FREQUENCY RESPONSE OF LTI SYSTEMS

Note: $e^{j\omega_o t} \rightarrow \boxed{h(t)} \rightarrow H(j\omega_o)e^{j\omega_o t}$ in sinusoidal steady-state.

where: $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \mathcal{F}\{h(t)\}$=**frequency response** function.

Huh? A single frequency into an LTI system→Same single frequency out.

Why? y(t)=h(t)*$e^{j\omega_o t}$=$\int h(\tau)e^{j\omega_o(t-\tau)}d\tau = e^{j\omega_o t}\int h(\tau)e^{-j\omega_o\tau}d\tau = e^{j\omega_o t}H(j\omega_o)$.

So? $e^{+j\omega_o t} \rightarrow \boxed{h(t)} \rightarrow H(+j\omega_o)e^{+j\omega_o t}$.

and: $e^{-j\omega_o t} \rightarrow \boxed{h(t)} \rightarrow H(-j\omega_o)e^{-j\omega_o t}$.

Add: $\cos(\omega_o t) \rightarrow \boxed{h(t)} \rightarrow |H(j\omega_o)|\cos(\omega_o t + \arg[H(j\omega_o)])$

since: LTI system and $H(\pm j\omega_o) = |H(j\omega_o)|e^{\pm j\arg[H(j\omega_o)]}$.

Gain: Amplitude increases by factor of $|H(j\omega_o)|$.

Phase: Shift by $\arg[H(j\omega_o)] = \tan^{-1}\frac{\text{Im}[H(j\omega_o)]}{\text{Re}[H(j\omega_o)]}$.

EX #1: $h(t) = e^{-2t}u(t)$ and $x(t) = 6\cos(2t - 20^o)$. Find steady-state $y(t)$.

Soln: $H(s) = \frac{1}{s+2} \rightarrow H(j\omega) = \frac{1}{j\omega+2} \rightarrow H(j2) = \frac{1}{j2+2} = 0.3536e^{-j45^o}$.

So: $y(t) = 2.121\cos(2t - 65^o)$ where 6(0.3536)=2.121 and -20^o-45^o=-65^o.

EX #2: $h(t) = [5e^{-t} - 16e^{-2t} + 13e^{-3t}]u(t)$ and $x(t) = 6\cos(2t - 20^o)$. y(t)=?

Soln: $H(s)=\mathcal{L}\{h(t)\}=\frac{5}{s+1}\frac{(s+2)(s+3)}{(s+2)(s+3)} - \frac{16}{s+2}\frac{(s+1)(s+3)}{(s+1)(s+3)} + \frac{13}{s+3}\frac{(s+1)(s+2)}{(s+1)(s+2)} = \frac{2s^2+8}{s^3+6s^2+11s+6}$.

Then: $H(j\omega) = \frac{2(j\omega)^2+8}{(j\omega)^3+6(j\omega)^2+11(j\omega)+6}$ by plugging $s = j\omega$ into $H(s)$.

And: $H(\omega) = \frac{8-2\omega^2}{(6-6\omega^2)+j(11\omega-\omega^3)}$ Don't confuse $H(j\omega)$ and $H(\omega)$!

Note: Re$[H(\omega)]$ is even function in ω and Im$[H(\omega)]$ is odd function in ω.
But $H(2) = 0 \rightarrow y(t) = 0$! (in steady-state). Did you see that coming?
Should: $H(s)$ has zeros at $s = \pm j2$, so sinusoid at $\omega = 2$ is rejected.

So? Use *superposition* property of LTI systems and Fourier transform:

1. Let $x(t)$ have Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$.
2. Then $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}$ (inverse Fourier transform)
3. Then $X(j\omega)e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow H(j\omega)X(j\omega)e^{j\omega t}$ (scaling of LTI)
4. Then $x(t) \rightarrow \boxed{h(t)} \rightarrow \int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}dt$ (superposition)

Huh? The frequency component of $x(t)$ at ω_o is now multiplied by $H(j\omega_o)$.

EX #3: $h(t) = [5e^{-t} - 16e^{-2t} + 13e^{-3t}]u(t)$ as in **EX #2** above.

And: $x(t) = 6 + 10\cos(t) + 13\cos(2t) + 30\cos(\sqrt{11}t)$. Compute $y(t)$ in SS.

Soln: $H(0) = \frac{8}{6}; H(1) = \frac{6}{10j}; H(2) = 0; H(\sqrt{11}) = \frac{14}{60}$ (plug into $H(\omega)$).

So: $y(t)$=8+6cos(t-90^o)+0cos(2t)+7cos($\sqrt{11}$t)=8+6sin(t)+7cos($\sqrt{11}$t).

EX #4: Compute $H(j\omega)$ for LTI system $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 4\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x$.

Soln: Inserting x=$e^{j\omega t}$ and y=$H(j\omega)e^{j\omega t}$ yields $H(j\omega) = \frac{4(j\omega)^2+5(j\omega)+6}{(j\omega)^2+2(j\omega)+3}$.