Forward: Laplace transform $X(s)$ of $x(t)$ is: $X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}dt$.

Inverse: $x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$. **You'll never use this formula–ever!**

Need: $x(t)$ piecewise continuous; of exponential order, for $X(s)$ to be defined.

EX: $\mathcal{L}\{e^{at}1(t)\} = \int_0^\infty e^{(a-s)t}dt = \frac{1}{s-a}$ **CRUCIAL FORMULA**. $a = 0 \to \mathcal{L}\{1(t)\} = \frac{1}{s}$.

EX: $\mathcal{L}\{\cos(\omega t)1(t)\} = \mathcal{L}\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})1(t)\} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega} = \frac{s}{s^2+\omega^2}$.

EX: $\mathcal{L}\{\sin(\omega t)1(t)\} = \mathcal{L}\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})1(t)\} = \frac{1/(2j)}{s-j\omega} - \frac{1/(2j)}{s+j\omega} = \frac{\omega}{s^2+\omega^2}$.

Props: $\mathcal{L}\{\delta(t)\} = 1$. $\mathcal{L}\{e^{at}x(t)\} = X(s-a)$. $\mathcal{L}\{e^{at}\cos(\omega t)1(t)\} = \frac{s-a}{(s-a)^2+\omega^2}$

Vital: $\mathcal{L}\{\frac{dx}{dt}\} = sX(s) - x(0)$.$\mathcal{L}\{\frac{d^n y}{dt^n}\} = s^n Y(s) - s^{n-1}y(0) - \ldots - \frac{d^{n-1}y}{dt^{n-1}}(0)$.

Use: The solution to the linear constant-coefficient differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y(t) = K\left(\frac{d^m x}{dt^m} + b_1\frac{d^{m-1}x}{dt^{m-1}} + \ldots + b_m x(t)\right)$$

and: Initial Conditions $y(0), \frac{dy}{dt}(0), \ldots \frac{d^{n-1}y}{dt^{n-1}}(0)$ for differential equation, is:

$$Y(s) = K\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n}X(s) + \frac{\sum_{i=0}^{n-1}\frac{d^{n-1-i}y}{dt^{n-1-i}}(0)\sum_{j=0}^{i}a_j s^{i-j}}{s^n+a_1s^{n-1}+\ldots+a_n} = \text{ZSR+ZIR}$$

INVERSE LAPLACE TRANSFORM OF RATIONAL FUNCTION

Goal: Compute $\mathcal{L}^{-1}\left\{\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n}\right\}$ of $\begin{smallmatrix}\text{RATIONAL}\\ \text{FUNCTION}\end{smallmatrix}$ if $\begin{smallmatrix}\text{STRICTLY}\\ \text{PROPER}\end{smallmatrix}$ $(m < n)$.

Note: If this is not $\begin{smallmatrix}\text{STRICTLY}\\ \text{PROPER}\end{smallmatrix}$ function, write $\begin{smallmatrix}\text{ORIGINAL}\\ \text{FUNCTION}\end{smallmatrix} = c + \begin{smallmatrix}\text{STRICTLY}\\ \text{PROPER}\end{smallmatrix}$.

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1. Compute the n **poles** $\{p_i\}$ which solve $s^n + a_1s^{n-1} + \ldots + a_n = 0$.

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2. Compute $\begin{smallmatrix}\text{PARTIAL}\\ \text{FRACTION}\end{smallmatrix}$ $\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n} = \frac{A_1}{s-p_1} + \ldots + \frac{A_n}{s-p_n}$

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where: $A_k = \frac{s^m+b_1s^{m-1}+\ldots+b_m}{(s-p_1)\ldots(s-p_n)}(s-p_k)|_{s=p_k} = \frac{p_k^m+b_1p_k^{m-1}+\ldots+b_m}{(p_k-p_1)\ldots(p_k-p_n)}$ (formula).

Note: Denominator$=(p_k - p_1)\ldots(p_k - p_{k-1})(p_k - p_{k+1})\ldots(p_k - p_n)$.

Note: Assumes poles distinct: $p_i \neq p_j$. $\begin{smallmatrix}\text{COMPLEX}\\ \text{CONJUGATE}\end{smallmatrix}$: $p_{k+1}=p_k^* \to A_{k+1}=A_k^*$

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3. Then $\mathcal{L}^{-1}\left\{\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n}\right\} = A_1e^{p_1t}1(t)+\ldots A_ne^{p_nt}1(t)$ (formula).

Note: $\mathcal{L}^{-1}\left\{c + \frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n}\right\} = c\delta(t) + A_1e^{p_1t}1(t) + \ldots A_ne^{p_nt}1(t)$.

Note: $\mathcal{L}^{-1}\left\{\frac{Ae^{j\theta}}{s-(a+j\omega)} + \frac{Ae^{-j\theta}}{s-(a-j\omega)}\right\} = 2Ae^{at}\cos(\omega t+\theta)1(t)$ (useful formula).

Several illustrative numerical examples are given overleaf.

EECS 216 INVERSE LAPLACE TRANSFORM EXAMPLES

Solve: $\frac{d^2y}{dt^2} + y = t$ for $t > 0$ with $y(0) = 1$ and $\frac{dy}{dt}(0) = -2$. Take $\mathcal{L}$:

Soln: $[s^2Y(s) - sy(0) - \frac{dy}{dt}(0)] + Y(s) = \frac{1}{s^2}$. Substitute initial conditions:

$\rightarrow [s^2Y - s + 2] + Y = \frac{1}{s^2} \rightarrow [s^2 + 1]Y = s - 2 + \frac{1}{s^2}$ (don't need partial

$\rightarrow Y(s) = \frac{s-2}{s^2+1} + \left[\frac{1}{s^2}\frac{1}{s^2+1}\right] = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$ fractions here)

$\rightarrow Y(s) = \frac{s}{s^2+1} - \frac{3}{s^2+1} + \frac{1}{s^2} \rightarrow y(t) = \cos(t) - 3\sin(t) + t$ for $t > 0$.

Solve: $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{3t}$ for $t > 0$ with $y(0) = -3$ and $\frac{dy}{dt}(0) = 5$.

Soln: $[s^2Y(s) - sy(0) - \frac{dy}{dt}(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s-3}$

$\rightarrow [s^2Y + 3s - 5] - 3[sY + 3] + 2Y = \frac{4}{s-3} \rightarrow [s^2 - 3s + 2]Y + (3s - 14) = \frac{4}{s-3}$

$\rightarrow Y(s) = \frac{4}{(s^2-3s+2)(s-3)} - \frac{3s-14}{s^2-3s+2}\frac{s-3}{s-3} = \frac{-3s^2+23s-38}{(s-1)(s-2)(s-3)}$.

PARTIAL FRACTION: $Y(s) = \frac{-9}{s-1} + \frac{4}{s-2} + \frac{2}{s-3} \rightarrow y(t) = -9e^t + 4e^{2t} + 2e^{3t}$ for $t > 0$.

Solve: $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin(t)$ for $t > 0$ with $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$.

Soln: $[s^2Y(s) - sy(0) - \frac{dy}{dt}(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2+1}$

$\rightarrow [s^2Y - 1] + 2[sY] + 5Y = \frac{1}{s^2+2s+2} \rightarrow [s^2 + 2s + 5]Y - 1 = \frac{1}{s^2+2s+2}$

$\rightarrow Y(s) = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}\frac{s^2+2s+2}{s^2+2s+2} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$.

Poles: $s^2+2s+2=0 \rightarrow \{-1 \pm j\}$ and $s^2+2s+5=0 \rightarrow \{-1 \pm 2j\}$.

PARTIAL FRACTION: $Y(s) = \frac{1/(6j)}{s+1-j} - \frac{1/(6j)}{s+1+j} + \frac{1/(6j)}{s+1-2j} - \frac{1/(6j)}{s+1+2j}$. Note $\frac{1}{6j} = \frac{1}{6}e^{-j\pi/2}$.

$\rightarrow y(t) = 2\frac{1}{6}e^{-t}\cos(t - \frac{\pi}{2}) + 2\frac{1}{6}e^{-t}\cos(2t - \frac{\pi}{2}) = \frac{1}{3}e^{-t}[\sin(t) + \sin(2t)]1(t)$.

OR: $Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5} = \frac{1/3}{(s+1)^2+1} + \frac{(1/3)2}{(s+1)^2+4}$

$\rightarrow y(t) = \frac{1}{3}e^{-t}\sin(t) + \frac{1}{3}e^{-t}\sin(2t)$ for $t > 0$ (this is usually too hard).

Solve: The *simultaneous* $\frac{dx}{dt} = 2x - 3y$ with $x(0) = 8$.
differential eqns. $\frac{dy}{dt} = -2x + y$ with $y(0) = 3$.

Huh? Using Laplace transforms, this is actually very easy!

Soln: $sX(s) - 8 = 2X(s) - 3Y(s) \rightarrow (s-2)X(s) + 3Y(s) = 8$.
$sY(s) - 3 = -2X(s) + Y(s) \rightarrow 2X(s) + (s-1)Y(s) = 3$.
Solving 2 simultaneous algebraic equations in 2 unknowns yields
$X(s) = \frac{8s-17}{s^2-3s-4} = \frac{5}{s+1} + \frac{3}{s-4} \rightarrow x(t) = 5e^{-t} + 3e^{4t}$ for $t > 0$.
$Y(s) = \frac{3s-22}{s^2-3s-4} = \frac{5}{s+1} - \frac{2}{s-4} \rightarrow y(t) = 5e^{-t} - 2e^{4t}$ for $t > 0$.