CONTINUOUS-TIME IMPULSES

Def: An *impulse* is the limiting case of a constant-area high and fast pulse. An impulse has *zero width, infinite height*, and *finite area* under it.

Math: Mathematicians: Impulses are *distributions* or *generalized functions.* Don't call them Dirac delta functions–Dirac would sue for defamation!

Def: The *impulse response* of a system is simply its response to an impulse: Impulse=$\delta(t) \rightarrow$ SYSTEM $\rightarrow h(t)$=impulse response (makes sense).

Def: The *step response* of a system is simply its response to a step function: step=$1(t) \rightarrow$ SYSTEM $\rightarrow s(t)$=step response (is easier to compute).

CONTINUOUS-TIME IMPULSE RESPONSE

Q: How to compute the response to an input that doesn't exist physically?

A: Consider a series RC circuit driven by a high and fast voltage pulse:

1. Source=step=$x(t)$=$1(t) \rightarrow y(t)$=$s(t)$=$[1-e^{-t/(RC)}]1(t)$ (familiar).
2. Step up, then down: $x(t)$=$1(t)-1(t-\Delta) \rightarrow y(t)=s(t)-s(t-\Delta)$.
3. Scale previous: $x(t)$=$\frac{1}{\Delta}[1(t)-1(t-\Delta)] \rightarrow y(t)=\frac{1}{\Delta}[s(t)-s(t-\Delta)]$.
4. For $t>\Delta$, this becomes $y(t)=\frac{1}{\Delta}[1-e^{-t/(RC)}]-\frac{1}{\Delta}[1-e^{-(t-\Delta)/(RC)}]$
   $=[e^{\Delta/(RC)}-1]\frac{1}{\Delta}e^{-t/(RC)} \approx \frac{\Delta}{RC}\frac{1}{\Delta}e^{-t/(RC)}=\frac{1}{RC}e^{-t/(RC)}$ for $t>0$
   using $(e^x-1)=(1+x+\frac{x^2}{2!}+\ldots-1)\approx x$ for $x=\Delta/(RC)<<1$.
5. Response $y(t)$ to a pulse $x(t)$ of width Δ, height $\frac{1}{\Delta}$ and area $\frac{\Delta}{\Delta}=1$ is **independent of** Δ and $\frac{1}{\Delta}$, as long as $\Delta << RC$ (note the units).

PHYSICAL INTERPRETATION OF IMPULSE AND IMPULSE RESPONSE

1. Take Norton equivalent: Impulsive current source $\frac{1}{R\Delta}$. Duration=Δ.
2. Shot of charge $\frac{1}{R\Delta}\Delta$ charges capacitor up to $\frac{1}{C}\frac{1}{R\Delta}\Delta=\frac{1}{RC}$. $(q=Cv)$
3. Capacitor voltage decays. Impulsive source like an initial condition!
4. Current=$\frac{1}{R\Delta}$ and duration=Δ don't matter–only charge=product.
5. How can the capacitor voltage jump? Because the current is infinite!

PROPERTIES OF IMPULSE AND IMPULSE RESPONSE

1. Impulse response=$h(t)=\frac{ds}{dt}$=$\frac{d}{dt}$(step response). For above circuit, $h(t)=\frac{d}{dt}(1-e^{-t/(RC)})=\frac{1}{RC}e^{-t/(RC)}$ for $t>0$. For above circuit,
2. $h(t)=\mathcal{L}^{-1}\{H(s)\}=\mathcal{L}^{-1}\{\frac{1/(sC)}{R+1/(sC)}\}=\mathcal{L}^{-1}\{\frac{1/(RC)}{s+1/(RC)}\}=\frac{1}{RC}e^{-t/(RC)}$.
3. $\int_a^b \delta(t-c)dt$ =area under $\delta(t)$=1 if $a<c<b$. $\int_{-\infty}^{\infty}\delta(t-a)dt=1$.
4. $\int_{-\infty}^{\infty}x(t)\delta(t-a)dt=x(a)$ (sifting property). $x(t)\delta(t-a)=x(a)\delta(t-a)$.
5. $\delta(at)=\frac{1}{|a|}\delta(t)$ (both have area $\frac{1}{|a|}$). $\delta(t)1(t)$ and $\frac{\delta(t)}{t}$ are undefined.

TEN RULES FOR COMPUTING CONVOLUTIONS

1. $h(t) * [ax(t) + by(t)) = a[h(t) * x(t)] + b[h(t) * y(t)]$
   Break up convolutions using linear combinations.
2. If $h(t) * x(t) = y(t)$ then $h(t-a) * x(t+b) = y(t-a+b)$
   This is *very* useful if the given functions have delays.
3. $x(t) * \delta(t-a) = x(t-a)$ and $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$
   $\frac{dh}{dt} * x(t) = h(t) * \frac{dx}{dt} = \frac{dy}{dt}$; similarly for integrals.
   #1-#3 greatly simplifies many convolutions in EECS 216.
4. If both $h(t)$ and $x(t)$ are causal, then
   $y(t) = h(t) * x(t) = [\int_0^t h(\tau)x(t-\tau)d\tau]u(t)$ is also causal.
5. If $x(t)$ is a more complicated function than $h(t)$, use
   $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ so you don't substitute $t-\tau$.
6. If $h(t)$=0 outside $0 < t < L$ and $x(t)$=0 outside $0 < t < M$,
   then $y(t) = h(t) * x(t) = 0$ outside $0 < t < (L+M)$. And use #2!
7. Even*even=even; Even*odd=odd; Odd*odd=even functions.
   Use #2 to shift symmetric and antisymmetric functions.
8. $\delta(t)*\delta(t) = \delta(t)$ and u(t)*u(t)=r(t)=tu(t) and rect(t)*rect(t)=triangle(t).
9. If $y(t) = h(t) * x(t)$ then $\int y(t)dt = [\int h(t)dt][\int x(t)dt]$. Good check.
10. Discretize $h(t)$ and $x(t)$ and use Matlab's `conv` to check form.
    You still need to set the scale factor properly (use #9).

EX #1: Compute $y(t) = e^{-t}u(t) * [u(t) - u(t-a)]$.

Sol'n: Using #1-#5, $y(t) = e^{-t}u(t) * u(t) - e^{-t}u(t) * u(t-a)$

1^{st} term=$\int_0^t e^{-\tau}d\tau\, u(t) = [1-e^{-t}]u(t)$. 2^{nd} term=$[1-e^{-(t-a)}]u(t-a)$.

$y(t) = [1-e^{-t}]u(t) - [1-e^{-(t-a)}]u(t-a)$=RC circuit pulse response.

EX #2: Compute $y(t) = e^{-t}u(t) * \frac{t}{T}[u(t) - u(t-T)]$.

Hint: Note $\frac{t}{T}[u(t) - u(t-T)] = \frac{1}{T}\int_0^t [u(t) - u(t-T) - T\delta(t-T)]dt$.

Sol'n: Abusing notation for clarity and using #3 and Ex #1 above, we have

$y(t) = \int_0^t [e^{-t}u(t) * \frac{1}{T}[u(t) - u(t-T) - T\delta(t-T)]]dt$

$y(t) = \frac{1}{T}\int_0^t [1-e^{-t}]dt\, u(t) - \frac{1}{T}\int_T^t [1-e^{-(t-T)}]dt\, u(t-T) - \int_T^t e^{-(t-T)}dt\, u(t-T)$

$y(t) = \frac{1}{T}[t+e^{-t}-1]u(t) - \frac{1}{T}[(t-T)+e^{-(t-T)}-1]u(t-T) + [e^{-(t-T)}-1]u(t-T)$

Compare to Soliman and Srinath p. 60–yes, they DO agree (try it!).