
TRANSFER (ALSO KNOWN AS SYSTEM) FUNCTIONS

DEF: $H(z) = N(z)/D(z) = Y(z)/X(z) = B(z)/A(z)$ as defined below.

Poles: Roots of $D(z) = 0$. **ARMA:** $\sum a[i]y[n-i] = \sum b[j]x[n-j]$.

Zeros: Roots of $N(z) = 0$. **Impulse response:** $\delta[n] \rightarrow \overline{h[n]} \rightarrow h[n]$.

Transfer functions are associated with zero initial conditions.

EX #1: $h[n] = 2^n u[n] + 4^n u[n] \rightarrow H(z) = \frac{z}{z-2} + \frac{z}{z-4} = \frac{2z^2-6z}{z^2-6z+8}$. ZEROS: $\{0,3\}$
 POLES: $\{2,4\}$.

EX #2: $y[n] - 3y[n-1] + 2y[n-2] = x[n] - 4x[n-1] \rightarrow H(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}}$.

where: $A(z) = \mathcal{Z}\{1, -3, 2\} = 1-3z^{-1}+2z^{-2}$ and $B(z) = \mathcal{Z}\{1, -4\} = 1-4z^{-1}$.

z not z^{-1} : $H(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}} \frac{z^2}{z^2} = \frac{z^2-4z}{z^2-3z+2}$. Use **this** to compute poles+zeros.

Zeros: Solve $N(z) = z^2 - 4z = 0 \rightarrow z = \{0, 4\}$. Common mistake: cite only 4.

Poles: Solve $D(z) = z^2 - 3z + 2 = 0 \rightarrow z = \{1, 2\}$. Use Matlab's **roots**.

Note: $H(z) = 0$ for z =any zero while $H(z) \rightarrow \infty$ for z =any pole.

There are 5 ways to describe an LTI system: (1) $h[n]$; (2) $H(z)$;

(3) Difference equation (4) Any input-output pair (5) Poles+zeros.

Using transfer functions, **can go from any of these to any other:**

INPUT-OUTPUT: Input $x[n] \rightarrow \overline{h[n]} \rightarrow$ output $y[n]$ \Leftrightarrow $\mathbf{H(z)}$ \Leftrightarrow $h[n] =$ $\frac{\text{IMPULSE}}{\text{RESPONSE}}$

$\underbrace{\hspace{10em}}_{Y(z)/X(z)} \quad \underbrace{\hspace{10em}}_{\mathcal{Z}\{h[n]\}}$

ARMA: $\sum a[i]y[n-i] = \sum b[j]x[n-j]$ \Leftrightarrow $\mathbf{H(z)}$ \Leftrightarrow **POLE ZERO** plot.

$\underbrace{\hspace{10em}}_{B(z)/A(z)} \quad \underbrace{\hspace{10em}}_{C \prod \frac{z-z_i}{z-p_i}}$

EX #3: $x[n] = (-2)^n u[n] \rightarrow \overline{h[n]} \rightarrow y[n] = \frac{2}{3}(-2)^n u[n] + \frac{1}{3}u[n]$. Find $h[n]$.

Soln: $X(z) = \frac{z}{z+2}$. $Y(z) = \frac{2z/3}{z+2} + \frac{z/3}{z-1} = \frac{z^2}{(z+2)(z-1)}$. $H(z) = \frac{z}{z-1} \cdot h[n] = u[n]$.

First term of $y[n]$ =forced response. Second term=natural response.

Difference eqn.: $\frac{Y(z)}{X(z)} = H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \rightarrow y[n] - y[n-1] = x[n]$.

EX #4: Find difference equation implementing $H(z) = \frac{z^3+7z^2}{z^3-z^2+2z-3}$.

Soln: Write $H(z) = \frac{Y(z)}{X(z)} = \frac{1+7z^{-1}}{1-z^{-1}+2z^{-2}-3z^{-3}}$ and **cross-multiply:**

$Y(z)(1-z^{-1}+2z^{-2}-3z^{-3}) = X(z)(1+7z^{-1}) \rightarrow$ difference equation:

z^{-1} : $y[n] - y[n-1] + 2y[n-2] - 3y[n-3] = x[n] + 7x[n-1]$.

EX #5: Find step response of system with zero at 1, pole at 3, and $H(0) = 1$.

Soln: $H(z) = 3 \frac{z-1}{z-3}$. $U(z) = \frac{z}{z-1} \rightarrow Y(z) = 3 \frac{z}{z-3} \rightarrow y[n] = 3 \cdot 3^n u[n]$.

$\frac{Y(z)}{X(z)} = H(z) = 3 \frac{z-1}{z-3} = 3 \frac{1-z^{-1}}{1-3z^{-1}} \rightarrow y[n] - 3y[n-1] = 3x[n] - 3x[n-1]$.

POLES & ZEROS AND FREQUENCY RESPONSE

Given: Locations of zeros $\{z_1 \dots z_M\}$ and poles $\{p_1 \dots p_N\}$ of a filter.

Goal: *Shape* of gain function $= |H(\omega)| = \text{magnitude of frequency response}$.

H(z): $H(z) = \frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$ (ignore any constant in front).

H(w): $|H(\omega)| = [|e^{j\omega} - z_1| \dots |e^{j\omega} - z_M|] / [|e^{j\omega} - p_1| \dots |e^{j\omega} - p_N|]$.

What does each of these terms contribute to gain $|H(\omega)|$?

Zeros: Let n^{th} zero $z_n = e^{j\omega_n}$. Then $|e^{j\omega} - z_n| = 0$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)| = 0$ if there is a zero at $e^{j\omega_o}$ (on the unit circle $|z| = 1$).
 - Gain $|H(\omega_o)| \approx 0$ if there's a zero at $Ae^{j\omega_o}$, $A \approx 1$ (near the unit circle).
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Poles: Let n^{th} pole $p_n = Ae^{j\omega_n}$, $A \approx 1$. Then $\frac{1}{|e^{j\omega} - p_n|} = \frac{1}{|A-1|}$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)|$ is large if there is a pole at $Ae^{j\omega_o}$ (near $|z| = 1$).
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1. Start on the unit circle at $\omega = 0 \rightarrow z = e^{j\omega} = 1$.
 2. Trace along the unit circle counterclockwise (increasing ω)
 3. When pass a zero at $Ae^{j\omega_m}$, $A \approx 1$, gain dips at ω_m .
 4. When pass a pole at $Ae^{j\omega_n}$, $A \approx 1$, gain peak at ω_n .
 5. On unit circle $z = e^{j\omega}$: $z = 1 \rightarrow DC$; $z = j \rightarrow \omega = \frac{\pi}{2}$; $z = -1 \rightarrow \omega = \pi$.
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EX: Zeros: $\{e^{\pm j\pi/4}, e^{\pm j\pi/2}, e^{j\pi}\}$. **Poles:** $\{0.8e^{\pm j\pi/3}, 0.8e^{\pm j2\pi/3}\}$. **Gain:**

1. Note gain dips to zero at $\omega = \pm\frac{\pi}{4} = \pm 0.785$ and $\omega = \pm\frac{\pi}{2} = \pm 1.57$.
2. Note peaks at $\omega = \pm\frac{\pi}{3} = \pm 1.05$ and $\omega = \pm\frac{2\pi}{3} = \pm 2.10$
3. The closer a zero or pole is to unit circle, the sharper the peak or dip.
4. The closer pole is to unit circle, the longer $h[n]$ takes to decay to zero.
5. Need all poles inside the unit circle for the system to be BIBO stable.

