INVERSE LAPLACE TRANSFORM: MULTIPLE POLES

Problem: Compute $\mathcal{L}^{-1}\{N(s)/D(s)\}$ when $D(s)$ has a multiple root (pole).

Solution: Let $\frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)^K(s-p_2)\dots(s-p_N)}$ (ratio of two polynomials). Then:

Partial Fraction Expansion $\frac{N(s)}{D(s)} = \frac{A_1}{s-p_1} + \frac{A_2}{(s-p_1)^2} + \dots + \frac{A_K}{(s-p_1)^K} + \frac{B_2}{s-p_2} + \dots + \frac{B_N}{s-p_N}$ where:

$B_2 = \frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)^K(s-p_2)\dots(s-p_N)}(s-p_2)|_{s=p_2} = \frac{(p_2-z_1)(p_2-z_2)\dots(p_2-z_M)}{(p_2-p_1)^K(p_2-p_3)\dots(p_2-p_N)}$

$B_3 = \frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)^K(s-p_2)\dots(s-p_N)}(s-p_3)|_{s=p_3} = \frac{(p_3-z_1)(p_3-z_2)\dots(p_3-z_M)}{(p_3-p_1)^K(p_3-p_2)(p_3-p_4)\dots}$

And: $A_K = \frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)^K(s-p_2)\dots(s-p_N)}(s-p_1)^K|_{s=p_1} = \frac{(p_1-z_1)(p_1-z_2)\dots(p_1-z_M)}{(p_1-p_2)(p_1-p_3)\dots(p_1-p_N)}$

And: $A_{K-1} = \frac{d}{ds}[\frac{N(s)}{D(s)}(s-p_1)^K]|_{s=p_1}$; $\quad A_{K-2} = \frac{1}{2!}\frac{d^2}{ds^2}[\frac{N(s)}{D(s)}(s-p_1)^K]|_{s=p_1}$;

And: $\dots A_1 = \frac{1}{(K-1)!}\frac{d^{K-1}}{ds^{K-1}}[\frac{N(s)}{D(s)}(s-p_1)^K]|_{s=p_1}$

Huh? Before you pass out, note that computing the residues A_n uses:

1. $\frac{d}{ds}\frac{N(s)}{D(s)} = [D(s)\frac{dN}{ds} - N(s)\frac{dD}{ds}]/[D(s)]^2$ (the basic calculus formula) $N(s)$ and $D(s)$ are polynomials, so computing the derivatives is easy!
2. Can compute the derivatives $\frac{d^n}{ds^n}\frac{N(s)}{D(s)}$ and A_{K-n} **recursively** in n.
3. **Watch the order:** Compute $A_K, A_{K-1}, A_{K-2}\dots A_1$ in that order!
4. Multiply $\frac{N(s)}{D(s)}$ by $(s-p_1)^K$ **before** taking derivatives! Makes sense: Cancel factor $(s-p_1)^K$ in the denominator before doing anything else.

Example: Compute $\mathcal{L}^{-1}\{\frac{2s^2-25s-33}{s^3-3s^2-9s-5}\} = \mathcal{L}^{-1}\{\frac{2s^2-25s-33}{(s+1)^2(s-5)}\}$ (S and S page 515).

Solution: $\frac{2s^2-25s-33}{(s+1)^2(s-5)} = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{B}{s-5}$ (partial fraction expansion) where:

Residues: $B = \frac{2s^2-25s-33}{(s+1)^2}|_{s=5} = -3$; $\qquad A_2 = \frac{2s^2-25s-33}{s-5}|_{s=-1} = 1$

And: $A_1 = \frac{1}{1!}\frac{d}{ds}[\frac{2s^2-25s-33}{s-5}]|_{s=-1} = \frac{(s-5)(4s-25)-(2s^2-25s-33)}{(s-5)^2}|_{s=-1} = 5$

So: $\frac{2s^2-25s-33}{s^3-3s^2-9s-5} = \frac{5}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{s-5}$ (partial fraction expansion)

And: $\mathcal{L}^{-1}\{\frac{2s^2-25s-33}{s^3-3s^2-9s-5}\} = 5e^{-t} + te^{-t} - 3e^{5t}$ for $t > 0$; $\quad 0$ for $t < 0$.

Note: There are several typos in Soliman and Srinath on page 515! Multiply out your answer, or plug in several values of s, to check work.

Matlab: `>> [R,P,K]=residue([2 -25 -33],[1 -3 -9 -5])` yields answers

`R=[-3 5 1]'    P=[5 -1 -1]'    K=[]`

PARTIAL FRACTIONS FOR MULTIPLE COMPLEX POLES

How? Do this the same way, only for complex poles, then use the formula:

$$Ae^{pt}+A^*e^{p^*t} = 2\text{Re}[Ae^{pt}] = 2e^{\text{Re}[p]t}[\text{Re}[A]\cos(\text{Im}[p]t) - \text{Im}[A]\sin(\text{Im}[p]t)]$$

Example: Compute $\mathcal{L}^{-1}\{\frac{5s^3-3s^2+7s-3}{(s^2+1)^2}\} = \frac{5s^3-3s^2+7s-3}{(s-j)^2(s+j)^2}$ (S and S on page 249).

Solution: $\frac{5s^3-3s^2+7s-3}{(s^2+1)^2} = \frac{A_1}{s-j} + \frac{A_2}{(s-j)^2} + \frac{A_1^*}{s+j} + \frac{A_2^*}{(s+j)^2}$ (partial fractions) where:

Residue: $A_2 = \frac{5s^3-3s^2+7s-3}{(s+j)^2}|_{s=j} = -\frac{j}{2}$; $\quad A_2^* = \frac{j}{2}$ (just a complex conjugate)

And: $A_1 = \frac{1}{1!}\frac{d}{ds}[\frac{5s^3-3s^2+7s-3}{(s+j)^2}]|_{s=j} = \frac{1}{2}(5+3j)$; $\quad A_1^* = \frac{1}{2}(5-3j)$ (ditto)

So: $\frac{5s^3-3s^2+7s-3}{(s^2+1)^2} = \frac{1}{2}\frac{5+3j}{s-j} + \frac{1}{2}\frac{5-3j}{s+j} - \frac{1}{2}\frac{j}{(s-j)^2} + \frac{1}{2}\frac{j}{(s+j)^2}$ (partial fractions)

And: $\mathcal{L}^{-1}\{\frac{5s^3-3s^2+7s-3}{(s^2+1)^2}\} = \frac{1}{2}(5+3j)e^{jt} + \frac{1}{2}(5-3j)e^{-jt} - \frac{1}{2}jte^{jt} + \frac{1}{2}jte^{-jt}$

Then: Setting $A = \frac{1}{2}(5+3j)$ and $p = j$ above, then $A = -\frac{1}{2}j$ and $p = j$:

$$\mathcal{L}^{-1}\{\frac{5s^3-3s^2+7s-3}{(s^2+1)^2}\} = 5\cos(t) - 3\sin(t) + 0t\cos(t) + t\sin(t) \text{ for } t > 0$$

which agrees with S and S on page 249 but is (to me) much simpler, since it does not require the solution of a linear system of equations.

Matlab: `>> [R,P,K]=residue([5 -3 7 -3],[1 0 2 0 1])` yields the answers `R=[2.5+1.5i 0-0.5i 2.5-1.5i 0+0.5i]; P=[0+1i 0+1i 0-1i 0-1i]`

OR: Use S and S approach: Requires solving linear system of equations:

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1. $\frac{5s^3-3s^2+7s-3}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$ for unknown constants A, B, C, D.
2. Multiply by $(s^2+1)^2$: $5s^3-3s^2+7s-3 = (s^2+1)(As+B)+(Cs+D)$
3. Equate coefficients of s^i: 5=A; –3=B; 7=A+C; –3=B+D→C=2; D=0.
4. Plug in: $\frac{5s^3-3s^2+7s-3}{(s^2+1)^2} = \frac{5s}{s^2+1} - \frac{3}{s^2+1} + \frac{2s}{(s^2+1)^2}$.
5. $\mathcal{L}^{-1} \to 5\cos(t) - 3\sin(t) + t\sin(t)$ for $t > 0$ since $\frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$.

BUT: Usually this procedure is *much* more complicated! See the following.

COMPLETING THE SQUARE APPROACH

Goal: Compute $\mathcal{L}^{-1}\{\frac{2s^3+14s^2+40s+64}{(s^2+2s+5)(s^2+4s+13)}\}$ using "completing the square."

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1. $\frac{2s^3+14s^2+40s+64}{(s^2+2s+5)(s^2+4s+13)} = \frac{As+B}{s^2+2s+5} + \frac{Cs+D}{s^2+4s+13}$ for constants A, B, C, D.
2. **s=1:** $\frac{120}{144} = \frac{1}{8}$(B+A)+$\frac{1}{18}$(D+C). **s=–1:** $\frac{36}{40} = \frac{1}{4}$(B–A)+$\frac{1}{10}$(D–C). **s=0:** $\frac{64}{65} = \frac{1}{5}$B+$\frac{1}{13}$D. **s=2:** $\frac{216}{325} = \frac{1}{13}$(2A+B)+$\frac{1}{25}$(2C+D) →[A,B,C,D]=[1,3,1,5].

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3. $\frac{2s^3+14s^2+40s+64}{(s^2+2s+5)(s^2+4s+13)} = \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2} + \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$.
4. $\mathcal{L}^{-1} \to e^{-t}\cos(2t) + e^{-t}\sin(2t) + e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$ for $t > 0$.