FORWARD OR DIRECT Z-TRANSFORM

DEF: $\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$ (compare to $\mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$).
Huh? *Finite*-duration signal $x[n]$ →polynomial $X(z)$ with coefficients=$x[n]$.
Or: *Infinite*-duration $x[n]$=coefficients of Laurent (power) series $X(z)$.

Finite $x[n] = \{\underline{3}, 1, 4, 2, 5\}$. Note $x[0] = 3$ and finite duration=5→
length $X(z) = 3 + 1z^{-1} + 4z^{-2} + 2z^{-3} + 5z^{-4} = (3z^4 + z^3 + 4z^2 + 2z + 5)/z^4$.
signal $\mathcal{Z}\{\delta[n-D]\} = z^{-D}$ for any integer delay $D \geq 0$. $\mathcal{Z}\{\delta[n]\} = 1$.

Expon- $\mathcal{Z}\{a^n u[n]\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n = 1/(1-az^{-1}) = z/(z-a)$.
ential EX: $\mathcal{Z}\{(\frac{1}{2})^n u[n]\} = \frac{1}{1-\frac{1}{2}z^{-1}}$. EX: $\mathcal{Z}\{(-\frac{1}{3})^n u[n]\} = \frac{1}{1+\frac{1}{3}z^{-1}}$.

Sinu- $\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \frac{1}{2}\mathcal{Z}\{e^{j\omega_0 n}u[n]\} + \frac{1}{2}\mathcal{Z}\{e^{-j\omega_0 n}u[n]\} =$ (since linear)
soids $\frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}} = \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} = \frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$.

Linear: $\mathcal{Z}\{\{\underline{3}, 1, 4\} + u[n]\} = \frac{3z^2+1z+4}{z^2} + \frac{z}{z-1} = \frac{4z^3-2z^2+3z-4}{z^3-z^2} = \frac{\text{RATIONAL}}{\text{FUNCTION}}$.
Delay: If $\mathcal{Z}\{x[n]\} = X(z)$ and $D \geq 0$, then $\mathcal{Z}\{x[n-D]\} = z^{-D}X(z)$.
EX: $\mathcal{Z}\{u[n] - u[n-2]\} = \frac{z}{z-1} - z^{-2}\frac{z}{z-1} = \frac{z}{z^2}\frac{z^2-1}{z-1} = 1 + z^{-1}$.
Note: Note that $u[n] - u[n-2] = \delta[n] + \delta[n-1]$, so these are consistent.

Also: $\mathcal{Z}\{\frac{1}{n}u[n-1]\} = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} = -\log(1 - z^{-1})$ if you recognize this.
Note: Convolution↔polynomial multiplication: $\mathcal{Z}\{x[n] * y[n]\} = X(z)Y(z)$.

Eigen- $z^n \rightarrow \boxed{h[n]} \rightarrow z^n H(z)$: z^n in→ scaled z^n out. $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$.
funcs $h[n]*z^n = \sum h[i]z^{n-i} = z^n \sum h[i]z^{-i} = z^n H(z)$. $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$.
of LTI Note z^n plays the same role that e^{st} plays in continuous time.
Note: $H(z)$=**transfer function**. $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$=**frequency response**.

EX: Compute step response (to $u[n]$) of LTI system with $h[n] = \{\underline{2}, -3, 1\}$.
Soln: We need to compute $y[n] = \{\underline{2}, -3, 1\} * u[n]$. **Do this two ways:**
#1: $y[n] = (2\delta[n] - 3\delta[n-1] + 1\delta[n-2]) * u[n]$. Using $u[n] * \delta[n] = u[n]$,
#1: $y[n] = 2u[n] - 3u[n-1] + u[n-2] = \{\underline{2}, -1\}$ (try it!) has duration=2.
#2: $Y(z) = H(z)U(z) = (2-3z^{-1}+z^{-2})\frac{1}{1-z^{-1}} = 2-z^{-1} \rightarrow y[n] = \{\underline{2}, -1\}$.
Note: This system has wiped out the step input! This seldom happens.

APPLICATIONS OF THE z-TRANSFORM

Given: $x[n]=3^n u[n] \rightarrow \boxed{y[n]-2y[n-1]=x[n-1]-x[n-2]} \rightarrow y[n]$

Goal: Compute the response $y[n]$ of the system to this particular input.

$\mathcal{Z}$: $Y(z)-2z^{-1}Y(z)=z^{-1}X(z)-z^{-2}X(z)$ and $X(z)=\frac{z}{z-3}$ here

$\rightarrow Y(z)=\frac{z^{-1}-z^{-2}}{1-2z^{-1}}\frac{z}{z-3}=\frac{z-1}{(z-2)(z-3)}=\frac{2}{z-3}-\frac{1}{z-2}$ $\quad [2=\frac{3-1}{3-2}; -1=\frac{2-1}{2-3}]$

$\rightarrow y[n]=[2(3)^{n-1}-(2)^{n-1}]u[n-1]=\substack{\text{FORCED}\\ \text{RESPONSE}}\left(\substack{\text{like}\\ x[n]}\right)+\substack{\text{NATURAL}\\ \text{RESPONSE}}\left(\substack{\text{like}\\ h[n]}\right)$.

Given: $x[n]=(\frac{1}{2})^n u[n]\rightarrow\boxed{\textbf{LTI}}\rightarrow y[n]=\{\underline{0},0,1\}=\delta[n-2]$

Goal: Compute the response of this system to $2\cos(\frac{\pi}{3}n)$.

H(z): $\substack{\text{TRANSFER}\\ \text{FUNCTION}}=H(z)=\mathcal{Z}\{y[n]\}/\mathcal{Z}\{x[n]\}=z^{-2}/[z/(z-\frac{1}{2})]=(z-\frac{1}{2})/z^3$.

h[n]: $\substack{\text{IMPULSE}\\ \text{RESPONSE}}=h[n]=\mathcal{Z}^{-1}\{(z-\frac{1}{2})/z^3\}=\mathcal{Z}^{-1}\{z^{-2}-\frac{1}{2}z^{-3}\}=\{\underline{0},0,1,-\frac{1}{2}\}$.

H(w): $\substack{\text{FREQUENCY}\\ \text{RESPONSE}}=H(\omega)=H(z)|_{z=e^{j\omega}}=(e^{j\omega}-\frac{1}{2})/e^{j3\omega}$.

$\omega=\frac{\pi}{3}$: $H(\frac{\pi}{3})=[e^{j\pi/3}-\frac{1}{2}]/e^{j3\pi/3}=(\frac{1}{2}+j\frac{\sqrt{3}}{2}-\frac{1}{2})/(-1)=-j\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}e^{-j\pi/2}$.

Sol'n: $2\cos(\frac{\pi}{3}n)\rightarrow\boxed{\textbf{LTI}}\rightarrow\sqrt{3}\cos(\frac{\pi}{3}n-\frac{\pi}{2})=\sqrt{3}\sin(\frac{\pi}{3}n)$.

Given: $x[n]\rightarrow\boxed{y[n]=x[n]-\frac{3}{4}x[n-1]+\frac{1}{8}x[n-2]}\rightarrow y[n]$

Huh? $x[n]$=cell phone signal. $y[n]$=multipath due to buildings.

Goal: Compute the **inverse filter** that recovers $x[n]$ from $y[n]$:

Huh? $x[n]\rightarrow\boxed{h[n]}\rightarrow y[n]\rightarrow\boxed{g[n]}\rightarrow x[n]$. That is, $g[n]$ *undoes* $h[n]$.

Idea: Systems in cascade (series) $\Leftrightarrow h[n]*g[n]=\delta[n]\Leftrightarrow H(z)G(z)=1$.

Here: $h[n]=\{\underline{1},-\frac{3}{4},\frac{1}{8}\}\rightarrow H(z)=1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}=(z^2-\frac{3}{4}z+\frac{1}{8})/z^2$.

$\rightarrow G(z)=1/H(z)=z^2/[z^2-\frac{3}{4}z+\frac{1}{8}]=z^2/[(z-\frac{1}{2})(z-\frac{1}{4})]$.

$\mathcal{Z}^{-1}$: $\frac{G(z)}{z}=\frac{z}{(z-1/2)(z-1/4)}=\frac{2}{z-1/2}-\frac{1}{z-1/4}\rightarrow G(z)=2\frac{z}{z-1/2}-1\frac{z}{z-1/4}$.

using: residues $2=(1/2)/[(1/2)-(1/4)]$ and $-1=(1/4)/[(1/4)-(1/2)]$.

g[n]: $g[n]=2(\frac{1}{2})^n u[n]-(\frac{1}{4})^n u[n]$=inverse filter for original system.

Note: Stable since zeros of H(z)=poles of G(z) are inside unit circle.

Note: $g[0]\neq 0$ since both $\substack{\text{numerator \&}\\ \text{denominator}}$ of $G(z)$ have the same degrees.