TRANSFER (ALSO KNOWN AS SYSTEM) FUNCTIONS

DEF: $H(s)=\mathcal{L}\{h(t)\}$. Also have $H(s)=\frac{N(s)}{D(s)}=\frac{Y(s)}{X(s)}=\frac{B(s)}{A(s)}$ as defined below.

Poles: Roots of $D(s)=0$. **Differential Eqn.:** $\sum_{n=0}^{N} a_n \frac{d^n y}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x}{dt^m}$.

Zeros: Roots of $N(s)=0$. **Impulse response:** $\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$.

Transfer functions are associated with zero initial conditions (ZSR).

There are 6 ways to describe LTI systems: (1) $h(t)$ (2) $H(s)$ (3) $H(j\omega)$ (4) Differential equation (5) Any input-output pair (6) Poles & zeros. Using transfer functions, **can go from any of these to any other:**

Input $x(t) \rightarrow \boxed{h(t)} \rightarrow$output $y(t)$ $\underbrace{\Longleftrightarrow}_{Y(s)/X(s)}$ $\mathbf{H(s)}$ $\underbrace{\Longleftrightarrow}_{\mathcal{L}\{h(t)\}}$ $h(t) =$ IMPULSE RESPONSE

$$\underbrace{\sum_{n=0}^{N} a_n \frac{d^n y}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x}{dt^m}}_{\text{DIFFERENTIAL EQUATION}} \quad \underbrace{\Longleftrightarrow}_{B(s)/A(s)} \quad \mathbf{H(s)} \quad \underbrace{\Longleftrightarrow}_{C\prod \frac{s-z_i}{s-p_i}}$$ **POLE ZERO** plot.

EX #1: $h(t)=[e^{-2t}+e^{-4t}]u(t) \rightarrow H(s)=\frac{1}{s+2}+\frac{1}{s+4}=\frac{2s+6}{s^2+6s+8}$. ZEROS:$\{-3\}$ POLES:$\{-2,-4\}$.

$\frac{Y(s)}{X(s)}=\frac{2s+6}{s^2+6s+8} \rightarrow Y(s)(s^2+6s+8)=X(s)(2s+6) \rightarrow \frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y=2\frac{dx}{dt}+6x$.

Forced ZSR to $x(t)=e^{-3t}u(t)$ is $y(t)=0$ due to zero at –3: "eats" $x(t)$!

Forced step response (to $x(t)=u(t)$) is $H(0)u(t)=\frac{6}{8}u(t)$. Use this below.

EX #2: $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=\frac{d^2x}{dt^2}+5\frac{dx}{dt}+4x$. Read off $H(s)=\frac{s^2+5s+4}{s^2+5s+6}$.

Zeros: $N(s)=s^2+5s+4=0 \rightarrow s=\{-1,-4\}$. Use Matlab's **roots**.

Poles: $D(s)=s^2+5s+6=0 \rightarrow s=\{-2,-3\}$. Use Matlab's **roots**.

$H(s)=0$ for s=any zero (see above) while $H(s)\rightarrow\infty$ for s=any pole.

EX #3: $x(t)=e^{-2t}u(t) \rightarrow \boxed{h(t)} \rightarrow y(t)=e^{-2t}u(t)-e^{t}u(t)$. Compute $h(t)$.

Soln: $X(s)=\frac{1}{s+2}$. $Y(s)=\frac{1}{s+2}-\frac{1}{s-1}=\frac{-3}{(s+2)(s-1)}$. $H(s)=\frac{-3}{s-1}$. $h(t)=-3e^{t}u(t)$.

First term of $y(t)$=forced response. Second term=natural response.

$H(s)=\frac{Y(s)}{X(s)}=\frac{-3}{s-1} \rightarrow sY(s)-Y(s)=-3X(s) \rightarrow \frac{dy}{dt}-y=-3x$. Read $h(t)$.

EX #4: Find the differential equation implementing system $H(s)=\frac{s+7}{s^3-s^2+2s-3}$.

Soln: Write $H(s)=\frac{Y(s)}{X(s)}=\frac{s+7}{s^3-s^2+2s-3}$, **cross-multiply**, and take $\mathcal{L}^{-1}$:

$Y(s)(s^3-s^2+2s-3)=X(s)(s+7) \rightarrow \frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\frac{dx}{dt}+7x$.

EX #5: A system has zero at 1, pole at –3, and a forced step response $-u(t)$.

Goal: Compute response to $x(t)=e^{t}u(t)$. Forced step response $\rightarrow H(0)=-1$.

Soln: $H(s)=3\frac{s-1}{s+3}$. $X(s)=\frac{1}{s-1} \rightarrow Y(s)=\frac{3}{s+3} \rightarrow y(t)=3e^{-3t}u(t)$. FORCED RESPON. $=0$.

Differential equation: $H(s)=\frac{Y(s)}{X(s)}=3\frac{s-1}{s+3}$. CROSS MULT. $\rightarrow \frac{dy}{dt}+3y=3\frac{dx}{dt}-3x$.

POLES AND ZEROS AND FREQUENCY RESPONSE

Given: Locations of zeros $\{z_1 \dots z_M\}$ and poles $\{p_1 \dots p_N\}$ of a filter.
Goal: *Shape* of gain function=$|H(\omega)|$=magnitude of frequency response.

H(s): $H(s) = \frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)(s-p_2)\dots(s-p_N)}$ (ignore any constant in front).
H(w): $|H(\omega)| = [|j\omega - z_1| \dots |j\omega - z_M|]/[|j\omega - p_1| \dots |\omega - p_N|]$.
What does each of these terms contribute to gain $|H(\omega)|$?

Zeros: Let n^{th} zero $z_n = j\omega_n$. Then $|j\omega - z_n| = 0$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)| = 0$ if there is a zero at $j\omega_o$ (on the imaginary axis).
- Gain $|H(\omega_o)| \approx 0$ if there is a zero at $r + j\omega_o, r \approx 0$ (near the axis).

Poles: Let n^{th} pole $p_n = r + j\omega_n, r \approx 0$. Then $\frac{1}{|j\omega - p_n|} = \frac{1}{|r|}$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)|$ is large if there is a pole at $r + j\omega_o, r \approx 0$.

1. Start on the imaginary axis at $\omega = 0 \rightarrow s = j\omega = 0$.
2. Trace along imaginary axis upwards (increasing ω).
3. When pass a zero at $r_m + j\omega_m, r_m \approx 0$, gain dips at ω_m.
4. When pass a pole at $r_n + j\omega_n$, $r_n \approx 0$, gain peak at ω_n.
5. You saw Bode plots in EECS 215, but it's worth reminding you of this.

Given: A series RLC circuit is driven by a 100 volt (rms) sinusoidal source.
Max. power=5 watts at 159 Hz; 2.5 watts at 0.159 Hz and 159 kHz.
Goal: From this information, compute R,L,C, assuming reasonable values.

Soln: Phasors: $I_{rms} = V_{rms}/[R + j\omega L + \frac{1}{j\omega C}] = 100/[R + j(\omega L - \frac{1}{\omega C})]$.
Power: In sinusoidal steady-state: $|I_{rms}|^2 R = 10000R/[R^2 + (\omega L - \frac{1}{\omega C})^2]$.
Max: 5=10000R/[R^2+0] at resonance ω=$\frac{1}{\sqrt{LC}} \rightarrow R$=2000Ω. 159≈ $\frac{1000}{2\pi}$.
Half: $2\pi\frac{1}{2\pi} = 1 = \frac{1}{RC} \rightarrow C = \frac{1}{2000}F = 500\mu F$ (lower half-power frequency).
$2\pi\frac{10^6}{2\pi} = 10^6 = \frac{R}{L} \rightarrow L = \frac{2}{1000}H$=2 mH (upper half-power frequency).

Exchanging $\frac{R}{L} \Leftrightarrow \frac{1}{RC} \rightarrow$ C=0.5nF and L=2000H (unreasonable!).
Note: 2000H is a completely unreasonable value for a real-world inductor.
Note: Identifying all three RLC circuit element values from so little data shows the power of looking at problems in the frequency domain.

FEEDBACK AMPLIFIER GAIN-BANDWIDTH PRODUCT

Given: 741 op-amp has DC gain=106 dB=$2 \cdot 10^5$ and break freq.=8 Hz(!).
Goal: See how feedback allow us to trade smaller gain for more bandwidth.

Open loop: $H(s)=10^7/(s+50)$: $2\pi 8 \approx 50$ and $H(j0)=10^7/50=2 \cdot 10^5$.
Closed loop: $\frac{H(s)}{1+KH(s)} = \frac{10^7/(s+50)}{1+K \cdot 10^7/(s+50)} \frac{s+50}{s+50} = 10^7/(s+50+K \cdot 10^7)$.

$50 << K \cdot 10^7 \rightarrow$ DC Gain=$\frac{1}{K}$; Bandwidth=$K \cdot 10^7$; (Gain)(BW)=10^7.