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**REVIEW (?) OF COMPLEX NUMBERS**


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**DEF:**  $j = \sqrt{-1}$ . Use  $j$  not  $i$  in EECS since  $i$ =current in EECS!

**DEF:** A complex number  $z$  can be written as  $z = x + jy = Me^{j\theta} = M\angle\theta$

**where:**  $x + jy$ =**rectangular** form of  $z$ ;  $Me^{j\theta} = M\angle\theta$ =**polar** form of  $z$ .

**AND:**  $x = \text{Re}[z]$ ;  $y = \text{Im}[z]$ ;  $M = |z|$ ;  $\theta = \text{arg}[z]$ .

Real part    Imag part    Magnitude    Argument or phase

**LEMMA: Euler's theorem:**  $e^{j\theta} = \cos\theta + j\sin\theta$ .

**WHY?** Insert  $x = j\theta$  into the infinite series  $e^x = 1 + x + x^2/2! + \dots$

**ALSO:**  $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ ;  $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ .

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**Converting rectangular form  $\leftrightarrow$  polar form:**


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**P  $\rightarrow$  R:**  $Me^{j\theta} = x + jy$  where  $x = M\cos\theta$  and  $y = M\sin\theta$ .

**R  $\rightarrow$  P:**  $x + jy = Me^{j\theta}$  where  $M = \sqrt{x^2 + y^2}$  and  $\tan\theta = y/x$ .

**Summary:** 
$$\begin{cases} x = M\cos\theta \\ y = M\sin\theta \end{cases}; \quad M = \sqrt{x^2 + y^2}; \quad \theta = \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0; \\ \arctan \frac{y}{x} \pm \pi & \text{if } x < 0. \end{cases}$$

**EX#1:**  $+3 + j4 = 5e^{+j0.927}$  since  $5 = \sqrt{3^2 + 4^2}$  and  $+0.927 = \arctan \frac{+4}{+3}$ .

**EX#2:**  $-3 - j4 = 5e^{-j2.214}$  since  $5 = \sqrt{3^2 + 4^2}$  and  $-2.214 = \arctan \frac{-4}{-3} - \pi$ .

**HINT:** Draw an *Argand diagram* ( $\text{Im}[z]$  vs.  $\text{Re}[z]$ ) to visualize.

**Matlab:** `abs(3+4j); angle(3+4j); real(3+4j); imag(3+4j)`

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**Adding, Subtracting, Multiplying, Dividing:**


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- **Add and subtract** complex numbers in **rectangular** form.

**since:**  $(a + jb) + (c + jd) = (a + c) + j(b + d)$ . **Note:**  $\text{Re}[z + w] = \text{Re}[z] + \text{Re}[w]$ .

**EX:**  $5e^{j0.927} + \sqrt{2}e^{j0.785} = (3 + j4) + (1 + j) = 4 + j5 = 6.4e^{j0.896}$ .

- **Multiply and divide** complex numbers in **polar** form.

**since:**  $(M_1e^{j\theta_1})(M_2e^{j\theta_2}) = (M_1M_2)e^{j(\theta_1+\theta_2)}$ . **Note:**  $|zw| = |z| \cdot |w|$ .

**EX:**  $(3 + j4)(1 + j) = (5e^{j0.927})(\sqrt{2}e^{j0.785}) = 5\sqrt{2}e^{j1.71} = -1 + j7$ .

- **Multiplying and dividing** in rectangular form:

$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$ ;  $\frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \frac{c - jd}{c - jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}$  (ugh)

**EX:**  $(3 + j4)(1 + j) = (3 \cdot 1 - 4 \cdot 1) + j((4 \cdot 1) + (3 \cdot 1)) = -1 + j7 = 5\sqrt{2}e^{j1.71}$ .

**EX:** Compute  $2\sqrt{3}e^{j\pi/6} + 2e^{-j\pi/3}$ . Convert polar to rectangular:

$= (2\sqrt{3}\cos(\pi/6) + j2\sqrt{3}\sin(\pi/6)) + (2\cos(\pi/3) + j2\sin(-\pi/3))$

$= (2\sqrt{3}(\sqrt{3}/2) + j2\sqrt{3}(1/2)) + (2(1/2) - j2(\sqrt{3}/2)) = 4. (!)$

**EX:**  $(2\sqrt{3}e^{j\pi/6}) / (2e^{-j\pi/3}) = \frac{2\sqrt{3}}{2}e^{j(\pi/6 - (-\pi/3))} = \sqrt{3}e^{j\pi/2} = j\sqrt{3}$ .

**PHASORS: COMPLEX NUMBERS REPRESENT SINUSOIDS**

**Phasors:** Represent sinusoid  $x(t) = M \cos(\omega t + \theta)$  with complex no.  $X = Me^{j\theta}$ .

**Note:**  $x(t) = \text{Re}[Xe^{j\omega t}] = \text{Re}[Me^{j\theta}e^{j\omega t}] = \text{Re}[Me^{j(\omega t + \theta)}] = M \cos(\omega t + \theta)$ .

**Why?**  $A \cos(\omega t + \theta) + B \cos(\omega t + \phi) = \text{Re}[Ae^{j(\omega t + \theta)} + Be^{j(\omega t + \phi)}]$   
 $= \text{Re}[e^{j\omega t}(Ae^{j\theta} + Be^{j\phi})] = \text{Re}[e^{j\omega t}(Ce^{j\psi})] = C \cos(\omega t + \psi)$ .

**where:**  $Ae^{j\theta} + Be^{j\phi} = Ce^{j\psi}$ . Add sinusoids  $\Leftrightarrow$  add complex numbers!

**EX#1:** Simplify  $x(t) = 3 \cos(\omega t) + 3 \cos(\omega t + 120^\circ) + 3 \cos(\omega t + 240^\circ)$ .

**Hard:** Use cosine addition formula  $\rightarrow$  mess. If do it right, get  $x(t) = 0$  (!)

**Easy:** Phasors:  $X = 3e^{j0} + 3e^{j120^\circ} + 3e^{j240^\circ} = 0 \rightarrow x(t) = \text{Re}[Xe^{j\omega t}] = 0!$

**Why?** Draw picture in complex plane: easy to see resultant of these=0!

**EX#2:** Show that  $5 \cos(\omega t + 53^\circ) + \sqrt{2} \cos(\omega t + 45^\circ) = 6.4 \cos(\omega t + 51^\circ)$ .

**Hard:** Use cosine addition formula twice. Left as exercise for student.

**Easy:**  $5e^{j53^\circ} + \sqrt{2}e^{j45^\circ} = (3 + j4) + (1 + j) = (4 + j5) = 6.4e^{j51^\circ}$ . QED.

1. Sinusoids must **all be at the same frequency** to add their phasors. If different frequencies: partition into sums at same frequency.
2. Use  $\sin(\omega t + \theta) = \cos(\omega t + \theta - \pi/2)$  as necessary.
3. Multiplying sinusoids is NOT equivalent to multiplying phasors!

**OTHER COMPLEX NUMBER FACTS:**

**DEF: Complex conjugate**  $z^*$  of  $z$  is  $z^* = x - jy = Me^{-j\theta} = M \angle -\theta$ .

1.  $|z^*| = |z|$ ;  $|z|^2 = zz^*$ ;  $\arg[z^*] = -\arg[z]$ ;  $(zw)^* = z^*w^*$ .
2.  $\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}$ . Compare to rectangular division formula.
3.  $z = Me^{j\theta} \rightarrow z^* = Me^{-j\theta}$ ;  $\frac{1}{z} = \frac{1}{M}e^{-j\theta}$ ;  $-z = Me^{j(\theta \pm \pi)}$ .
4.  $A \cos(\omega t + \theta) = \frac{1}{2}(X + X^*)$  where  $X = Ae^{j(\omega t + \theta)} = Ae^{j\theta}e^{j\omega t}$ .
5.  $\text{Re}[z] = \frac{1}{2}(z + z^*)$  and  $|e^{j|z|}| = 1$  for **any** complex  $z$ .

**OTHER COMPLEX NUMBER TRICKS:**

1.  $\text{Im}[3 + j4] = 4$  **NOT**  $j4$ ! **VERY common mistake!**
2.  $e^{\pm j\pi} = -1$ .  $j = e^{j\pi/2}$  and  $-j = e^{j3\pi/2} \Leftrightarrow \cos(\omega t \pm 90^\circ) = \mp \sin(\omega t)$ .
3.  $j(3 + j4) = -4 + j3 = (1e^{j90^\circ})(5e^{j53^\circ}) = 5e^{j143^\circ}$  :  $90^\circ$  rotation.
4. Compute  $|\frac{5(8+j)(8+j6)(5+j12)(5+j10)}{26(7+j4)(7+j24)(2+j11)}| = \sqrt{\frac{(25)(65)(100)(169)(125)}{(26)^2(65)(625)(125)}} = 1$ .
5. The following identity is true to an overall sign. Why?

$$\frac{(a+jb)(c+jd)}{(e+jf)(g+jh)} = \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(e^2+f^2)(g^2+h^2)}} \exp j[\tan^{-1} \frac{b}{a} + \tan^{-1} \frac{d}{c} - \tan^{-1} \frac{f}{e} - \tan^{-1} \frac{h}{g}]$$