SUMMARY OF VARIOUS SYSTEM RESPONSES

The complete response of a system can be decomposed in 2 different ways:

$$\underbrace{\text{ZERO INPUT RESPONSE}}_{\text{initial conds}} + \underbrace{\text{ZERO STATE RESPONSE}}_{\text{from input}} = \underbrace{\text{TRANSIENT RESPONSE}}_{\text{decays to 0}} + \underbrace{\text{STEADY STATE RESPONSE}}_{\text{doesn't decay to 0}}$$

The zero-state response (ZSR) can be further decomposed as follows:

$$\underbrace{\text{ZERO STATE RESPONSE}}_{\text{from input}} = \underbrace{\text{FORCED RESPONSE}}_{\text{like input}} + \underbrace{\text{NATURAL RESPONSE}}_{\text{like h(t)}}$$ where $h(t) =$ IMPULSE RESPONSE.

If all poles are in left-half-plane, then FORCED= STEADY STATE and TRANSIENT= NATURAL+ZIR.

Given: System described by the linear constant-coefficient differential eqn.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y(t) = K\left(\frac{d^m x}{dt^m} + b_1 \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_m x(t)\right)$$

and: **Initial Conditions** $y(0), \frac{dy}{dt}(0), \ldots \frac{d^{n-1}y}{dt^{n-1}}(0)$ for differential eqn.

Goal: Compute the response $y(t)$ to a *causal* input $x(t) = 0$ for $t < 0$.

Using: $\mathcal{L}\{\frac{d^n y}{dt^n}\} = s^n Y(s) - s^{n-1}y(0) - \ldots - \frac{d^{n-1}y}{dt^{n-1}}(0)$ we obtain the formula

$$Y(s) = K\frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n} X(s) + \frac{\sum_{i=0}^{n-1} \frac{d^{n-1-i}y}{dt^{n-1-i}}(0) \sum_{j=0}^{i} a_j s^{i-j}}{s^n + a_1 s^{n-1} + \ldots + a_n} = \text{ZSR+ZIR}$$

ZERO-INPUT RESPONSE (ZIR): SET INPUT=0

Def: Characteristic eqn.: $s^n + a_1 s^{n-1} + \ldots + a_n = 0$ (denominator of $Y(s)$).

Modes: Let $\{p_i\}$ be the n roots of $s^n + a_1 s^{n-1} + \ldots + a_n = 0$. $\{p_i\}$ are *modes*.

Note: The *modes* of a system are also called *natural frequencies* of system.

ZIR: $zir(t) = B_1 e^{p_1 t} + \ldots + B_n e^{p_n t}$ for $t > 0$. Decays to 0 if $Re[p_i] < 0$.

EX: Parallel RLC circuit with R=3Ω, L=4H, C=$\frac{1}{12}$F, no circuit source.

Goal: Compute the ZIR v if $v_c(0) = 2$ and $i_c(0^+) = -1$ ($i_c(t)$ *can* jump).

KCL: $\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(t')dt' + C\frac{dv}{dt} = 0$. Take $\frac{1}{C}\frac{d}{dt} \rightarrow \frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$.

Here: R=3Ω, L=4H, C=$\frac{1}{12}F \rightarrow \frac{d^2 v}{dt^2} + 4\frac{dv}{dt} + 3v = 0$. $\frac{dv}{dt}(0) = \frac{i(0)}{C} = -12$.

Soln: $V(s) = \frac{(s+4)v(0) + \frac{dv}{dt}(0)}{s^2+4s+3} = \frac{2s-4}{s^2+4s+3} = \frac{5}{s+3} - \frac{3}{s+1} \rightarrow v(t) = 5e^{-3t}1(t) - 3e^{-t}1(t)$.

Now: If $i(0^+) = -2$, find initial $v(0)$ (I.C.) so that $v(t) = Ce^{-3t}$ for $t > 0$.

Huh? By choosing I.C., we can cancel out one of the modes of the circuit.

Soln: $(s+4)v(0) - 24 = C(s+1)$ to cancel $(s+1)$ factor in denominator.

$\rightarrow$ Equate coeff. of $s \rightarrow$v(0)=C=4v(0)–24→v(0)=8→ $v(t) = 8e^{-3t}1(t)$.

ZERO-STATE RESPONSE (ZSR): INITIAL CONDITIONS=0

Def: *Transfer function* for system described by differential eqn. overleaf is $H(s) = K\frac{s^m+b_1 s^{m-1}+\ldots+b_m}{s^n+a_1 s^{n-1}+\ldots+a_n}$ (from first term of the $Y(s)$ expression).

Zeros: *Zeros* $\{z_i\}$ of $H(s)$ are roots of $s^m+b_1 s^{m-1}+\ldots+b_m=0$ (numerator)
Poles: *Poles* $\{p_i\}$ of $H(s)$ are roots of $s^n+a_1 s^{n-1}+\ldots+a_n=0$ (denominator)
Modes: What's the difference between poles and modes? $\{\text{poles}\}\subset\{\text{modes}\}$.

EX: $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=\frac{dx}{dt}-3x \rightarrow \{\text{modes}\}=\{3,4\}$ $(s^2-7s+12=0)$.
But: Get $H(s)=\frac{s-3}{s^2-7s+12}=\frac{1}{s-4}\rightarrow\{\text{poles}\}=\{4\}$ $(s-4=0)$. No zeros.
Note: Doesn't happen in the real world, due to roundoff. So use p_i for both.

Note: System is *stable* iff all poles have $Re[p_i]<0$ (all p_i in left-half-plane).
Note: Transfer function $H(s)=\frac{Y(s)}{X(s)}=\mathcal{L}\{h(t)\}$ where $h(t)$=impulse response.

EX: $e^{-3t}1(t)\rightarrow$ SYSTEM $\rightarrow e^{-5t}1(t)$. Compute $y(t)$ if $x(t)=e^{-7t}1(t)$.
Soln: $H(s)=\frac{Y(s)}{X(s)}=\frac{1/(s+5)}{1/(s+3)}=\frac{s+3}{s+5}$. Zeros:$\{-3\}$. Poles:$\{-5\}$. Hence stable.
Then: $Y(s)=H(s)X(s)=\frac{s+3}{s+5}\frac{1}{s+7}=\frac{2}{s+7}-\frac{1}{s+5}\rightarrow y(t)=\underbrace{2e^{-7t}1(t)}_{\text{FORCED}}-\underbrace{e^{-5t}1(t)}_{\text{NATURAL}}$.
Then: $h(t)=\mathcal{L}^{-1}\{1-\frac{2}{s+5}\}=\delta(t)-2e^{-5t}1(t)$. Read off $\frac{dy}{dt}+5y=\frac{dx}{dt}+3x$.

FORMS OF THE FORCED AND NATURAL PARTS OF ZSR

h(t): $h(t)=C_1e^{p_1t}1(t)+\ldots+C_ne^{p_nt}1(t)\Leftrightarrow H(s)=K\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$.
x(t): $x(t)=B_1e^{q_1t}1(t)+\ldots+B_ne^{q_nt}1(t)\Leftrightarrow X(s)=L\frac{(s-w_1)\ldots(s-w_m)}{(s-q_1)\ldots(s-q_n)}$.
y(t): Formula: $y(t)=\underbrace{\sum_{i=1}^{n}B_iH(q_i)e^{q_it}1(t)}_{\text{FORCED:like x(t)}}+\underbrace{\sum_{i=1}^{n}C_iX(p_i)e^{p_it}1(t)}_{\text{NATURAL:like h(t)}}$.

since: $Y(s)=H(s)X(s)=K\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}L\frac{(s-w_1)\ldots(s-w_m)}{(s-q_1)\ldots(s-q_n)}$. Take residues.
Assume: $\{p_1\ldots p_n,q_1\ldots q_n\}$ distinct. $w_i=p_j\rightarrow X(p_j)=0$ and $z_i=q_j\rightarrow H(q_j)=0$.

EX: Voltage source $e^{-4t}1(t)$ drives series RLC with R=3Ω, L=1H, C=0.5F.
Goal Compute ZSR for resistor voltage. No initial conditions since ZSR.

VOLTAGE DIVIDER: $H(s)=\frac{R}{R+sL+\frac{1}{sc}}=\frac{sR/L}{s^2+\frac{R}{L}s+\frac{1}{LC}}=\frac{3s}{s^2+3s+2}=\frac{6}{s+2}-\frac{3}{s+1}$ (use s-plane circuit).

Forced: $v_{\text{FORCED}}=H(-4)e^{-4t}=-2e^{-4t}$ and $h(t)=6e^{-2t}-3e^{-t}$ all for $t>0$.
Natural: $v_{\text{NATURAL}}=X(-2)6e^{-2t}+X(-1)(-3e^{-t})=3e^{-2t}-e^{-t}$ all for $t>0$.
ZSR: $v_{ZSR}(t)=v_{\text{FORCED}}+v_{\text{NATURAL}}=-2e^{-4t}+3e^{-2t}-e^{-t}$ all for $t>0$.
OR: $Y(s)=H(s)X(s)=\frac{3s}{s^2+3s+2}\frac{1}{s+4}=\frac{-2}{s+4}+\frac{3}{s+2}+\frac{-1}{s+1}\rightarrow v_{ZSR}(t)$ above.