

CORRELATION BETWEEN SINUSOIDAL SIGNALS

Given: $x(t) = A \cos(\omega t + \theta_x)$ and $y(t) = B \cos(\omega t + \theta_y)$, $-\infty < t < \infty$.

Goal: To determine the *phase difference* $|\theta_x - \theta_y|$ from data $x(t)$ and $y(t)$.

Soln: Compute the $\text{CORRELATION COEFFICIENT} = \rho_{xy} = C_N(x, y) = C(x, y) / \sqrt{E(x)E(y)}$

where: $C(x, y) = \int_0^T x(t)y(t)dt$ and $E(x) = C(x, x)$ and $T = \frac{2\pi}{\omega}$ = period.

Then: $\rho(x, y) = C_N(x, y) = \cos(\theta_x - \theta_y) \rightarrow |\theta_x - \theta_y|$.

Proof: $E(x) = \int_0^T A^2 \cos^2(\omega t + \theta_x) = \int_0^T \frac{A^2}{2} (1 + \cos(2\omega t + 2\theta_x)) = T \frac{A^2}{2}$.

using: $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$. Compare to **rms** derivation.

$$C(x, y) = \int_0^T AB \cos(\omega t + \theta_x) \cos(\omega t + \theta_y) dt = \frac{AB}{2} \int_0^T \cos(2\omega t + \theta_x + \theta_y) dt \\ + \frac{AB}{2} \int_0^T \cos(\theta_x - \theta_y) dt = T \frac{AB}{2} \cos(\theta_x - \theta_y)$$

using: $\cos(x) \cos(y) = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y)$ and $M(\text{sinusoid})=0$.

Then: $\rho_{xy} = C_N(x, y) = \frac{C(x, y)}{\sqrt{E(x)E(y)}} = \frac{(TAB/2) \cos(\theta_x - \theta_y)}{\sqrt{(TA^2/2)(TB^2/2)}} = \cos(\theta_x - \theta_y)$.

Note: (1) $\theta_x = \theta_y \Leftrightarrow \rho_{xy} = 1$; (2) $|\theta_x - \theta_y| = \frac{\pi}{2} \Leftrightarrow \rho_{xy} = 0$:

The sin and cos functions are *orthogonal* over one period.

Given: $x(t) = A \cos(2\pi f_x t)$ and $y(t) = B \cos(2\pi f_y t)$, $-\infty < t < \infty$.

Goal: To determine the *frequency difference* $|f_x - f_y|$ from data $x(t)$ and $y(t)$.

Soln: $\text{CORRELATION COEFFICIENT} = \rho_{xy} = C_N(x, y) = \frac{C(x, y)}{\sqrt{E(x)E(y)}} = \text{sinc}((f_x - f_y)T)$

where: $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ has peak at $x = 0$; smaller at half-integer x .

Proof: $E(x) = T \frac{A^2}{2}$ and $E(y) = T \frac{B^2}{2}$ as above. But now we have:

$$C(x, y) = AB \int_{-T/2}^{T/2} \cos(2\pi f_x t) \cos(2\pi f_y t) dt = \frac{AB}{2} \int_{-T/2}^{T/2} \cos(2\pi(f_x + f_y)t) dt \\ + \frac{AB}{2} \int_{-T/2}^{T/2} \cos(2\pi(f_x - f_y)t) dt = T \frac{AB}{2} [\text{sinc}((f_x + f_y)T) + \text{sinc}((f_x - f_y)T)]$$

using: $\frac{1}{T} \int_{-T/2}^{T/2} \cos(\omega t) dt = \frac{\sin(\omega t)}{\omega T} \Big|_{-T/2}^{T/2} = \frac{\sin(\omega T/2)}{\omega T/2} = \text{sinc}(fT)$ ($\omega = 2\pi f$).

Now: $(f_x + f_y) \gg (f_x - f_y) \rightarrow \text{sinc}((f_x + f_y)T) \ll \text{sinc}((f_x - f_y)T)$.

Then: $\rho_{xy} = C_N(x, y) = \frac{C(x, y)}{\sqrt{E(x)E(y)}} = \frac{(TAB/2) \text{sinc}(f_x - f_y)T}{\sqrt{(TA^2/2)(TB^2/2)}} = \text{sinc}(f_x - f_y)T$.

Note: (1) $f_x = f_y \Leftrightarrow \rho_{xy} = 1$; (2) Including phase in this \rightarrow mess.