DEF: $\mathcal{Z}^+\{x(n)\} = X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$, *even if x(n) is noncausal.*
Compare to *2-sided* z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$.

WHY?: To solve difference equations with nonzero initial conditions.

PROPERTIES OF 1-SIDED Z-TRANSFORM $X^+(z)$:

1. $x(n)$ causal→1-sided z-transform $X^+(z)$=2-sided z-transform $X(z)$
2. Unique inverse only for causal signals: no info on $\{x(n), n < 0\}$
   Equal causal parts but unequal anticausal parts→same $X^+(z)$
3. ROC: Always $|z| > |p_{max}|$ (largest pole of $X^+(z)$), so don't bother

4a. $D > 0 \rightarrow \mathcal{Z}^+\{x(n-D)\} = z^{-D}(X^+(z) + \sum_{n=1}^{D} x(-n)z^n)$

4b. $D > 0 \rightarrow \mathcal{Z}^+\{x(n+D)\} = z^{+D}(X^+(z) - \sum_{n=0}^{D-1} x(n)z^{-n})$

   i. Shift right→must add in terms *previously* anticausal
   ii. Shift left→must subtract off terms *now* anticausal
   iii. Compare to $\mathcal{L}\{\frac{d^2x}{dt^2}\} = s^2X(s) - sx(0) - \frac{dx}{dt}(0)$

SOLVING DIFFERENCE EQNS WITH INITIAL CONDITIONS

Solve: $2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2)$ (Chen p.264)

Take $\mathcal{Z}^+$: $2Y^+(z) + 3z^{-1}(Y^+(z) + y(-1)z) + z^{-2}(Y^+(z) + y(-1)z + y(-2)z^2) =$
$(2 + 3z^{-1} + z^{-2})Y^+(z) + [(3y(-1) + y(-2)) + z^{-1}y(-1)] = (1 + z^{-1} - z^{-2})U(z)$

Solve:
$$Y^+(z) = \underbrace{\frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}}U(z)}_{\text{zero-state response}} - \underbrace{\frac{(3y(-1) + y(-2)) + z^{-1}y(-1)}{2 + 3z^{-1} + z^{-2}}}_{\text{zero-input response}}$$

Plug in: $u(n)$=unit step→ $U(z) = U^+(z) = \frac{1}{1-z^{-1}}$; $y(-1) = 2, y(-2) = -1$:

$$Y^+(z) = \frac{1+z^{-1}-z^{-2}}{(2+3z^{-1}+z^{-2})(1-z^{-1})} - \frac{5+2z^{-1}}{2+3z^{-1}+z^{-2}} = \frac{-4+4z^{-1}+z^{-2}}{2(1+\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})}$$

Partial fraction: $Y^+(z) = \frac{4/3}{1+\frac{1}{2}z^{-1}} - \frac{7/2}{1+z^{-1}} + \frac{1/6}{1-z^{-1}}$. Note no constant term.

$\mathcal{Z}^{-1}$:
$$y(n) = \underbrace{\frac{4}{3}(-\frac{1}{2})^n u(n) - \frac{7}{2}(-1)^n u(n)}_{\text{natural response}} + \underbrace{\frac{1}{6}u(n)}_{\text{forced}}.$$

Given: $2y(n)+3y(n-1)+y(n-2)=u(n)+u(n-1)-u(n-2)$ (previous)

Soln: $$Y(z)=\underbrace{\frac{1+z^{-1}-z^{-2}}{2+3z^{-1}+z^{-2}}U(z)}_{\text{zero-state response(ZSR)}}-\underbrace{\frac{[3y(-1)+y(-2)]+z^{-1}y(-1)}{2+3z^{-1}+z^{-2}}}_{\text{zero-input response(ZIR)}}$$

Poles: Roots of $2+3z^{-1}+z^{-2}=0\rightarrow -1,-\frac{1}{2}$ from denominator of ZSR.

Modes: Roots of $2+3z^{-1}+z^{-2}=0\rightarrow -1,-\frac{1}{2}$ from denominator of ZIR.

Huh? $\{\text{poles}\}\subseteq\{\text{modes}\}$ since poles can be canceled by zeros of $H(z)$.

Note: Can also cancel poles by proper choice of input $u(n)$.

EX #1: $u(n)=\delta(n)+\delta(n-1)\rightarrow U(z)=1+z^{-1}\rightarrow ZSR(z)=\frac{1+z^{-1}-z^{-2}}{2+z^{-1}}$.

Note: Can also cancel modes by proper choice of initial conditions $y(-1),y(-2)$.

EX #2: $y(-1)=1,y(-2)=-2\rightarrow ZIR(z)=-\frac{1}{2+z^{-1}}\rightarrow zir(n)=-\frac{1}{2}(-\frac{1}{2})^n u(n)$.

How? Choose initial conditions $y(-1),y(-2)$ to cancel the mode at -1.

Soln: Need $3y(-1)+y(-2)=y(-1)$ so ZIR numerator is $y(-1)(1+z^{-1})$.

EX #3: Change one MA-part coefficient so $\{\text{poles}\}\subset\{\text{modes}\}$.

Soln: Change $1+z^{-1}-z^{-2}$ so $\frac{1+z^{-1}-z^{-2}}{2+3z^{-1}+z^{-2}}$ has pole-zero cancellation.

EX: Change ZIR numerator to $1+0z^{-1}-z^{-2}=(1+z^{-1})(1-z^{-1})$.

CAUSALITY AND BIBO STABILITY

Fact: BIBO stable$\Leftrightarrow\sum|h(n)|<\infty\Leftrightarrow\{|z|=1\}\subset$ ROC.

Fact: Causal$\Leftrightarrow$ ROC $=\{|z|>A\}$ for some constant A. Also Stable if $|A|<1$.

1. Stable AND Causal$\Leftrightarrow$Poles inside unit circle$\Leftrightarrow|p_i|<1,i=1\dots N$.
2. Stable+Anticausal$\Leftrightarrow$Poles outside unit circle$\Leftrightarrow|p_i|>1,i=1\dots N$.
3. Stable+2-sided$\Leftrightarrow\{|z|=1\}\subset$ ROC = annulus.

INITIAL AND FINAL VALUE THEOREMS AND CORRELATION

IVT: $x(n)$ causal$\rightarrow \underset{z\rightarrow\infty}{\text{LIM}}X(z)=x(0)$ since $X(z)=x(0)+x(1)z^{-1}+\dots$

FVT: $\underset{n\rightarrow\infty}{\text{LIM}}x(n)=\underset{z\rightarrow 1}{\text{LIM}}(z-1)X^+(z)$ (1-sided z-xform)(step).

Auto: $r(n)=x(n)*x(-n)=\sum x(i)x(i\pm n)=r(-n)$ (even function).

Cross: $r_{xy}(n)=x(n)*y(-n)=\sum x(i)y(i-n)=r_{yx}(-n)\neq r_{xy}(-n)$.