SAMPLING THEOREM FOR PERIODIC SIGNALS

NOTE: See *DFT: Discrete Fourier Transform* for more details.

GIVEN: A periodic continuous-time signal $x(t)$ such that:

1. $x(t)$ is **periodic** and real: $x(t) = x(t+T)$ for all t;
2. $x(t)$ is **bandlimited**: No frequencies above F Hz;
3. $x(t)$ is **sampled**: Given samples $x[n] = x(t = n\Delta)$.

GOAL: We can reconstruct $x(t)$ from its samples $x[n] = x(t = n\Delta)$

IF: $\Delta < 1/(2F) \Leftrightarrow$ Sampling rate $> 2F\ \frac{\text{SAMPLES}}{\text{SECOND}}$.

DERIVATION WITHOUT USING THE FOURIER TRANSFORM

1. $x(t)$ periodic with period $T \rightarrow x(t)$ has the Fourier series expansion $x(t) = X_0 + X_1 e^{j\frac{2\pi}{T}t} + X_2 e^{j\frac{4\pi}{T}t} + \ldots X_N e^{j\frac{2\pi}{T}Nt} + X_1^* e^{-j\frac{2\pi}{T}t} + \ldots X_N^* e^{-j\frac{2\pi}{T}Nt}$

where: $X_k = \frac{1}{T}\int_0^T x(t) e^{-j2\pi\frac{kt}{T}} dt$ and $\frac{N}{T} < F < \frac{N+1}{T}$. Say $F = \frac{N+1/2}{T}$.

Note: We will not need to use the formula for X_k! No integrals here!

2. Hence $x(t)$ is specified by $2N+1$ complex numbers $\{X_{-N} \ldots X_0 \ldots X_N\}$.
3. Sample $x(t)$ at $t = n\Delta$ so there are $(2N+1)$ samples per period T.

$\rightarrow$ $(2N+1)\Delta = T$. This and $(N + \frac{1}{2}) = FT \rightarrow \Delta = \frac{1}{2F}$.

4. Then setting $t = n\Delta = \frac{n}{2F}, n = 0 \ldots 2N$ in the Fourier series gives $(2N+1)$ linear equations in $(2N+1)$ unknowns $\{X_k\}$:

5. $x(n\Delta) = \sum_{k=-N}^{N} X_k e^{\frac{j2\pi nk}{2N+1}}, n = 0 \ldots 2N$. Sum over different period:

$\rightarrow$ $x(n\Delta) = \sum_{k=0}^{2N} X_k e^{\frac{j2\pi nk}{2N+1}}, n = 0 \ldots 2N$ ($(2N+1)$-point DFT).

6. We can solve this linear system for the $\{X_k\}$ and insert these $\{X_k\}$ into the Fourier series in #1 above to get $x(t)$ for ALL values of t.

BANDLIMITED SIGNAL INTERPOLATION FORMULA

7. Or, we can note that the solution to this linear system is:

$X_k = \frac{1}{2N+1}\sum_{n=0}^{2N} x(n\Delta) e^{-\frac{j2\pi nk}{2N+1}}, k = -N \ldots N$ ($(2N+1)$-point DFT)

8. Inserting this into the Fourier series and using $\frac{t}{T} - \frac{n}{2N+1} = \frac{t-n\Delta}{T}$ gives

$$x(t) = \sum_{k=-N}^{N} \frac{1}{2N+1} \sum_{n=0}^{2N} x(n\Delta) e^{-\frac{j2\pi nk}{2N+1}} e^{j\frac{2\pi}{T}kt} = \sum_{n=0}^{2N} x(n\Delta) s(t - n\Delta)$$

where $s(t) = \frac{1}{2N+1}\sum_{k=-N}^{N} e^{j2\pi kt/T} = \frac{\sin[(2N+1)\pi t/T]}{(2N+1)\sin(\pi t/T)}$ (text p.145).

9. **NOTE:** This holds for **any** value of T, e.g., T=1 century!
10. Shannon proved this for aperiodic signals (think of this as $T \rightarrow \infty$).

Note: As $T \rightarrow \infty$, $s(t) \rightarrow \frac{\sin(\pi t/\Delta)}{\pi t/\Delta} = \text{sinc}(t/\Delta) = p(t)$ in the lecture notes

since: $\frac{2N+1}{T} = \frac{1}{\Delta}$ and $\sin(\pi\frac{t}{T}) \approx (\pi\frac{t}{T})$ as $(\pi\frac{t}{T}) \rightarrow 0 \Leftrightarrow (T \rightarrow \infty)$.

EX: $x(t)$ has period=T=10 sec; bandlimit=F=100 Hz. Then $\Delta = \frac{1}{200.1}$.

since: Fourier series of $x(t)$ has 2001 terms$\rightarrow$2001 samples per period=T.

DIGITAL SIGNAL PROCESSING: COMPLETE SYSTEM

$x(t)$ =Continuous-time (analog) signal.
EX: Audio (from microphone) signal.
Goal: *Digitally* filter this signal $x(t)$.

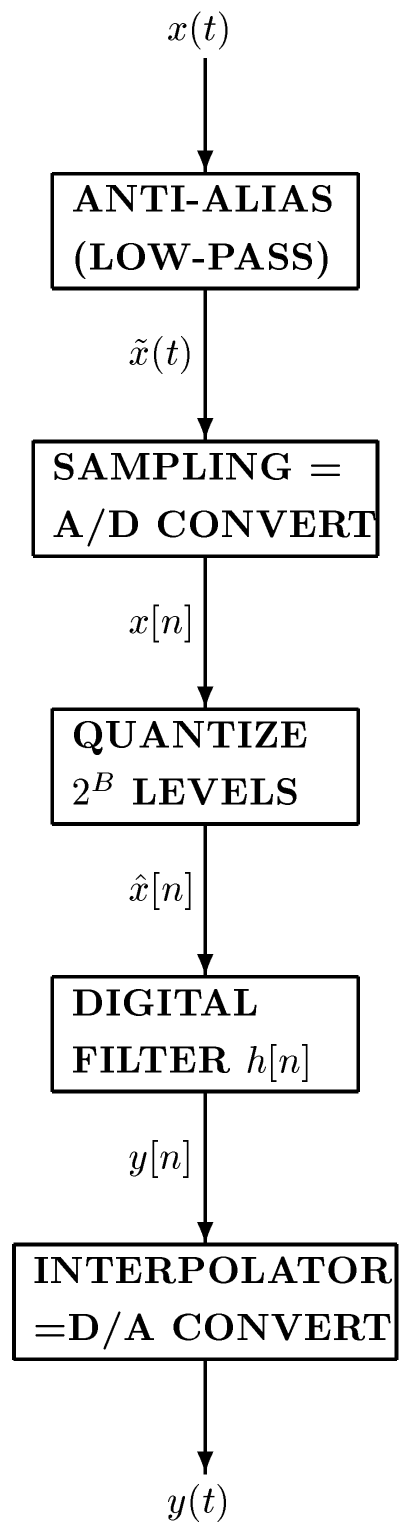
$\tilde{x}(t)$ =Lowpass-filtered version of $x(t)$.
How? Analog filter; use EECS 215 ideas.
Why? Remove frequencies $> \frac{1}{2}\left(\begin{smallmatrix}\text{SAMPLING}\\ \text{FREQUENCY}\end{smallmatrix}\right)$
$\rightarrow$ ensures there will be no aliasing.

$x[n]$ =Discrete-time (sampled) signal.
How? $x[n] = \tilde{x}(t = n\Delta); \Delta = \begin{smallmatrix}\text{SAMPLING}\\ \text{INTERVAL}\end{smallmatrix}$.
Why? We can now use EECS 216 ideas.
Note: Can recover $\tilde{x}(t)$ from $x[n]$ exactly, due to the anti-alias filter.

$\hat{x}[n]$ =Quantized version of $x[n]$.
How? Round $x[n]$ to nearest of 2^B levels. B=#bits used to represent numbers.
Why? To send/store bits, not numbers.
Note: Can't recover $x[n]$ from $\hat{x}[n]$, but the error is usually negligible.

$y[n]$ =Filtered version of input $\hat{x}[n]$.
How? $y[n] = h[n] * \hat{x}[n] = \sum_i h[i]\hat{x}[n-i]$.
$h[n]$ =impulse response of digital filter.
Why? Lowpass filter→remove some noise.
Notch filter→remove 60 Hz "hum."

$y(t)$ =Interpolated $y[n]$=analog output.
How? Use zero-order hold (constant interpolation)
OR: Linear interpolation (between samples $y[n]$)
OR: Exact formula (*Sampling* handout).



ALIASING IN A COMPLETE DSP SYSTEM

Given: $x(t) \rightarrow |\overline{\underline{\textbf{SAMPLE AT 5 HZ}}}| \rightarrow x[n] \rightarrow |\overline{\underline{\textbf{IDEAL (SINC) INTERPOLATOR}}}| \rightarrow \hat{x}(t)$

where: $x(t) = \cos(2\pi t) + 2\cos(8\pi t)$ (1 Hz, 4 Hz). **GOAL:** Compute $\hat{x}(t)$.

Ideal Interpolator: $\hat{x}(t) = \sum x[n]p(t - nT_s)$ where $p(t) = \text{sinc}(t/T_s)$.
sinc: $\text{sinc}(t) = (\sin(\pi t))/(\pi t)$=decaying sinusoid as $t \rightarrow \pm\infty$.

Nyquist: Sampling rate=5 Hz<2(max. frequency of $x(t)$)=2($4Hz$) $\rightarrow$ aliasing.
Interval: Sampling rate=5 Hz$\rightarrow$ T_s=Sampling interval=1/(5 Hz)=$\frac{1}{5}$second.
Sample: $t = nT_s = n(\frac{1}{5}) \rightarrow x[n] = x(\frac{t}{5}) = \cos(0.4\pi n) + 2\cos(1.6\pi n)$.

Alias: $2\cos(1.6\pi n) = 2\cos(0.4\pi n) \rightarrow x[n] = 3\cos(0.4\pi n)$. Note tripled!
Ideal: $n = \frac{t}{T_s} = 5t \rightarrow \hat{x}(t) = x[n = 5t] = 3\cos(2\pi t)$ (1 Hz, but **tripled**).
Note: Original 4 Hz$\rightarrow$aliased 1 Hz (folded across folding freq.=$\frac{5}{2}$=2.5 Hz).

Now: Change $x(t)$ to $x(t) = \cos(2\pi t) - \cos(8\pi t)$ (1 Hz, 4 Hz).
Alias: $x[n] = \cos(0.4\pi n) - \cos(0.4\pi n) = 0$! 1 Hz **eliminated**!
Aliasing: adds false signals, interferes with actual signal!

Now: Insert ideal antialias filter: Lowpass; pass<2.5 Hz, reject>2.5 Hz.

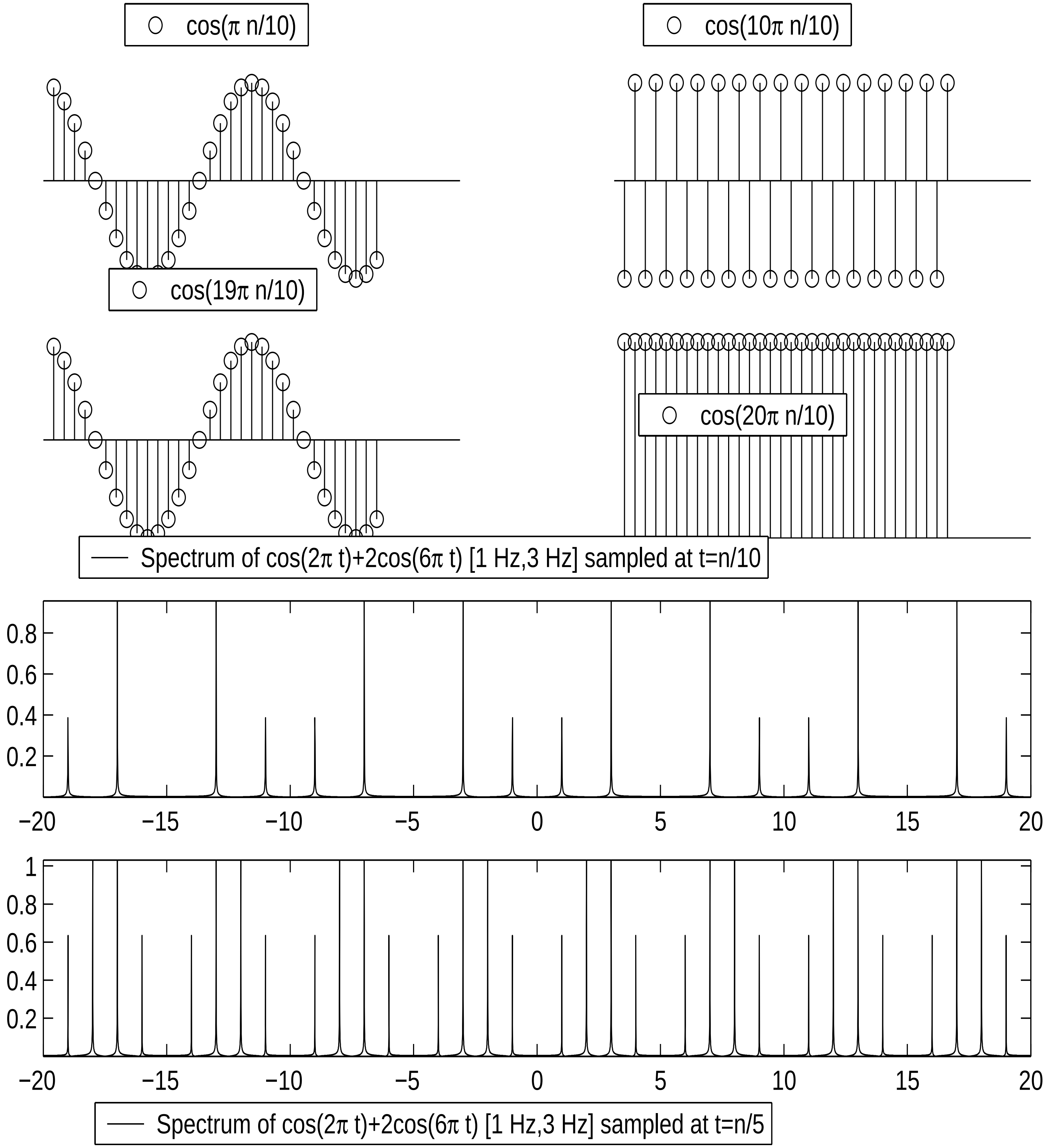
Given: $x(t) \rightarrow |\overline{\underline{\textbf{ANTI-ALIAS}}}| \rightarrow |\overline{\underline{\textbf{SAMPLE AT 5 HZ}}}| \rightarrow x[n] \rightarrow |\overline{\underline{\textbf{IDEAL (SINC) INTERPOLATOR}}}| \rightarrow \hat{x}(t)$

Now: Antialias filter eliminates original $2\cos(8\pi t)$ (4 Hz) component.
Get: $x[n] = \cos(0.4\pi n)$ and $\hat{x}(t) = \cos(2\pi t)$ (1 Hz).
Note: Aliased (false) 1 Hz eliminated. Original 1 Hz **unaffected**, at least.

Alias: Use $A\cos((\pi + \omega_o)n + \theta) = A\cos((\omega_o - \pi)n + \theta) = A\cos((\pi - \omega_o)n - \theta)$
since: $\cos(t)$ is an even function, and also periodic with period 2π.
EX: $3\cos(1.7\pi n + \frac{\pi}{6}) = 3\cos(0.3\pi n - \frac{\pi}{6})$. $\sin(1.8\pi n) = -\sin(0.2\pi n)$.

- Use to reduce all discrete-time signals **resulting from sampling**.
- **For non-sinusoidal signals:** Apply to Fourier series harmonics.

MSD: $\text{MSD}(x, \hat{x}) = \frac{1}{T}\int_0^T (x(t) - \hat{x}(t))^2 dt$=Mean Square Error.
How? Use Parseval's theorem to add average power in each harmonic:
Note: Average power of $A\cos(\omega_o n + \theta)$ is $A^2/2$.
IF: (1) $\omega_0 = 2\pi(\frac{\text{RATIONAL}}{\text{NUMBER}}) \rightarrow$ periodic; (2) $\omega_0 \neq 0, \pi$.


cos(π n/10)
cos(10π n/10)
cos(19π n/10)
cos(20π n/10)
Spectrum of cos(2π t)+2cos(6π t) [1 Hz,3 Hz] sampled at t=n/10
0.8
0.6
0.4
0.2
−20
−15
−10
−5
0
5
10
15
20
1
0.8
0.6
0.4
0.2
−20
−15
−10
−5
0
5
10
15
20
Spectrum of cos(2π t)+2cos(6π t) [1 Hz,3 Hz] sampled at t=n/5