THM: Let $x(t)$ be a bounded periodic signal with period T. Then $x(t)$ can be expanded as a weighted sum of sinusoids with angular frequencies that are integer multiples of $\omega_0 = \frac{2\pi}{T}$:

$$x(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$
$$x(t) = b_0 + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \ldots$$

This is the *trigonometric Fourier series expansion* of $x(t)$.

Or, we can use only cosines with phase shifts (using Problem Set #1):

$$x(t) = a_0 + c_1 \cos(\omega_0 t - \phi_1) + c_2 \cos(2\omega_0 t - \phi_2) + c_3 \cos(3\omega_0 t - \phi_3) + \ldots$$

PROOF: Take EECS 316 or Math 450 or a higher-level math course.

COMPUTATION OF FOURIER SERIES COEFFICIENTS:

THM: Coefficients a_n, b_n, c_n and ϕ_n can be computed using:

$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\omega_0 t) dt$ and $b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\omega_0 t) dt$.

For $n = 0$ we have: $a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$=average value of $x(t)$.

For cosines with phase shifts: $c_n = \sqrt{a_n^2 + b_n^2}; \phi_n = \tan^{-1}(\frac{b_n}{a_n}), n \neq 0$

EXAMPLE OF FOURIER SERIES DECOMPOSITION:

Let $x(t)$ be signal at right:

Period=$T = 2\pi \rightarrow \omega_0 = \frac{2\pi}{2\pi} = 1$.

$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} x(t) \cos(n\omega_0 t) dt = 0$ by inspection
(set $t_0 = -\frac{T}{2} = -\pi$) since $\int_{-\pi}^{0}$ and $\int_0^{\pi}$ cancel (think about it).

$$b_n = \frac{2}{2\pi} \int_0^{2\pi} x(t) \sin(nt) dt = \frac{1}{\pi} [\int_0^{\pi} (\frac{\pi}{4}) \sin(nt) dt + \int_{\pi}^{2\pi} (-\frac{\pi}{4}) \sin(nt) dt]$$
$$= \begin{cases} \frac{1}{n}, & \text{if n is odd;} \\ 0, & \text{if n is even.} \end{cases} \rightarrow x(t) = \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \ldots$$

PROOF OF THM: First we need the following lemma:

LEMMA: The sine and cosine functions are *orthogonal* functions:

$$\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\cos(j\omega_0 t)dt=\int_{t_0}^{t_0+T}\sin(i\omega_0 t)\sin(j\omega_0 t)dt=\begin{cases}T/2, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}$$

$\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\sin(j\omega_0 t)dt=0$ (even if $i=j$). These assume $i,j>0$.

PROOF OF LEMMA: Adding and subtracting the cosine addition formula $\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$ gives

$2\cos(x)\cos(y)=\cos(x-y)+\cos(x+y)$ and
$2\sin(x)\sin(y)=\cos(x-y)-\cos(x+y)$.

Setting $x=i\omega_0 t$ and $y=j\omega_0 t$ and $\int_{t_0}^{t_0+T}dt$ gives

$$2\int_{t_0}^{t_0+T}\cos(i\omega_0 t)\cos(j\omega_0 t)dt=\int_{t_0}^{t_0+T}\cos((i+j)\omega_0 t)+\cos((i-j)\omega_0 t)dt.$$

Since the integral of a sinusoid with nonzero frequency over an integer number $i\pm j$ of periods is zero, the first part of the lemma follows. The other two parts follow similarly. QED.

PROOF: Multiply the Fourier series by $\cos(n\omega_0 t)$ and $\int_{t_0}^{t_0+T}dt$:

$\int_{t_0}^{t_0+T}x(t)\cos(n\omega_0 t)dt=\int_{t_0}^{t_0+T}a_0\cos(n\omega_0 t)dt$
$+\int_{t_0}^{t_0+T}a_1\cos(n\omega_0 t)\cos(\omega_0 t)dt+\int_{t_0}^{t_0+T}a_2\cos(n\omega_0 t)\cos(2\omega_0 t)dt+\ldots$
$+\int_{t_0}^{t_0+T}b_1\cos(n\omega_0 t)\sin(\omega_0 t)dt+\int_{t_0}^{t_0+T}b_2\cos(n\omega_0 t)\sin(2\omega_0 t)dt+\ldots$
$=0+0+\ldots+0+a_n\frac{T}{2}+0+\ldots$ from which the a_n formula follows.
The formulae for a_0 and b_n follow similarly. QED.

COMMENTS:

1. t_0 is arbitrary; all integrals are over one period.
2. $\omega_0=\frac{2\pi}{T}$ is angular frequency in $\frac{radians}{sec}$; this is $\frac{1}{T}$ Hertz.
3. The sinusoid at frequency ω_0 is called the *fundamental*;
   the sinusoid at frequency $(n+1)\omega_0$ is the n^{th} *harmonic.*
   Harmonics are also called *overtones.*
4. The more terms (i.e., harmonics) we keep in the Fourier series,
   the better the approximation the truncated series is to $x(t)$.
5. If $x(t)$ has a discontinuity at $t=t_1$, the Fourier series converges to $\frac{1}{2}(x(t_1^-)+x(t_1^+))$, where $x(t^-)$ and $x(t^+)$ are the values of $x(t)$ on either side of the discontinuity.

TRANSFER FUNCTIONS AND EFFECT ON FREQUENCY CONTENT

Q: Why think about audio signals in terms of their frequency content?
A: Because any circuit or system composed of resistors, inductors, and capacitors has the following property:

$$A_{in}\cos(2\pi f_0 t+\phi_{in}) \rightarrow \boxed{H(f_0)} \rightarrow |H(f_0)|A_{in}\cos(2\pi f_0 t+\phi_{in}+\angle H(f_0))$$

$|H(f_0)| = gain$ and $\angle H(f_0) = phase$ at f_0 of **transfer function** $H(f)$.

1. Note output frequency f_0 is same as input frequency f_0.
   Amplitude and phase change, but frequency stays the same.
2. The *gain* is the ratio of the output amplitude to the input amplitude.
3. The *phase* is the output phase minus the input phase.
4. Both gain and phase vary with input frequency f_0.

EXAMPLE #1:

The circuit shown at right has
transfer function gain and phase
$|H(f)| = \frac{1}{\sqrt{1+(2\pi f RC)^2}}$
$\angle H(f) = -\tan^{-1}(2\pi f RC)$.
(we will learn how to derive this later in the term).

Let $V_{IN}(t)=\cos(2\pi t)+\cos(2\pi 10^6 t)$ and $RC=0.001$.
So $V_{IN}(t)$ is the sum of sinusoids at 1 Hz and 10^6 Hz=1 MHz.

Since $|H(1)|\approx 1$, $|H(10^6)|\approx 0.00016$, $\angle H(10^6)\approx -90^o$,
$V_{OUT}(t)\approx\cos(2\pi t)+0.00016\cos(2\pi 10^6 t-90^o)$
(these numbers are very close to the actual values).

POINT: We have *filtered* the two-tone signal $V_{IN}(t)$: We kept the low-frequency component and eliminated the high-frequency component.

EXAMPLE #2:

Let $V_{IN}(t)$ be the square wave in the Fourier series handout.
We found $V_{IN}(t)=\sin(t)+\frac{1}{3}\sin(3t)+\frac{1}{5}\sin(5t)+\ldots$ Then

$$V_{OUT}(t)=\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{\sqrt{1+(0.001n)^2}}\frac{1}{n}\sin[nt-\tan^{-1}(0.001n)]$$

1. Try plotting this using MATLAB.
   The circuit has "rounded off" the edges of the square wave.
   Those sharp edges require high frequencies that have been filtered.
2. What happened to the 2π?
   This is a good exercise in keeping track of $\omega = 2\pi f$ vs. f:
   angular frequency ($\frac{radians}{sec}$) vs. circular frequency (Hertz).

We can also describe the *effects* (possibly undesirable) of a circuit or system in terms of its frequency response:

1. Your stereo amplifier has its gain $|H(f)| \approx$constant for $20Hz < f < 20kHz$ but the gain falls off sharply at lower and higher frequencies.
2. Loudspeakers have similar characteristics; they don't reproduce well sounds with frequencies outside this band.
3. You can *alter* the frequency response of your stereo amplifier with your bass and treble knobs or your graphic equalizer.
4. See "Additional Course Notes" pp.15-18 for more details.

DECIBELS

The human ear perceive an amazing range of intensities (over 1 million!).
So gain is usually measured in *decibels*, a *logarithmic scale*:

DEF: Gain in decibels (dB)=$20\log_{10}|H(f)|$.

EXAMPLES: $|H(f)| = 1 \rightarrow 0$ dB; $|H(f)| = 100 \rightarrow 40$ dB;
$|H(f)| = 0.000001 \rightarrow -120$ dB; $|H(f)| = 1000 \rightarrow 60$ dB.

$|H(f)| = 10^n \rightarrow 20n$ dB; $|H(f)| = 2^n \rightarrow 6n$ dB (VERY close).

Note that *octaves* are a logarithmic scale for frequency.
Another logarithmic scale: the Richter scale for earthquakes.

POINT: Instead of MULTIPLYING the amplitude of the input sinusoid by the gain to get the output sinusoid, just ADD the decibels, since logarithms convert multiplication to addition. This can be a real time-saver, since this must be done for EACH input frequency.

Try plotting $20\log_{10}|H(f)|$ vs $\log_{10}(f)$ for the above using MATLAB.
You will get a horizontal line which changes to a sloping line!