and A. Lapidoth "An Improved Achievable Region for the Discrete Memoryless Two-User Multiple-Access Channel with Noiseless Feedback", IT, March 2005.