# *Optimal Real-time communication*

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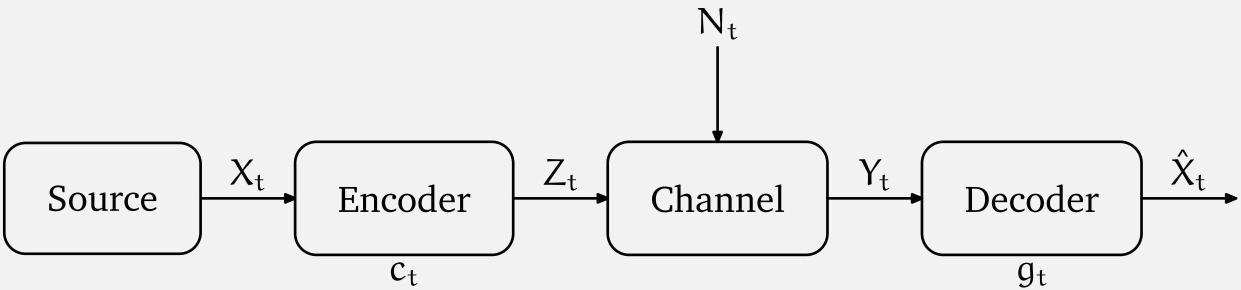
# Motivation

**Real-Time Communication** Communication systems in which information should be transmitted and decoded within a fixed delay constraint.

- *Applications*
  - Sensor networks
  - QoS over communication networks
  - Vehicular traffic control
  - Surveillance networks
  - Networked controlled systems
- *Features*
  - Informationally decentralized systems
  - Communication is delay sensitive
  - Channels may be noisy

# Model




**Encoder** $Z_t = c_t(X_1, \dots, X_t)$

**Decoder** $\hat{X}_t = g_t(Y_1, \dots, Y_t)$

**Delay** $\delta$

**Distortion** $\rho(X_{t-\delta}, \hat{X}_t)$

**Total Cost** $\mathbf{E}\left\{ \sum_{t=\delta+1}^{T} \rho(X_{t-\delta}, \hat{X}_t) \,\middle|\, c_1, \dots, c_T, g_1, \dots, g_T \right\}$

# Model

## Salient Features

- *Sequential operation*

$$\cdots \to \text{encoder at } t \to \text{decoder at } t \to \text{encoder at } t+1 \to \text{decoder at } t+1 \to \cdots$$

- *Decentralized information*

$$\underbrace{\sigma(X_1, X_2, \dots, X_t)}_{\text{info. at encoder}} \quad \begin{matrix} \not\subseteq \\ \not\supseteq \end{matrix} \quad \underbrace{\sigma(Y_1, Y_2, \dots, Y_t)}_{\text{info. at decoder}}$$

*Non-classical information structure*

# Comparison with Info Thy

| | Shannon Formulation | | Real-time communication |
|---|---|---|---|
| Encoder | $Z_t = c_t(X_1, \ldots X_{\color{red}T})$ | | $Z_t = c_t(X_1, \ldots X_{\color{red}t})$ |
| Decoder | $\hat{X}_t = g_t(Y_1, \ldots Y_{\color{red}T})$ | | $\hat{X}_t = g_t(Y_1, \ldots Y_{\color{red}t})$ |
| Distortion | $\rho(X^T, \hat{X}^T) = \sum_{t=1}^{T} \rho(X_t, \hat{X}_t)$ | | $\rho(X^T, \hat{X}^T) = \sum_{t=\delta+1}^{T} \rho(X_{t-\delta}, \hat{X}_t)$ |

- *Information theoretic results are not applicable*
  - Cannot use asymptotic equipartition theorem.
  - No concentration of measure on typical sequences.
  - Separate source channel coding is not optimal
- *Asymptotic concepts not appropriate*
  - Source entropy
  - Transmission rate
  - Channel capacity

# Our Approach

*Formulate the real-time communication problem as a stochastic optimization problem*

# Stochastic Opt --- MDP

- *Markov decision theory*
  - Classical methodology for solving stochastic optimization problems
- *Assumption: Centralized system*
  - One controller
  - Perfect recall at the controller
- *Real-time communication*
  - Has two "controllers"
  - Decentralized information

*Markov decision theory is not applicable to real-time communication problem*

# Real-time communication

## Conceptual difficulties with real-time comm

- Information theory does not apply
- Markov decision theory does not apply
- Brute force search is computationally very difficult

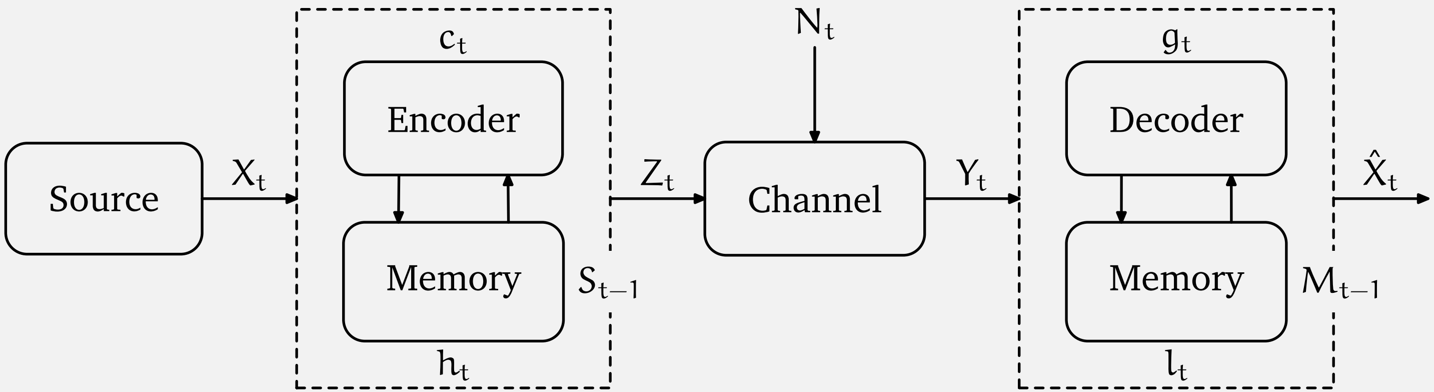
## Our Contribution

- *Provide sequential decomposition*
  - Break one shot optimization problem into sequence of nested optimization problems
  - Exponential reduction in the search complexity

$$O\left((2^{A})^{T}\right) \longrightarrow O(T \cdot K \cdot 2^{A})$$

## *Solution Methodology: An Example*

# Example




**Encoder**

$$Z_t = c_t(X_t, S_{t-1})$$

$$S_t = h_t(X_t, S_{t-1})$$

**Decoder**

$$\hat{X}_t = g_t(Y_t, M_{t-1})$$

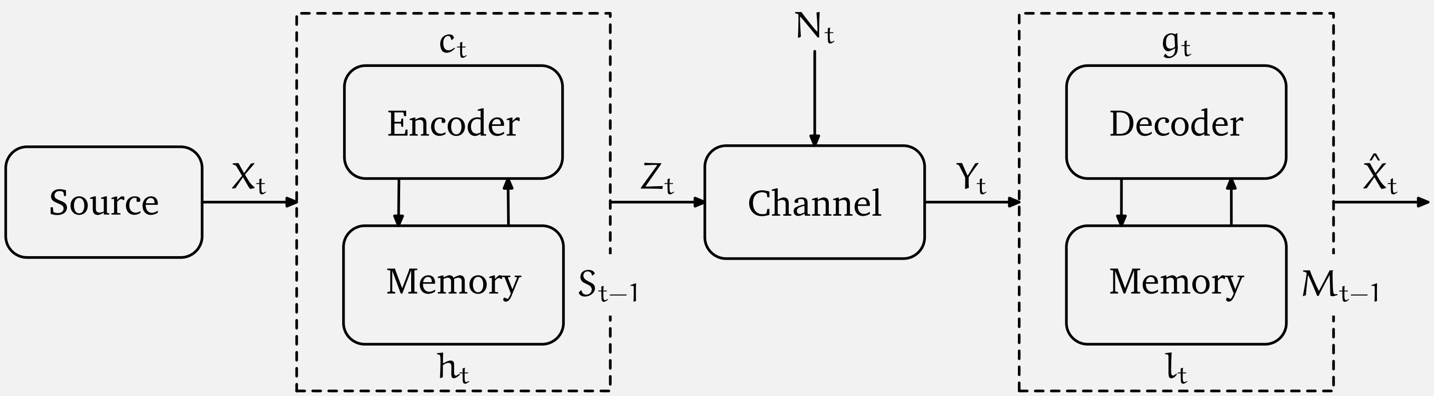
$$M_t = l_t(Y_t, M_{t-1})$$

$$T = 10$$

$$\delta = 1$$

**Distortion** $\rho(X_{t-1}, \hat{X}_t) =$ hamming distortion

# Example




**Communication Scheme**

$$C \coloneqq (c_1, \dots, c_{10}) \qquad H \coloneqq (h_1, \dots, h_{10})$$

$$G \coloneqq (g_1, \dots, g_{10}) \qquad L \coloneqq (l_1, \dots, l_{10})$$

**Performance**

$$\mathcal{J}(C, H, G, L) \coloneqq \mathbf{E}\left\{ \sum_{t=2}^{10} \rho(X_{t-1}, \hat{X}_t) \,\middle|\, C, H, G, L \right\}$$

# Brute Force Approach

- Fix a communication scheme $(C, H, G, L)$
- Evaluate $\Pr\left(X_1, \dots, X_{10}, \hat{X}_1, \dots, \hat{X}_{10} \,\middle|\, C, H, G, L\right)$
- Evaluate $\mathbf{E}\left\{\sum_{t=2}^{10} \rho(X_{t-1}, \hat{X}_t) \,\middle|\, C, H, G, L\right\}$
- Repeat for all choices of $(C, H, G, L)$
- Pick the scheme with best performance

## Complexity

- Possible choices for $c_t = 2^{2\times 2} = 16.$
- Possible choices for $(c_t, h_t, g_t, l_t) = 16^4 = 65,536$
- Possible choices for $(C, H, G, L) = (16^4)^{10} \approx 1.5 \times 10^{48}$

*Recall, this is for a "simple" example.*

*Solution Approach: Sequential Decomposition*

*Key Idea: Information state*

# Information State

## Requirements on information state ($\pi_t$)

- *$\pi_t$ should be a "state"*
  - $\pi_t = \text{fn}(c^T_{t-1}, h^T_{t-1}, g^T_{t-1}, l^T_{t-1})$
  - $\pi_{t+1} = \text{fn}(\pi_t, c_t, h_t, g_t, l_t)$
- *$\pi_t$ should absorb the effect of past functions on future performance*
  - $\mathbf{E}\left\{\rho(X_{t-1}, \hat{X}_t) \,\middle|\, c^t_1, h^t_1, g^t_1, l^t_1\right\} = \mathbf{E}\left\{\rho(X_{t-1}, \hat{X}_t) \,\middle|\, \pi_t, c_t, h_t, g_t, l_t\right\}$

## Sequential Decomposition

$$V_t(\pi_t) = \min_{\gamma_t}\left\{\hat{\rho}(\pi_t, \gamma_t) + V_{t+1}\big(\pi_{t+1}(\pi_t, \gamma_t)\big)\right\}$$

$$\mathcal{J}^* = V_1(\pi_1)$$

where $\gamma_t = (c_t, h_t, g_t, l_t)$.

*Identifying appropriate information states is highly non-trivial*

# Seq. Decomposition

## Information State

$$\pi_t = \Pr\left(X_t, S_{t-1}, Y_{t-1}, M_{t-1}\right)$$

$$\pi_t \in \Delta(\mathcal{X} \times \mathcal{S} \times \mathcal{Y} \times \mathcal{M})$$

$$\begin{tikzcd}
{} \arrow[r] & \pi_t \arrow[r, "(c_t, h_t, l_t)"] \arrow[d, "(c_t, g_t)"] & \pi_{t+1} \\
& \mathbf{E}\left\{ \rho(X_{t-1}, \hat{X}_t) \,\middle|\, \pi_t, c_t, g_t \right\} &
\end{tikzcd}$$

## Refine time

$$^1\pi_t = \Pr\left(X_t, S_{t-1}, Y_{t-1}, M_{t-1}\right)$$

$$^2\pi_t = \Pr\left(X_t, S_{t-1}, Y_t\phantom{{}_{-1}}, M_{t-1}\right)$$

$$^3\pi_t = \Pr\left(X_t, S_t\phantom{{}_{-1}}, Y_t\phantom{{}_{-1}}, M_{t-1}\right)$$

$$\begin{tikzcd}
{^1\pi_t} \arrow[r, "c_t"] & {^2\pi_t} \arrow[r, "h_t"] & {^3\pi_t} \arrow[r, "l_t"] \arrow[d, "g_t"] & {^1\pi_{t+1}} \\
& & \mathbf{E}\left\{ \rho(X_{t-1}, \hat{X}_t) \,\middle|\, {^3\pi_t}, g_t \right\} &
\end{tikzcd}$$

# Seq. Decomposition

(Initialization) $${}^3V_{T+1}({}^3\pi) = 0$$

(Recursion) $$\begin{cases} {}^1V_t({}^1\pi_t) = \min\limits_{c_t} {}^2V_t\big({}^2\pi_t({}^1\pi_t, c_t)\big) \\ {}^2V_t({}^2\pi_t) = \min\limits_{h_t} {}^3V_t\big({}^3\pi_t({}^2\pi_t, h_t)\big) \\ {}^3V_t({}^3\pi_t) = \min\limits_{g_t} \hat{\rho}({}^3\pi_t, g_t) + \min\limits_{l_t} {}^1V_{t+1}\big({}^4\pi_{t+1}({}^3\pi_t, l_t)\big) \end{cases}$$

- *Functional optimization problem*
  - Different from Markov decision theory
- *Two step solution*
  - Step One: Computations — The backward step (off-line)
  - Step Two: Implementation — The forward step (off-line or on-line)

# Seq. Decomposition

- *Computations — The backward step*
  - For each time instant $t$ and each ${}^{i}\pi_t \in \Pi$
    - $\star$ evaluate the cost to go ${}^{i}V_t({}^{i}\pi_t)$
    - $\star$ and store the corresponding arg minimum ${}^{i}\Phi_t({}^{i}\pi_t)$
  - $\mathcal{J}^* = V_1(\pi_1^\circ)$
- *Implementation — The forward step*
  - Start at time 1. We know ${}^{1}\pi_1^\circ$. Look-up $c_1^\circ = {}^{1}\Phi_t({}^{1}\pi_1^\circ)$.
  - ${}^{1}\pi_1^\circ$ and $c_1^\circ$ determine ${}^{2}\pi_1^\circ$. Look-up $h_1^\circ = {}^{2}\Phi_t({}^{2}\pi_2^\circ)$.
  - And so on ...

*Determine optimal* $(c_1^\circ, h_1^\circ, \dots, g_T^\circ, l_T^\circ)$ *off line*

# Seq. Decomposition

## Complexity

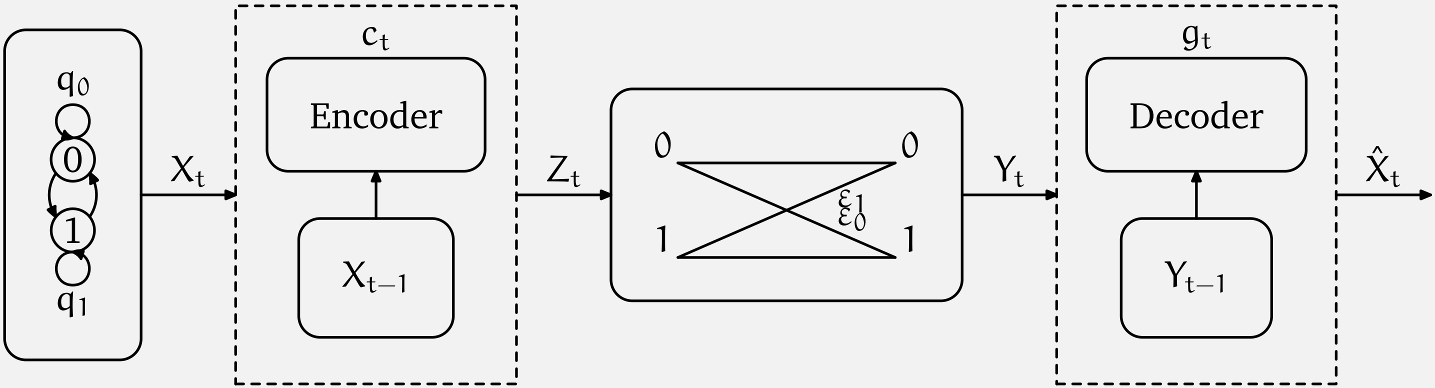
- Each equation is non-convex in function space.
- For a fixed $t$ and $\pi_t$, there are $2^4$ alternatives.
- There are $3 \times T = 30$ nested optimality equations. (linear in $T$)
- However, $\pi$ takes value in a continuous space.
- Suppose we partition $\Pi$ into $10^6$ points.

$$\text{Number of calculations} = 5 \times 10^8 \qquad (\text{cf. } 10^{48} \text{ for brute force})$$

# Numerical Computations

- *Reachability Analysis*
  - Find all reachable $\pi_t$ and solve the nested optimality equations for them
- *Smallwood & Sondik-like Algorithm*
  - ${}^{i}V_t(\cdot)$ is piecewise linear and convex
  - Can be represented as pointwise minimum of a finite family of affine functions
  - These affine functions can be computed by linear programming
- *Approximation Algorithms*
  - Grid based solutions
  - Rust's probabilistic algorithm
- *Specialized Algorithms ??*

# Numerical Example




Source $(0.7, 0.1)$

Channel $(0.2, 0.1)$

**Encoder**

| $X_t$ | $X_{t-1}$ | $Z_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

**Decoder**

| $Y_t$ | $Y_{t-1}$ | $\hat{X}_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

# Numerical Example

Source $(0.7, 0.1)$ Channel $(0.2, 0.2)$

Encoder 1

| $X_t$ | $X_{t-1}$ | $Z_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

Decoder 1

| $Y_t$ | $Y_{t-1}$ | $\hat{X}_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

Encoder 2

| $X_t$ | $X_{t-1}$ | $Z_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

Decoder 2

| $Y_t$ | $Y_{t-1}$ | $\hat{X}_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

Encoder 3

| $X_t$ | $X_{t-1}$ | $Z_t$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

Decoder 3

| $Y_t$ | $Y_{t-1}$ | $\hat{X}_t$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

$$\text{Encoder 1} \longrightarrow \text{Encoder 2} \longrightarrow \text{Encoder 3} \longrightarrow \text{Encoder 1} \longrightarrow \cdots$$

# Summary

## So far . . .

- Formulated real-time communication as a stochastic optimization problem.
- Obtained a methodology for sequential decomposition

## Key Ideas

- Information state
- Information structures

*The key ideas are fundamental and are also applicable to other problems*

- *Three different explanations of how to choose information states*
  - Related to Aumann's notion of common knowledge
  - Resolve the second guessing argument

# Other problems

- *Variations of real-time communication problem*
  - Arbitrary (but finite) delay
  - Channels with memory
  - Active noisy feedback
- *Control and communication*
  - Optimal feedback control over noisy communication channels.
- *Decentralized diagnosis with communication*
  - Fault diagnosis in discrete event systems with communication between diagnosers.

# Future Directions

- *Connections with classical information theory*
  - ▷ Smooth transition from real-time to asymptotic
  - ▷ Sequential information theory problems as stochastic optimization problem
- *Connections with other approaches to real-time communication*

  (e.g. Linder & Lugosi, Matloub & Weissman)
- *Connections with mathematical economics*
  - ▷ Mechanism design
  - ▷ Games with communication
- *Decentralized systems with a communication component*
  - ▷ Networks: Communication, control, and detection

# References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*Thank you*