FREQUENCY RESPONSE OF LTI SYSTEMS

Note: $e^{j\omega_o n} \rightarrow \boxed{h[n]} \rightarrow H(e^{j\omega_o})e^{j\omega_o n}$

where: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$=**frequency response** function.

Huh? Single frequency into LTI system→Same single frequency out.

Why? $y[n] = h[n] * e^{j\omega_o n} = \sum h[i]e^{j\omega_o(n-i)} = e^{j\omega_o n}\sum h[i]e^{-j\omega_o i} = e^{j\omega_o n}H(e^{j\omega_o})$.

So? $e^{+j\omega_o n} \rightarrow \boxed{h[n]} \rightarrow H(e^{+j\omega_o})e^{+j\omega_o n}$.

and: $e^{-j\omega_o n} \rightarrow \boxed{h[n]} \rightarrow H(e^{-j\omega_o})e^{-j\omega_o n}$.

Add: $\cos(\omega_o n) \rightarrow \boxed{h[n]} \rightarrow |H(e^{j\omega_o})|\cos(\omega_o n + arg[H(e^{j\omega_o})])$

since: LTI system and $H(e^{\pm j\omega_o}) = |H(e^{j\omega_o})|e^{\pm j arg[H(e^{j\omega_o})]}$.

Gain: Amplitude increases by factor of $|H(e^{j\omega_o})|$.

Phase: Shift by $arg[H(e^{j\omega_o})] = \tan^{-1}\frac{Im[H(e^{j\omega_o})]}{Re[H(e^{j\omega_o})]}$.

EX #1: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = \{\ldots -1, 0, 1, 0, -1 \ldots\} = \cos(\frac{\pi n}{2})$.
$H(e^{j\omega}) = 1/(1 - \frac{1}{2}e^{-j\omega}) = 1/(1 + \frac{j}{2}) = 0.89e^{-j26.6^o}$ at $\omega = \frac{\pi}{2}$
$y[n] = 0.89\cos(\frac{\pi n}{2} - 26.6^o) = \{\ldots 0.8, 0.4, -0.8, -0.4, 0.8, 0.4 \ldots\}$.

EX #2: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = \{\ldots -1, 1, -1, 1, -1 \ldots\} = \cos(\pi n)$.
$H(e^{j\omega}) = 1/(1 + \frac{1}{2}) = \frac{2}{3}$ at $\omega = \pi \rightarrow y[n] = \frac{2}{3}\cos(\pi n) = \frac{2}{3}(-1)^n$.

Why $\frac{2}{3}$? $y[n] = h[n] * x[n] = \sum(\frac{1}{2})^i(-1)^{n-i} = (-1)^n\sum(-\frac{1}{2})^i = (-1)^n\frac{1}{1-(-\frac{1}{2})}$.

EX #3a: $h[n] = \{\frac{1}{2}, +\frac{1}{2}\} \rightarrow H(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) = \cos(\frac{\omega}{2})e^{-j\omega/2}$. Lowpass.

EX #3b: $h[n] = \{\frac{1}{2}, -\frac{1}{2}\} \rightarrow H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = \sin(\frac{\omega}{2})e^{j(\pi-\omega)/2}$. High.

Notch: $H(e^{j\omega}) = 2\cos(\omega) - 2\cos(\omega_o)$. $h[n] = \{1, -2\cos(\omega_o), 1\}$.

Why? This **filters** out a frequency component at frequency ω_o.

So? Use *superposition* property of LTI systems and Fourier series:

1. Let $x[n]$ be **any** periodic signal. Let $x[n]$ have period N.
2. Then $x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$ for some constants X_k.
3. Then $X_k e^{j2\pi nk/N} \rightarrow \boxed{h[n]} \rightarrow H(e^{j2\pi k/N})X_k e^{j2\pi nk/N}$ (note $\omega_o = \frac{2\pi k}{N}$).
4. Then $x[n] \rightarrow \boxed{h[n]} \rightarrow \sum_{k=0}^{N-1} H(e^{j2\pi k/N})X_k e^{j2\pi nk/N}$.

Huh? The k^{th} frequency component of $x[n]$ is multiplied by $H(e^{j2\pi k/N})$.

EX: $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = 1 + 2\cos(\frac{\pi}{2}n) + 3\cos(\pi n)$.

Then: $y[n] = \frac{1}{1-\frac{1}{2}}1 + |\frac{1}{1+\frac{j}{2}}|2\cos(\frac{\pi}{2}n + arg[\frac{1}{1+\frac{j}{2}}]) + \frac{1}{1+\frac{1}{2}}3\cos(\pi n)$

$\rightarrow y[n] = 2 + 1.78\cos(\frac{\pi}{2}n - 26.6^o) + 2\cos(\pi n)$.

NOTE: Plots of **several periods** of frequency response for $h[n]$.
Remember: $H(e^{j\omega})$ is periodic in ω with period 2π!