EECS 216 DISCRETE-TIME FOURIER TRANSFORM

DTFT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(z)|_{z=e^{j\omega}}$ (z-xform on unit circle).

Inverse: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$. $(-\pi,\pi) \to (p-\pi, p+\pi)$ for any p.

Period: $X(e^{j\omega})$ is **periodic** with period 2π. Highest frequency: $\omega = \pi$.

Dual: **Fourier series:** Expand $X(e^{j\omega})$ as a Fourier series with period 2π: $x(n)$=Fourier coefficients; computed using $DTFT^{-1}$ formula above.

Uniform converge $\sum |x(n)| < \infty$ (absolutely summable) $\to$ *uniform* convergence: $\lim_{N\to\infty} \max |X_N(e^{j\omega}) - X(e^{j\omega})| = 0$ where $X_N(e^{j\omega}) = \sum_{n=-N}^{N} x(n)e^{j\omega n}$

Mean-square $\sum |x(n)|^2 < \infty$ (finite energy)$\to$ $mean - square$ convergence: $\lim_{N\to\infty} \int_{-\pi}^{\pi} |X_N(e^{j\omega}) - X(e^{j\omega})|^2 d\omega = 0$. Weaker than uniform:

Sinc: $x(n) = \frac{\sin(Bn)}{\pi n} \to X(e^{j\omega}) = \begin{cases} 1 & \text{if } 0 \le |\omega| < B \\ 0 & \text{if } B < |\omega| \le \pi \end{cases}$

Finite length signal $x(n) = \{\ldots 0, 0, 3, 1, 4, 2, 5, 0, 0 \ldots\} (x(0) = 4; \text{finite length} = 5) \to$
$X(e^{j\omega}) = 3e^{j2\omega} + 1e^{j\omega} + 4 + 2e^{-j\omega} + 5e^{-j2\omega} = X(z)|_{z=e^{j\omega}}$
$X(e^{j\omega}) = [4 + 3\cos(\omega) + 8\cos(2\omega)] - j[\sin(\omega) + 2\sin(2\omega)]$

Exponential $x(n) = a^n u(n) \to X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = 1/(1 - ae^{-j\omega})$.
$x(n) = a^n u(n) + b^n u(-n-1) \to X(e^{\omega}) = \frac{b-a}{a+b-e^{j\omega}-abe^{-j\omega}}$

provided: $|a| < 1 < |b|$ (stable $x(n)$ $\Leftrightarrow$ROC must include the unit circle $|z| = 1$).

DISCRETE-TIME FOURIER SERIES (DTFS)

DTFS: $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$; $\quad x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$

Discrete+periodic in time domain$\Leftrightarrow$Discrete+periodic in frequency.

Basis: $\sum_{n=0}^{N-1} e^{j2\pi nk/N} = \begin{cases} N & \text{if N divides k} \\ 0 & \text{otherwise} \end{cases}$. Orthogonal function.

Periodic: $x(n), X_k, e^{j2\pi nk/N}$ are all periodic in n and k with periods N.

Parseval: $\frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X_k|^2$=power in the periodic $x(n)$.

Square: $x(n) = \begin{cases} 1 & \text{if } 0 \le n \le L-1 \\ 0 & \text{if } L \le n \le N-1 \end{cases} \to X_k = \begin{cases} \frac{L}{N} \text{ if N divides k; else} \\ \frac{1}{N}\frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N} \end{cases}$

$$\begin{array}{ccccc} Continuous & \mathcal{L} & \Leftrightarrow & \mathcal{F} & \\ z = e^s & \Updownarrow & s = j\omega & \Updownarrow & \mathcal{F}\{\sum x(n)\delta(t-n)\} \\ Discrete & \mathcal{Z} & \Leftrightarrow & DTFT & \\ Time & & z = e^{j\omega} & & \end{array}$$

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

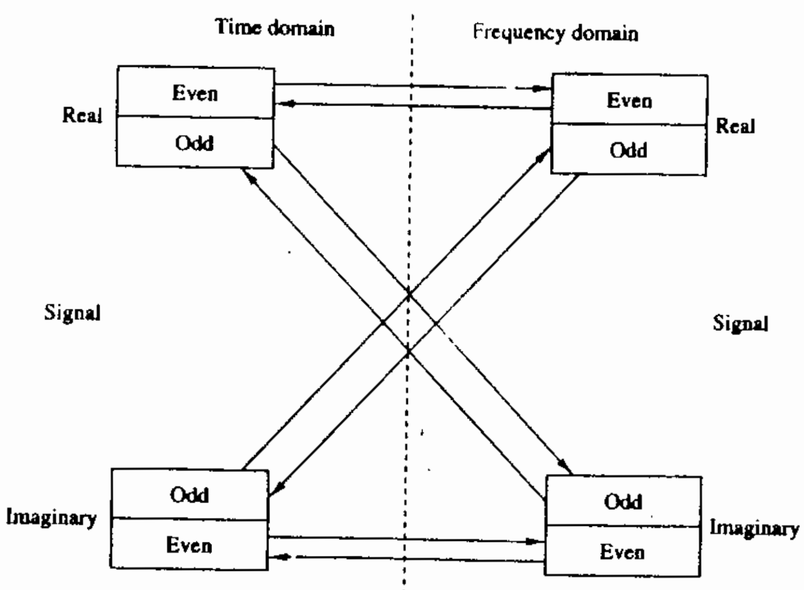

Figure 4.29 Summary of symmetry properties for the Fourier transform.

Property	Time domain	Frequency domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega + \omega_0) + \frac{1}{2}X(\omega - \omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$